GANIT

Available online at https://ganitjournal.bdmathsociety.org/

GANIT: Journal of Bangladesh Mathematical Society

GANIT *J. Bangladesh Math. Soc.* 44.1 (2024) 1–22
DOI: https://doi.org/10.3329/ganit.v44i1.71859

Lyapunov Functions and Qualitative Analysis of an Epidemic Model with Vaccination

Md. Saiful Islam[1], Kazi Mehedi Mohammad*[2], Md. Mashih Ibn Yasin Adan[1], and Md. Kamrujjaman[2]

[1]Department of Mathematics, Bangabandhu Sheikh Mujibur Rahman University, Kishoreganj, Bangladesh
[2]Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh

ABSTRACT

The spatial-temporal diffusion dynamics of infectious disease with vaccination therapy are studied through a mathematical model. We have investigated the well-posedness, disease-free equilibrium, disease equilibrium, the existence and the uniqueness of solutions, and the calculation of basic reproduction numbers by Jacobian matrix. After that, the positivity, as well as boundedness of solutions, are also established. The global stability of disease-free and steady-state disease results is established by utilizing compatible Lyapunov functions and LaSalle's invariance principle. Illustration of the numerical examples to show the dynamics of different population groups over time. The effects of different parameters on the compartments are shown in detail.

Received: February 07, 2024 **Accepted:** February 27, 2024 **Published Online:** March 13, 2024 (Online First)

Keywords: Lyapunov functions; LaSalle's invariance; global stability; uniform persistence; vaccination.
AMS Subject Classifications 2020: 92D30, 92D25, 93D05, 93D20, 93C20, 76E30.

1 Introduction

To control infectious diseases, vaccination can play a vital role. Early studies showed that mathematical model helps to understand the effect of vaccination on controlling the disease. The aim of proposing mathematical models is to understand epidemiological patterns and predict the outcome of the disease. In our daily routine life, we face many serious types of infectious diseases, thus reminding us to construct a reasonable mathematical model. Control the infectious diseases has become increasingly important in recent years. Including all the strategies, vaccinations are the most important strategy to control the diseases such as avian influenza, polio, diphtheria, and measles.

Vaccination plays an important role in eliminating infectious diseases. Several studies reflects that mathematical modeling can help understand the role of the effect of vaccination in controlling infectious diseases. The following epidemic model of vaccination with non-linearity was suggested in [1, 2].

$$\frac{dS}{dt} = \Pi - \frac{c\beta_1 I}{1+I}S - \varphi S + \alpha I - \mu S$$

*Corresponding author: K. M. Mohammad, Email: mehedimim.me@gmail.com

$$\begin{aligned}\frac{dV}{dt} &= \varphi S - \frac{c\beta_2 I}{1+I}V - \mu V \\ \frac{dI}{dt} &= \frac{c\beta_1 I}{1+I}S + \frac{c\beta_2 I}{1+I}V - \alpha I - \mu I.\end{aligned} \tag{1.1}$$

In this model, $S(t), V(t)$, and $I(t)$ represent susceptible, vaccinated and infectious individuals, respectively. The parameter Π represents the recruitment rate of susceptible entity, β_1 and β_2 represent transmission probabilities of susceptible and vaccinated entities, α represents the therapeutic treatment coverage of infectious entity. Whereas vaccination can prevent or eradicate the spread of infectious, the authors also observed (realistically) that $\beta_2 \leq \beta_1$. The parameters φ, and μ stands for the vaccination coverage of susceptibles and the natural deaths.

In [1], demonstrated the local stability analysis of the disease and disease-free equilibrium of the model (1.1). The findings revealed the vaccine coverage threshold for disease control and elimination. In [3, 14] established compatible Lyapunov functions and utilized LaSalle's invariance principle for the global stability analysis. The study also highlighted the most effective immunization and treatment techniques for reducing disease burden and requiring intervention. Recently numerous studies [4, 5, 6, 7] showed that spatial is also the key issue in infectious disease modeling. In [8] Smith introduced basic definitions and tools regarding to dynamical systems and suggested basic relations to partially ordered metric space. Lyapunov Mappings and Analysis of a Nonlinear Spatio-temporal Epidemic Model are carried out by Kamrujjaman et al. in [10]. Further in [11], Ahmed et al. examined the optimal treatment strategies to control acute HIV infection. Lou and Zhao have investigated several theories and numerical results regarding reaction-diffusion malaria model with incubation period in the vector population in [12]. After that, analysis of the global asymptotics in some quasimonotone reaction–diffusion systems with delays are examined by Freedman and Zhao in [13]. This study has constructed the modified spatial-temporal diffusive vaccination model, where individual movements are considered for all three compartments. The following model is

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = \mu_1 \Delta S + \lambda - (\epsilon+\gamma)S - \beta\dfrac{I}{1+I}S + \delta I \\ \dfrac{\partial V(t,x)}{\partial t} = \mu_2 \Delta V + \gamma S - \alpha\beta\dfrac{I}{1+I}V - \epsilon V \\ \dfrac{\partial I(t,x)}{\partial t} = \mu_3 \Delta I + \beta(S+\alpha V)\dfrac{I}{1+I} - \epsilon I - \delta I \end{cases} \tag{1.2}$$

here $t \in [0, \infty)$ and $x \in \Omega$. The zero Neumann boundary conditions of this system is

$$\frac{\partial S(t,x)}{\partial n} = \frac{\partial V(t,x)}{\partial n} = \frac{\partial I(t,x)}{\partial n} = 0, \quad \forall\, x \in \partial\Omega \tag{1.3}$$

as well as the initial conditions are

$$S(0,x) = \psi_1(x) \geq 0, \quad V(0,x) = \psi_2(x) \geq 0, \quad I(0,x) = \psi_3(x) \geq 0, \quad x \in \Omega \tag{1.4}$$

where Ω represent the domain where the smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$; moreover $\psi_i(x)$ represents the non-negative Hölder continuous bounded functions that defined on Ω. We have assumed the susceptible class $S(t,x)$, consists of the individuals who are capable of becoming infected, the vaccinated class $V(t,x)$, consists of individuals who have been vaccinated, and the infective class $I(t,x)$, consists of the individuals who are capable to transmit the disease at time t and Ω. Here μ_1, μ_2, μ_3 represent the diffusion coefficients for susceptible, infected and vaccinated populations, respectively. All over the paper, we have assumed that the parameter λ is the recruitment rates, ϵ is the natural death rates, and β is the transmission rates of infections. The parameter, δ represents the constant recovery rate. The vaccination rates infection rates by a factor of α, thus $\alpha = 0$ indicates that the vaccine is totally efficient in preventing infections, while $\alpha = 1$ indicates that the vaccine is completely ineffective. The parameter γ represents the susceptible entity that is vaccinated at a constant rate. and n is an outward normal vector on the boundary $\partial\Omega$.

The following figure represents the schematic diagram of the model (1.2).

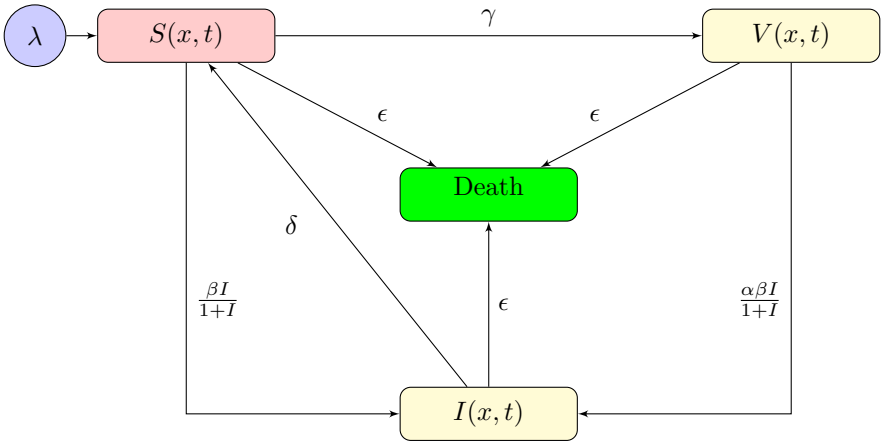

Figure 1.1: Schematic diagram of the model (1.2).

Table 1: The system parameters and their interpretations

Parameter	Interpretations	Parameter	Interpretations
μ_i	Diffication rate	ϵ	Natural death rate
λ	Recruitment rate of S class	δ	Constant recovery rate
β	Transmission rate of infection from I class	γ	Susceptible individual is vaccinated at a constant rate
φ	Vaccination coverage of susceptible individuals	α	Effect of the vaccine to reduce the infection rate

The main novelties in this study are both analytical and theoretical results to establish and present in terms of reproductive ratio, $\mathcal{R}_0$ to understand the severity. To complete this paper, firstly, we have studied the well-posedness, existence, and uniqueness of our solution. After covering preliminaries, we determine the disease-free equilibrium, calculating the basic reproduction number; we have rigorously utilized the next generation matrix and disease equilibrium. Moreover, we have established two compatible Lyapunov functions of the model (1.2) and utilized LaSalle's principle for global stability of the two constant equilibria E_0 and E^*. We also obtained the stable disease-free equilibrium for $\mathcal{R}_0 < 1$ and endemic solution as long as $\mathcal{R}_0 > 1$. The existence of solutions and the model's uniform persistence outcomes are investigated. Finally, we have illustrated numerical examples to justify the theoretical results. Our results demonstrate that the threshold value $\mathcal{R}_0$ determines the model's global dynamics.

2 Preliminaries

The goal of the following section is to present the well-posedness of our model (1.2), disease-free equilibrium (DFE), basic reproduction number, and disease equilibrium (DE), which are further considered in the following sections of the study, allowing us to study in the field.

2.1 The well-posedness

The well-posedness of the initial value problem of the model (1.2) where the following initial values are

$$\begin{cases} S(0,x) = S^{\theta}(x) \geq 0 & \text{in } \Omega, \\ V(0,x) = V^{\theta}(x) \geq 0 & \text{in } \Omega, \\ I(0,x) = I^{\theta}(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

with the zero Neumann boundary conditions,

$$\frac{\partial S}{\partial n}(t,x) = \frac{\partial V}{\partial n}(t,x) = \frac{\partial I}{\partial n}(t,x) = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

The boundary condition has the following meaning

- The population is separated from the rest of the world by the bounded territory, which means they cannot get in or go out.
- Individuals going in and out at any location from the boundary stay equal at all times.

2.2 Disease-free steady state

Using the disease-free equilibrium $E_0(S_0, V_0, I_0)$ of the model (1.2), we get

$$\begin{aligned}
&\mu_1 \Delta S_0 + \lambda - (\epsilon+\gamma)S_0 - \beta\frac{I_0}{1+I_0}S_0 + \delta I_0 = 0\\
&\mu_2 \Delta V_0 + \gamma S_0 - \alpha\beta\frac{I_0}{1+I_0}V_0 - \epsilon V_0 = 0\\
&\mu_3 \Delta I_0 + \beta(S_0 + \alpha V_0)\frac{I_0}{1+I_0} - (\epsilon+\delta)I_0 = 0.
\end{aligned}$$

Take into account the diffusion rates $\mu_i = 0$ for $i = 1, 2, 3$, moreover infectious entity $I_0 = 0$ for determining the disease-free equilibrium, yields

$$\begin{aligned}
\lambda &= (\epsilon+\gamma)S_0\\
\gamma S_0 &= \epsilon V_0
\end{aligned}$$

Therefore, the disease-free steady state of the model (1.2) is

$$E_0 = \left(\frac{\lambda}{\epsilon+\gamma}, \frac{\lambda\gamma}{\epsilon(\epsilon+\gamma)}, 0\right). \tag{2.3}$$

2.3 Determination of basic reproduction ratio

At the E_0 the Jacobian matrix of the linearized model (1.2) is

$$J = \begin{pmatrix}
-(\epsilon+\gamma) & 0 & -\dfrac{\lambda\beta}{\epsilon+\gamma} + \delta\\
\gamma & -\epsilon & -\dfrac{\lambda\alpha\beta\gamma}{\epsilon(\epsilon+\gamma)}\\
0 & 0 & \dfrac{\lambda\beta}{\epsilon+\gamma} + \dfrac{\lambda\alpha\beta\gamma}{\epsilon(\epsilon+\gamma)} - (\epsilon+\delta)
\end{pmatrix}$$

with eigenvalues

$$\begin{aligned}
\lambda_1 &= -(\epsilon+\lambda) < 0\\
\lambda_2 &= -\epsilon < 0\\
\lambda_3 &= \frac{\lambda\beta}{\epsilon+\gamma} + \frac{\lambda\alpha\beta\gamma}{\epsilon(\epsilon+\gamma)} - (\epsilon+\delta).
\end{aligned}$$

Whereas the model parameters are positive, thus it is easily seen that $\lambda_1, \lambda_2 < 0$. Hence the steady state E_0 is locally asymptotically stable which provides $\lambda_3 < 0$. Therefore, we have obtained the basic reproduction ratio or number of the model (1.2) is

$$\mathcal{R}_0 = \frac{\lambda\beta}{(\epsilon+\gamma)(\epsilon+\delta)} + \frac{\lambda\alpha\beta\gamma}{\epsilon(\epsilon+\gamma)(\epsilon+\delta)}. \tag{2.4}$$

2.4 Disease steady state

To evaluate the disease steady state of the system (1.2), we have set the partial derivative and diffusion coefficients to zero and replaced the variable as $(S, V, I) \equiv (S^*, V^*, I^*)$

$$\lambda - (m+\gamma)S^* - \beta\frac{I^*}{1+I^*}S^* + \delta I^* = 0 \tag{2.5}$$

$$\gamma S^* - \alpha\beta\frac{I^*}{1+I^*}V^* - \epsilon V^* = 0 \tag{2.6}$$

$$\beta(S^* + \alpha V^*)\frac{I^*}{1+I^*} - (\epsilon+\delta)I^* = 0 \tag{2.7}$$

From equation (2.5) we have obtain

$$\begin{aligned}
&\lambda - (\epsilon+\gamma)S^* - \beta\frac{I^*}{1+I^*}S^* + \delta I^* = 0\\
&\therefore S^* = \frac{(1+I^*)(\lambda+\delta I^*)}{\beta I^* + (\epsilon+\gamma)(1+I^*)}
\end{aligned} \tag{2.8}$$

From equation (2.6) we have get

$$\begin{aligned}
&\gamma S^* - \alpha\beta\frac{I^*}{1+I^*}V^* - \epsilon V^* = 0\\
\Rightarrow &V^* = \frac{\gamma S^*(1+I^*)}{\alpha\beta I^* + \epsilon(1+I^*)}\\
&\therefore V^* = \frac{\gamma(\lambda+\delta I^*)(1+I^*)^2}{(\beta I^* + (m+\gamma)(1+I^*))(\alpha\beta I^* + \epsilon(1+I^*))}
\end{aligned} \tag{2.9}$$

From equation (2.7) we have found

$$\begin{aligned}
&\beta(S^* + \alpha V^*)\frac{I^*}{1+I^*} - (\epsilon+\delta)I^* = 0\\
\Rightarrow &\beta\frac{S^*}{1+I^*} + \alpha\beta\frac{V^*}{1+I^*} = \epsilon+\delta\\
\Rightarrow &\frac{\beta(\lambda+\delta I^*)}{\beta I^* + (\epsilon+\gamma)(1+I^*)} + \frac{\alpha\beta\gamma(\lambda+\delta I^*)(1+I^*)}{(\beta I^* + (\epsilon+\gamma)(1+I^*))(\alpha\beta I^* + \epsilon(1+I^*))} = \epsilon+\delta\\
\Rightarrow &\frac{\beta(\lambda+\delta I^*)(\alpha\beta I^* + \epsilon(1+I^*)) + \alpha\beta\gamma(\lambda+\delta I^*)(1+I^*)}{(\beta I^* + (m+\gamma)(1+I^*))(\alpha\beta I^* + \epsilon(1+I^*))} = \epsilon+\delta
\end{aligned} \tag{2.10}$$

The following polynomial of degree two obtain from the above equation (2.10)

$$\tau_2(I^*)^2 + \tau_1 I^* + \tau_0 = 0, \tag{2.11}$$

$$\begin{aligned}
\tau_2 &= -\epsilon(\epsilon^2 + \gamma\epsilon + \beta\epsilon + \alpha\beta\epsilon + \alpha\beta\gamma + \alpha\beta^2) - \delta(\epsilon^2 + \gamma\epsilon + \alpha\beta\epsilon),\\
\tau_1 &= \lambda(\beta m + \alpha\beta\gamma + \alpha\beta^2) - \epsilon(2\epsilon^2 + 2\gamma\epsilon + \beta\epsilon + \alpha\beta\epsilon + \alpha\beta\gamma) - \delta(2\epsilon^2 + 2\gamma\epsilon + \alpha\beta\epsilon),\\
\tau_0 &= \gamma\beta(\epsilon + \alpha\gamma) - \epsilon(\epsilon+\delta)(\epsilon+\gamma).
\end{aligned}$$

The count of infectious persons I^* is defined by the real positive roots of (2.11). As a result, we have obtained the model (1.2) disease steady state $E^*(S^*, V^*, I^*)$.

3 Existence and uniqueness of solution

Let us assume that $\mathcal{X}$ is a Banach space and be defined as $\mathcal{X} := \mathcal{C}(\overline{\Omega}, \mathbb{R}^3)$ with the supremum norm $\|\cdot\|_{\mathcal{X}}$. The notation $\mathcal{C}(\overline{\Omega}, \mathbb{R}^3)$ represents the space of continuous functions from the closure of Ω to $\mathbb{R}^3$. Now, in the study, the notation $\mathcal{X}_+$ is used to represent the positive part of a Banach space $\mathcal{X}$. If $\mathcal{X}$ is a Banach space, then $\mathcal{X}_+$ would typically refer to the subset of $\mathcal{X}$ consisting of all non-negative elements in $\mathcal{X}$. When ν is non-negative, let us define

$$C := C([-\nu, 0], \mathcal{X}) \text{ and } \|\varphi\| = \max_{\vartheta \in [-\nu, 0]} \|\varphi(\vartheta)\|_{\mathcal{X}},$$

thus, we can say that C is also a Banach space. Also define,

$$\mathcal{X}^+ := C(\overline{\Omega}, \mathcal{X}_+) \text{ and } C^+ := C([-\tau, 0], \mathbb{R}_+).$$

Therefore both (C, C^+) and $(\mathcal{X}, \mathcal{X}^+)$ are strongly order space. Assume

$$\mathcal{H}_i(t) : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R}) \qquad \forall\, i = 1, 2, 3$$

is the C_0 strongly continuous semigroups where $\mu_1\Delta - (m + \gamma), \mu_2\Delta - m$ and $\mu_3\Delta - (m + c)$ are attached with the Neumann boundary condition. Hence, we obtain $\mathcal{H}_i(t) : \mathcal{X} \to \mathcal{X}, i = 1, 2, 3$ is compact and strongly positive where $t > 0$. Clearly, for each $\varphi \in C(\Omega, \mathbb{R})$ where $t \geq 0$, yields

$$\begin{aligned}
\mathcal{H}_1(t)\varphi(x) &= e^{-(m+\gamma)t} \int_\Omega \Gamma_1(t, x, y)\varphi(y)\mathrm{d}x, \\
\mathcal{H}_2(t)\varphi(x) &= e^{-mt} \int_\Omega \Gamma_2(t, x, y)\varphi(y)\mathrm{d}x, \\
\mathcal{H}_3(t)\varphi(x) &= e^{-(m+c)t} \int_\Omega \Gamma_3(t, x, y)\varphi(y)\mathrm{d}x,
\end{aligned}$$

here Γ_1, Γ_2 and Γ_3 are the Green functions which are associated with $\mu_1\Delta$, $\mu_2\Delta$ and $\mu_3\Delta$ respectively. From [8], we obtain $\mathcal{H}_i(t) : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})$, $\forall\, i = 1, 2, 3,$ and $t > 0$, is compact as well as strongly positive. In particular, $\mathcal{H}_i(t) : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R}), \forall\, i = 1, 2, 3,$ and $t \geq 0$, be a strongly continuous semigroup.

Since, $\mathcal{J}_i : M(\mathcal{J}_i) \to \mathcal{X}$ is the generator of N_i, $\forall\, i = 1, 2, 3$, then $\mathcal{H}(t) = (\mathcal{H}_1(t), \mathcal{H}_2(t), \mathcal{H}_3(t)) : \mathcal{X} \to \mathcal{X}$ be a semigroup formed by the operator $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ and defined on $M(\mathcal{J}) := M(\mathcal{J}_1) \times M(\mathcal{J}_2) \times M(\mathcal{J}_3)$. For each $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{X}$, we will define $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) : \mathcal{X}^+ \to \mathcal{X}$ as follows:

$$\begin{aligned}
\mathcal{W}_1(\varphi)(x) &= \lambda - \beta\frac{\varphi_3(x)}{1 + \varphi_3(x)}\varphi_1(x) - (m + \gamma)\varphi_1(x) + \delta\varphi_3(x), && \forall\, x \in \Omega \\
\mathcal{W}_2(\varphi)(x) &= \gamma\varphi_1(x) - \alpha\beta\frac{\varphi_3(x)}{1 + \varphi_3(x)}\varphi_2(x) - \epsilon\varphi_1(x), && \forall\, x \in \Omega \\
\mathcal{W}_3(\varphi)(x) &= \beta\frac{\varphi_3(x)}{1 + \varphi_3(x)}\varphi_1(x) + \alpha\beta\frac{\varphi_3(x)}{1 + \varphi_3(x)}\varphi_2(x) - (\epsilon + \delta)\varphi_3(x), && \forall\, x \in \Omega.
\end{aligned}$$

for $x \in \overline{\Omega}$ as well as $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{X}$. We can now write (1.2) and (1.3)-(1.4) by the following way

$$u(t) = \mathcal{H}(t)\varphi + \int_0^t \mathcal{H}(t - s)\mathcal{W}(u(s))\mathrm{d}s,$$

here,

$$u(t) = \begin{pmatrix} S(t) \\ V(t) \\ I(t) \end{pmatrix}, \mathcal{H}(t) = \begin{pmatrix} \mathcal{H}_1(t) & 0 & 0 \\ 0 & \mathcal{H}_2(t) & 0 \\ 0 & 0 & \mathcal{H}_3(t) \end{pmatrix}.$$

Alternatively, we can write in the following way

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathcal{J}\mathbf{u} + F(\mathbf{u}), \quad t > 0, \\ \mathbf{u}_0 = \varphi \in \mathcal{X}^+, \end{cases} \tag{3.1}$$

here, $\mathbf{u} = (S, V, I)$ and $\varphi = (S_0, V_0, I_0)$. Because, $F(\varphi)$ is local Lipschitz continuous on $\mathcal{X}^+$, then it follows that each $\varphi \in \mathcal{X}^+$, (3.1) provides for such an another unique non-continuous mild solution $u(\cdot, t, \varphi)$ as a result $u(\cdot, t, \varphi) \in X$ for all t in its maximum interval of existence. Furthermore, ([9], Corollary 2.2.5) shows that for all $t > 0, u(\cdot, t, \varphi)$ is a class solution of (1.2) with Neumann boundary condition (1.3). Further, from ([9], Corollary 2.2.5), we obtain $u(\cdot, t, \varphi)$ is a class solution of (1.2). Additionally, we can conclude from the equation in (1.2) that S, V and I are non-negative using the scalar parabolic maximum principle.

As a consequence, on the solution of systems (1.2) and (1.3)-(1.4), we gain the following fundamental results:

Lemma 1. *If $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{X}^+$ is the initial value function then the system (1.2) has a unique solution $u(x, t, \varphi)$ on $[0, \sigma_\varphi)$ with $u(x, t, \varphi) = \varphi$ and $u(\cdot, t, \varphi) \in \mathcal{X}^+$, for all $t \in [0, \sigma_\varphi)$, where $\sigma_\varphi \leq \infty$.*

Now, we demonstrate that the solution to systems (1.2) with the initial value function $\varphi \in \mathcal{X}^+$ persists worldwide, i.e. $\sigma = \infty$. Providing it, we shall utilize the following ([12], Lemma 1).

Take into account the following system

$$\begin{cases} v_t(t, x) = D\Delta v(t, x) - \mathrm{d}v(t, x) + \mathcal{J}, & x \in \Omega \;\; t > 0, \\ \dfrac{\partial v}{\partial n}(t, x) = 0, & x \in \partial\Omega \;\; t > 0, \end{cases} \tag{3.2}$$

where $\mathcal{J}, D$ and d are positive constants.

Lemma 2. *The above system (3.2) obeys the unique positive equilibrium point $u^* = \frac{\mathcal{J}}{d}$ that is globally attractive in $C(\Omega, \mathbb{R})$.*

Proof. Denote $\mathcal{J}_1 = \min_{x \in \delta\Omega}$ and $\mathcal{J}_2 = \max_{x \in \delta\Omega}$. It is simple to understand that for every $\varphi \in C(\partial\Omega, \mathbb{R}_+)$, (3.2) has the unique solution $v(t, \varphi)$ on $[0, \infty)$ with $v(0, \varphi) = \varphi$. Consider $q(t)$ is the solution semiflow concerned with (3.2), which is, $Q(t)\varphi = v(t, \varphi)$. According to the standard comparison arguments, the omega limit set $n(\varphi)$ gratifies for every $\varphi \in Y$,

$$n(\varphi) \subset \left\{ \varphi : \frac{\mathcal{J}_1}{d} \leq \varphi \leq \frac{\mathcal{J}_2}{d} \right\} \tag{3.3}$$

Using the comparison principle once more, we have $Q(t)\varphi = Q(t)\varphi$, $t > 0$, for all $\varphi > \varphi$.

Note that $\aleph(x, w) := k(x) - dw$ in the sense that it is strictly subhomogeneous $\aleph(x, \alpha w) > \alpha\aleph(x, w)$ for all $\alpha \in (0, 1)$ and $w >> 0$. In the analogous manner, $\aleph(x, \alpha w) > \alpha\aleph(x, w)$ for all $\alpha \in (0, 1)$ as in ([13], Theorem 2.2), we clearly see that $Q(t)\varphi$ is strictly subhomogeneous such that $Q(t)\alpha\varphi > \alpha Q(t)\varphi$ for all $\alpha \in (0, 1)$ and $\varphi >> 0$. Then from ([15] Theorem 2.3.1) $Q(t)$ has a positive steady state $w^*(x)$ which implies $n(\varphi) = w^* \in Y, \forall \varphi \in Y$. Typically, for $k(x) \equiv k$, $\forall\, x \in \partial n$, we obtain $w^* = \frac{k}{d}$. □

Theorem 1. *If $\varphi \in \mathcal{X}^+$ is the initial value, then the system (1.2) has the unique solution $u(\cdot, t, \varphi)$ on $[0, \infty)$, moreover the solution semiflow $\Phi(t) := u(\cdot, t) : \mathcal{X}^+ \to \mathcal{X}^+$ where $t \geq 0$, has a global compact attractor in $\mathcal{X}^+$.*

Proof. From Lemma 1, the model (1.2) has the unique solution $u(\cdot, t, \varphi)$ on $[0, \sigma_\varphi)$ and $u(x, t, \varphi) \geq 0$ where $t \in [0, \sigma_\varphi)$ and $x \in \Omega$. For the reaction-diffusion equation below, Lemma(2) shows that $\frac{\lambda}{m+\gamma}$ is a global attractor.

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x) = \mu_1 \Delta v(t, x) + \lambda - (\epsilon + \gamma)v(t, x) + \delta v(t, x), & x \in \Omega \;\; t > 0 \\ \dfrac{\partial v}{\partial n}(t, x) = 0, & x \in \partial\Omega \;\; t > 0. \end{cases}$$

When,

$$\frac{\partial S}{\partial t}(t,x) \leq \mu_1 \Delta S(t,x) + \lambda - (\epsilon+\gamma)S(t,x), \quad t \in [0,\sigma_\varphi),\ x \in \Omega, \tag{3.4}$$

$S(\cdot, t.\varphi)$ would be bounded on $[0,\sigma_\varphi)$, according to the basic parabolic comparison theorem [8]. As a result, there is an

$$M_1 := \max\left\{\frac{\lambda}{\epsilon+\gamma}+1, \max_{x\in\overline{\Omega}}\varphi_1(x)\right\} > 0.$$

After that, we have

$$\frac{\partial V}{\partial t}(t,x) \leq \mu_2 \Delta V(t,x) + \gamma M_1 - \epsilon V(t,x), \quad t \in [0,\sigma_\varphi),\ x \in \Omega.$$

As a result, there exist an M_2 such that

$$V(t,x) \leq M_2, \quad t \in [0,\sigma_\varphi),\ x \in \Omega.$$

where $M_2 := \max\left\{\frac{\gamma M_1}{\epsilon}+1, \max_{x\in\overline{\Omega}}\ \varphi_2(x)\right\}$. Further,

$$\frac{\partial I}{\partial t}(t,x) \leq \mu_3 \Delta I(t,x) + \beta M_1 + \alpha\beta M_2 - \epsilon I(t,x) - \delta I(t,x), \quad t \in [0,\sigma_\varphi),\ x \in \Omega.$$

Thus, there exist an $M_3 := \left\{\frac{\beta M_1 + \alpha\beta M_2}{\epsilon+\delta}+1, \max_{x\in\overline{\Omega}}\ \varphi_3\right\}$ such that

$$I(t,x) \leq M_3, \quad t \in [0,\sigma_\varphi),\ x \in \Omega.$$

Thus, on $[0,\infty)$, the solution exists globally. From (3.4), there exists $t_1 = t_1(\varphi) > 0$ which implies that

$$S(t,x) \leq \frac{\lambda}{\epsilon+\gamma} + 1 := B_1\ t \in t_1,\ x \in \overline{\Omega}.$$

Now, using arguments analogous to those given previously, we have proved that there exists $B_2 > 0$ which is independent of $\varphi \in \mathcal{X}_+$ and $t_2 = t_2(\varphi) > 0$, such that

$$V(t,x) \leq B_2, \quad t \in t_2,\ x \in \overline{\Omega},$$
$$I(t,x) \leq B_2, \quad t \in t_2,\ x \in \overline{\Omega}.$$

As a consequence, (1.2) and (1.3)-(1.4) the non-negative solution are finally bounded with respect to the maximum norm which suggests that the point dissipative solution $\Phi(t) : \mathcal{X}^+ \to \mathcal{X}^+$ which is defined by $(\Phi(t)\varphi)(x) = u(x,t,\varphi),\ x \in \Omega$ is semi-flow. From [9], $\Phi(t)$ is compact for any $t > 0$. Hence, [16] which delineates that $\Phi(t) : \mathcal{X}^+ \to \mathcal{X}^+,\ t \geq 0$, has a global compact attract in $\mathcal{X}^+$ which complete the proof. □

4 Positivity and boundedness of model solutions

The positivity and boundedness of the solutions are important characteristics of an epidemiologically relevant model. As a result, it's critical to demonstrate that every variables are non-negative when $t \geq 0$, implying that any solution with positive beginning values will remain positive for $t \geq 0$.

Theorem 2. *Let us assume that $\mu_1 = \mu_2 = \mu_3$ in model (1.2). Then for any initial function $(\psi_1(x), \psi_2(x), \psi_3(x)) \in \mathcal{X}_+$ model (1.2) has a unique, non-negative and bounded solution on $[0,\infty) \times \overline{\Omega}$.*

Proof. It is obvious that model (1.2)'s right hand functions satisfy Lipschitz for (S, I, R) in $\mathcal{X}^+$. Using reasoning similar to those in [17] or ([18], Lemma 2.2), along with [[20], Corollary 4], we have the system (1.2) permits a single local mild solution (S, V, I) defined on the maximum existence interval $[0, S_{max})$, where $0 < S_{max} < \infty$. In addition, solution (S, V, I) is a classic model solution of (1.2). From [[19], Section 10], We can also acquire that $(S(t,x), V(t,x), I(t,x))$ is nonnegative for $t \in [0, S_{max})$.

Define $N(t,x) = S(t,x) + V(t,x) + I(t,x)$ when $\mu_1 = \mu_2 = \mu_3$ we have

$$\begin{cases} \dfrac{\partial N(t,x)}{\partial t} = \mu_1 \Delta N(t,x) + a - \epsilon N(t,x) \\ \dfrac{\partial N(t,x)}{\partial t} = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases} \tag{4.1}$$

According to ([17], Lemma 1), for each solution $N(t,x)$ of equation (4.1), we get $\lim\limits_{x\to\infty} N(t,x) = \dfrac{a}{\epsilon}$ uniformly, whenever $x \in \Omega$. Moreover, $N(t,x)$ is bounded on $[0,\infty) \times \overline{\Omega}$. It follows from the standard theory for semi-linear parabolic systems (see [21]) that $S_{max} = \infty$. This demonstrates that (S, V, I) is defined whenever $(t,x) \in [0,\infty) \times \overline{\Omega}$, as well as being unique and non-negative.
Since $\lim\limits_{x\to\infty} N(t,x) = \dfrac{a}{\epsilon}$, then we have

$$N(t,x) \le \frac{a}{\epsilon} + 1, \qquad \forall (t,x) \in [0,\infty) \times \overline{\Omega} \tag{4.2}$$

Finally, we may deduce from inequality (4.2) that the solution of the model (1.2) is bounded, which completes the proof. □

5 Stability analysis

5.1 Local stability analysis

In this section, we have studied the local stability analysis of the steady state.

Theorem 3.

(i) Disease-free steady state E_0 of the model (1.2) is locally asymptotically stable, if $\mathcal{R}_0 < 1$;

(ii) Disease steady state E^* of the model (1.2) is locally asymptotically stable, if $\mathcal{R}_0 > 1$.

Proof. Linearize the model (1.2) at E_0, yields

$$\frac{\partial \mathbf{v}(t,x)}{\partial t} = \mu \Delta \mathbf{v}(t,x) + \mathcal{M}_1 \mathbf{v}(t,x),$$

here,

$$\mu = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \mathcal{M}_1 = \begin{pmatrix} -\epsilon - \gamma & 0 & -\beta S_0 + \delta \\ \gamma & -\epsilon & -\alpha\beta V_0 \\ 0 & 0 & \beta S_0 + \alpha\beta V_0 - \epsilon - \delta \end{pmatrix}.$$

The following characteristic polynomial can then be obtained

$$|\tilde{\lambda}\mathcal{I} + d\mathcal{L}^2 - \mathcal{M}_1| = 0,$$

here, $\mathcal{I}$ represents 3×3 identity matrix, λ represents the eigenvalue which estimates temporal growth and $\mathcal{L}$ represents the wave-number [4]. Thus,

$$(\tilde{\lambda} + \mu_1\mathcal{L}^2 + \epsilon + \gamma)(\tilde{\lambda} + \mu_2\mathcal{L}^2 + \epsilon)(\tilde{\lambda} + \mu_3\mathcal{L}^2 + \epsilon + \delta - \beta S_0 - \alpha\beta V_0) = 0. \tag{5.1}$$

Now, it is clear that

$$\begin{aligned}\tilde{\lambda}_1 &= -(\mu_1\mathcal{L}^2+\epsilon+\gamma)<0,\\ \tilde{\lambda}_2 &= -(\mu_1\mathcal{L}^3+\epsilon)<0,\\ \text{and}\quad \tilde{\lambda}_3 &= -(\mu_3\mathcal{L}^2+\epsilon+\delta-\beta S_0-\alpha\beta V_0)\\ &= -(\mu_3\mathcal{L}^2+(\epsilon+\delta)(1-\mathcal{R}_0)).\end{aligned}$$

From the eigenvalue $\tilde{\lambda}_3$, we can conclude that when $\mathcal{R}_0<1$, the eigenvalue become $\tilde{\lambda}_3<0$, which completes the first portion of the theorem. For the second portion of the theorem, linearize the model (1.2) at E^*, yields

$$\frac{\partial \mathbf{v}(t,x)}{\partial t} = d\Delta\mathbf{v}(t,x)+\mathcal{M}_2\mathbf{v}(t,x),$$

here,

$$\mathcal{M}_2=\begin{pmatrix} -\epsilon-\gamma-\dfrac{\beta I^*}{1+I^*} & 0 & \delta-\dfrac{\beta S^*}{(1+I^*)^2}\\ \gamma & -\epsilon-\dfrac{\beta\gamma I^*}{1+I^*} & -\dfrac{\beta\gamma V^*}{(1+I^*)^2}\\ \dfrac{\beta I^*}{1+I^*} & \dfrac{\beta\gamma I^*}{1+I^*} & -\epsilon-\delta-\dfrac{\beta(S^*+\alpha V^*)}{(1+I^*)^2}\end{pmatrix}.$$

Then we get the characteristic equation as follows

$$\tilde{\lambda}^3+\mathcal{Z}_1(\mathcal{L}^2)\tilde{\lambda}^2+\mathcal{Z}_2(\mathcal{L}^2)\tilde{\lambda}+\mathcal{Z}_3(\mathcal{L}^2)=0 \tag{5.2}$$

here,

$$\begin{aligned}\mathcal{Z}_1(\mathcal{L}^2) =&\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}+\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\\ &+\mu_3\mathcal{L}^2+\epsilon+\delta-\beta\left(S^*+\alpha V^*\right)\frac{I^*}{1+I^*},\end{aligned}$$

$$\begin{aligned}\mathcal{Z}_2(\mathcal{L}^2) =&\left(\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\right)\left(\mu_3\mathcal{L}^2+\epsilon+\delta\right)+\frac{\beta S^*}{(1+I^*)^2}\times\frac{\beta I^*}{1+I^*}\\ &+\left(\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}\right)\left(\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}+\mu_3\mathcal{L}^2+\epsilon+\delta\right)\\ &+\frac{\alpha\beta V^*}{(1+I^*)^2}\times\frac{\alpha\beta I^*}{1+I^*}-\left(\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}+\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\right)\\ &\times\left(\frac{\beta S^*}{(1+I^*)^2}+\frac{\alpha\beta V^*}{(1+I^*)^2}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{Z}_3(\mathcal{L}^2) =&\left(\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}\right)\left(\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\right)\left(\mu_3\mathcal{L}^2+\epsilon+\delta\right)\\ &+\frac{\alpha\beta V^*}{(1+I^*)^2}\times\frac{\alpha\beta I^*}{1+I^*}(\mu_3\mathcal{L}^2+\epsilon+\delta)+\frac{\gamma\beta S^*}{(1+I^*)^2}\times\frac{\alpha\beta I^*}{1+I^*}\\ &+\frac{\beta S^*}{(1+I^*)^2}\times\beta\frac{I^*}{1+I^*}\left(\mu_2\mathcal{L}^2+\epsilon+\alpha\beta\frac{I^*}{1+I^*}\right)\\ &-\left(\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}\right)\left(\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\right)\left(\frac{\beta S^*}{(1+I^*)^2}+\frac{\alpha\beta V^*}{(1+I^*)^2}\right)\end{aligned}$$

$$-\frac{\alpha\beta I^*}{1+I^*}\times\frac{\alpha\beta}{(1+I^*)^2}\left(\frac{\beta S^*}{(1+I^*)^2}+\frac{\alpha\beta V^*}{(1+I^*)^2}\right).$$

Now, let us take

$$\frac{\beta S^*}{(1+I^*)^2}+\frac{\alpha\beta V^*}{(1+I^*)^2}\leq\beta(S^*+\alpha V^*)\frac{1}{1+I^*}=\epsilon+\delta\ ,$$

then we can get

$$\begin{aligned}
&\mathcal{Z}_1(\mathcal{L}^2)\geq\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}+\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}+\mu_3\mathcal{L}^2>0\ ,\\
&\mathcal{Z}_2(\mathcal{L}^2)>\mu_3\mathcal{L}^2\left(\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\right)>0\ ,\\
&\mathcal{Z}_3(\mathcal{L}^2)>\left(\mu_1\mathcal{L}^2+\epsilon+\gamma+\frac{\beta I^*}{1+I^*}\right)\left(\mu_2\mathcal{L}^2+\epsilon+\frac{\alpha\beta I^*}{1+I^*}\right)\mu_3\mathcal{L}^2>0.
\end{aligned}$$

Hence, we obtain

$$\mathcal{Z}_1(\mathcal{L}^2)\mathcal{Z}_2(\mathcal{L}^2)-\mathcal{Z}_3(\mathcal{L}^2)>\frac{\alpha\beta I^*}{1+I^*}\times\frac{\alpha\beta V^*}{(1+I^*)^2}\times\frac{\beta(S^*+\alpha V^*)}{(1+I^*)^2}>0\ .$$

From Routh-Hurwitz criteria, we get every eigenvalues of (5.2) have negative real parts. Therefore, disease steady state E^* of model (1.2) is locally asymptotically stable, if $\mathcal{R}_0>1$. □

5.2 Global stability analysis

In the following section, we investigate the global stability analysis of the steady-state.

Theorem 4. *Disease-free steady state E_0 of the model (1.2) is globally asymptotically stable, whenever $\mathcal{R}_0\leq 1$.*

Proof. Take into account a compatible Lyapunov function in the following way

$$V_1(t):=\int_\Omega W_1(t,x)dx,$$

here,

$$W_1(t,x)=S_0\left(\frac{S}{S_0}-1-\ln\frac{S}{S_0}\right)+V_0\left(\frac{V}{V_0}-1-\ln\frac{V}{V_0}\right)+I.$$

Now,

$$\frac{\partial W_1}{\partial t}=\left(1-\frac{S_0}{S}\right)\frac{\partial S}{\partial t}+\left(1-\frac{V_0}{V}\right)\frac{\partial V}{\partial t}+\frac{\partial I}{\partial t}.$$

Then we may write, from (1.2),

$$\begin{aligned}
\frac{\partial W_1}{\partial t}=&\left(1-\frac{S_0}{S}\right)\left(\mu_1\Delta S+\lambda-(\epsilon+\gamma)S-\beta\frac{I}{1+I}S+\delta I\right)\\
&+\left(1-\frac{V_0}{V}\right)\left(\mu_2\Delta V+\gamma S-\alpha\beta\frac{I}{1+I}V-\epsilon V\right)\\
&+\left(\mu_3\Delta I+\beta(S+\alpha V)\frac{I}{1+I}-(m+\delta)I\right).
\end{aligned}$$

But, as $\lambda=(\epsilon+\gamma)S_0$ and $\epsilon V_0=\gamma S_0$, we can write,

$$\frac{\partial W_1}{\partial t}=\left(1-\frac{S_0}{S}\right)\left\{\mu_1\Delta S+(\epsilon+\gamma)S_0-(\epsilon+\gamma)S-\beta\frac{I}{1+I}S+\delta I\right\}$$

$$
\begin{aligned}
&+\left(1-\frac{V_0}{V}\right)\left\{\mu_2\Delta V+\gamma S-\alpha\beta\frac{I}{1+I}V-\left(\frac{V}{V_0}\right)\epsilon V_0\right\}\\
&+\left(\mu_3\Delta I+\beta(S+\alpha V)\frac{I}{1+I}-(\epsilon+\delta)I\right)\\
=&\left(1-\frac{S_0}{S}\right)\left\{\mu_1\Delta S+(\epsilon+\gamma)S_0\left(1-\frac{S}{S_0}\right)-\beta\frac{I}{1+I}S+\delta I\right\}\\
&+\left(1-\frac{V_0}{V}\right)\left\{\mu_2\Delta V+\gamma S_0\left(\frac{S}{S_0}-\frac{V}{V_0}\right)-\alpha\beta\frac{I}{1+I}V\right\}\\
&+(\mu_3\Delta I+\beta\frac{I}{1+I}S+\alpha\beta\frac{I}{1+I}V-\epsilon I-\delta I)\\
=&\left(1-\frac{S_0}{S}\right)\mu_1\Delta S+\left(1-\frac{V_0}{V}\right)\mu_2\Delta V+\mu_3\Delta I+\epsilon S_0\left(2-\frac{S}{S_0}-\frac{S_0}{S}\right)\\
&+\gamma S_0\left(3-\frac{S_0}{S}-\frac{V}{V_0}-\frac{S}{S_0}\frac{V_0}{V}\right)-(\epsilon+\delta)(1+I-\mathcal{R}_0)\frac{I}{1+I}+\left(1-\frac{S_0}{S}\right)\delta I.
\end{aligned}
$$

Utilizing Green's formula as well as Neumann boundary conditions (1.3), we obtain

$$\int_\Omega \Delta S\mathrm{d}x=\int_{\partial\Omega}\frac{\partial S}{\partial\eta}\mathrm{d}S=0. \tag{5.3}$$

Analogously,

$$\int_\Omega \Delta V\mathrm{d}x=\int_\Omega \Delta I\mathrm{d}x=0. \tag{5.4}$$

Further, utilizing Green's formula as well as Neumann boundary conditions, we have get Green's first identity in the following way

$$\int_\Omega\left(\frac{\Delta S}{S}-\frac{\|\nabla S\|^2}{S^2}\right)\mathrm{d}x=\int_{\partial\Omega}\frac{1}{S}(\nabla S\cdot n)\mathrm{d}S=0.$$

Thus

$$\int_\Omega\frac{\Delta S}{S}\mathrm{d}x=\int_\Omega\frac{\|\nabla S\|^2}{S^2}\mathrm{d}x. \tag{5.5}$$

Analogously,

$$\int_\Omega\frac{\Delta V}{V}\mathrm{d}x=\int_\Omega\frac{\|\nabla V\|^2}{V^2}\mathrm{d}x, \tag{5.6}$$

$$\text{and}\int_\Omega\frac{\Delta I}{I}\mathrm{d}x=\int_\Omega\frac{\|\nabla I\|^2}{I^2}\mathrm{d}x. \tag{5.7}$$

Hence, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}V_1}{\mathrm{d}t}=&-\mu_1S_0\int_\Omega\frac{\|\nabla S\|^2}{S^2}\mathrm{d}x-\mu_2V_0\int_\Omega\frac{\|\nabla V\|^2}{V^2}\mathrm{d}x+\epsilon S_0\int_\Omega\left(2-\frac{S}{S_0}-\frac{S_0}{S}\right)\mathrm{d}x\\
&+\gamma S_0\int_\Omega\left(3-\frac{S_0}{S}-\frac{V}{V_0}-\frac{S}{S_0}\frac{V_0}{V}\right)\mathrm{d}x-(\epsilon+\delta)\int_\Omega\left((1+I-\mathcal{R}_0)\frac{I}{1+I}\right)\mathrm{d}x\\
&+\delta\int_\Omega I\left(1-\frac{S_0}{S}\right)\mathrm{d}x,\\
=&-\mu_1S_0\int_\Omega\frac{\|\nabla S\|^2}{S^2}\mathrm{d}x-\mu_2V_0\int_\Omega\frac{\|\nabla V\|^2}{V^2}\mathrm{d}x+\epsilon S_0\int_\Omega\left(2-\frac{(S-S_0)^2+2S_0S}{S_0S}\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma S_0\int_\Omega\left(3-\frac{S_0}{S}-\frac{V}{V_0}-\frac{SV_0}{S_0V}\right)\mathrm{d}x-(\epsilon+\delta)\int_\Omega\frac{I(1+I-\mathcal{R}_0)}{1+I}\mathrm{d}x+\int_\Omega\delta I\left(1-\frac{S_0}{S}\right)\mathrm{d}x,\\
=&-\mu_1S_0\int_\Omega\frac{\|\nabla S\|^2}{S^2}\mathrm{d}x-\mu_2V_0\int_\Omega\frac{\|\nabla V\|^2}{V^2}\mathrm{d}x-\epsilon S_0\int_\Omega\frac{(S-S_0)^2}{S_0S}\mathrm{d}x\\
&+\gamma S_0\int_\Omega\left(3-\frac{S_0}{S}-\frac{V}{V_0}-\frac{SV_0}{S_0V}\right)\mathrm{d}x-(\epsilon+\delta)\int_\Omega\frac{I(1+I-\mathcal{R}_0)}{1+I}\mathrm{d}x+\int_\Omega\delta I\left(1-\frac{S_0}{S}\right)\mathrm{d}x
\end{aligned}
$$

Thus, whenever $\mathcal{R}_0\leq 1$, we get $\frac{\mathrm{d}V_1}{\mathrm{d}t}\leq 0$. If $S=S_0,\ V=V_0,\ I=0$, we get $\frac{\mathrm{d}V_1}{\mathrm{d}t}=0$ as well as vice-versa. Thus, in $\{(S,V,I)\in\mathcal{C}(\Omega,\mathbb{R}^3_+):\frac{\mathrm{d}V_1}{\mathrm{d}t}=0\}$, the singleton E_0 would be the greatest compact invariant set. The LaSalle's invariance principle [21] implies that $\lim_{t\to\infty}(S,V,I)\to E_0$, it delineates that $E_0=(S_0,V_0,0)$ is globally asymptotically stable if $\mathcal{R}_0\leq 1$. □

Theorem 5. *Disease steady state $E^*(S^*,V^*,I^*)$ of the model (1.2) is globally asymptotically stable whenever $\mathcal{R}_0>1$.*

Proof. Take into account a compatible Lyapunov function in the following way

$$V_2(t):=\int_\Omega W_2(t,x)dx,$$

here,

$$W_2(t,x)=S^*\left(\frac{S}{S^*}-1-\ln\frac{S}{S^*}\right)+V^*\left(\frac{V}{V^*}-1-\ln\frac{V}{V^*}\right)+I^*\left(\frac{I}{I^*}-1-\ln\frac{I}{I^*}\right).$$

Now

$$\frac{\partial W_2}{\partial t}=\left(1-\frac{S^*}{S}\right)\frac{\partial S}{\partial t}+\left(1-\frac{V^*}{V}\right)\frac{\partial V}{\partial t}+\left(1-\frac{I^*}{I}\right)\frac{\partial I}{\partial t}.$$

Hence

$$
\begin{aligned}
\frac{\partial W_2}{\partial t}=&\left(1-\frac{S^*}{S}\right)(\mu_1\Delta S+\lambda-\beta\frac{I}{1+I}S-(\epsilon+\gamma)S+\delta I)\\
&+\left(1-\frac{V^*}{V}\right)(\mu_2\Delta V+\gamma S-\alpha\beta\frac{I}{1+I}V-\epsilon V)\\
&+\left(1-\frac{I^*}{I}\right)\left(\mu_3\Delta I+\beta\frac{I}{1+I}S+\alpha\beta\frac{I}{1+I}V-(\epsilon+\delta)I\right),\\
=&\left(1-\frac{S^*}{S}\right)(\mu_1\Delta S+\lambda-\beta\frac{I}{1+I}S-(\epsilon+\gamma)S+\delta I)\\
&+\left(1-\frac{V^*}{V}\right)(\mu_2\Delta V+\gamma S-\alpha\beta\frac{I}{1+I}V-\epsilon V)\\
&+(I-I^*)\left(\frac{\mu_3\Delta I}{I}+\beta\frac{1}{1+I}S+\alpha\beta\frac{1}{1+I}V-(\epsilon+\delta)\right).
\end{aligned}
\tag{5.8}
$$

But, from (2.5-2.7) we have

$$
\begin{aligned}
\lambda&=\beta\frac{I^*}{1+I^*}S^*+(\epsilon+\gamma)S^*-\delta I^*,\\
\gamma S^*&=\alpha\beta\frac{I^*}{1+I^*}V^*+\epsilon V^*,\\
(\epsilon+\delta)I^*&=\beta(S^*+\alpha V^*)\frac{I^*}{1+I^*}.
\end{aligned}
$$

Now, substituting these in (5.8), we get

$$\begin{aligned}\frac{\partial W_2}{\partial t} =& \left(1-\frac{S^*}{S}\right)\mu_1\Delta S + \left(1-\frac{V^*}{V}\right)\mu_2\Delta V + \left(1-\frac{I^*}{I}\right)\mu_3\Delta I \\ &+ \left(1-\frac{S^*}{S}\right)\left\{\beta\frac{I^*}{1+I^*}S^* + (\epsilon+\gamma)S^* - \delta I^* - \beta\frac{I}{1+I}S - (\epsilon+\gamma)S + \delta I\right\} \\ &+ \left(1-\frac{V^*}{V}\right)\left\{\gamma S^*\left(\frac{S}{S^*}-\frac{V}{V^*}\right) + \gamma S^*\frac{V}{V^*} - \alpha\beta\frac{I}{1+I}V - \epsilon V\right\} \\ &+ \left(\frac{I}{I^*}-1\right)\left\{\beta\frac{I^*}{1+I}S + \alpha\beta\frac{I^*}{1+I}V - (\epsilon+\delta)I^*\right\}\end{aligned}$$

For writing convenience, let assume, $f(I) = \frac{I}{1+I}$. Then,

$$\begin{aligned}\frac{\partial W_2}{\partial t} =& \left(1-\frac{S^*}{S}\right)\mu_1\Delta S + \left(1-\frac{V^*}{V}\right)\mu_2\Delta V + \left(1-\frac{I^*}{I}\right)\mu_3\Delta I \\ &+ \left(1-\frac{S^*}{S}\right)\left\{(\epsilon+\gamma)S^*\left(1-\frac{S}{S^*}\right) + \beta f(I^*)S^*\left(1-\frac{S}{S^*}\frac{f(I)}{f(I^*)}\right) - \delta I^*\left(1-\frac{I}{I^*}\right)\right\} \\ &+ \left(1-\frac{V^*}{V}\right)\left\{\gamma S^*\left(\frac{S}{S^*}-\frac{V}{V^*}\right) + (\alpha\beta f(I^*)V^* + \epsilon V^*)\frac{V}{V^*} - \alpha\beta f(I)V - \epsilon V\right\} \\ &+ \left(\frac{I}{I^*}-1\right)\left\{\beta\frac{I^*}{1+I}S + \alpha\beta\frac{I^*}{1+I}V - \beta(S^*+\alpha V^*)\frac{I^*}{1+I^*}\right\}, \\ =& \left(1-\frac{S^*}{S}\right)\mu_1\Delta S + \left(1-\frac{V^*}{V}\right)\mu_2\Delta V + \left(1-\frac{I^*}{I}\right)\mu_3\Delta I \\ &+ \left(1-\frac{S^*}{S}\right)\left\{(\epsilon+\gamma)S^*\left(1-\frac{S}{S^*}\right) + \beta f(I)S^*\left(1-\frac{S}{S^*}\frac{f(I)}{f(I^*)}\right)\right\} \\ &+ \left(1-\frac{V^*}{V}\right)\left\{\gamma S^*\left(\frac{S}{S^*}-\frac{V}{V^*}\right) + \alpha\beta f(I^*)V^*\frac{V}{V^*}\left(1-\frac{f(I)}{f(I^*)}\right)\right\} \\ &+ \left(\frac{I}{I^*}-1\right)\left\{\beta f(I^*)S^*\left(\frac{S}{S^*}\frac{1+I^*}{1+I}-1\right) + \alpha\beta f(I^*)V^*\left(\frac{V}{V^*}\frac{1+I^*}{1+I}-1\right)\right\} \\ &- \delta I^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{I}{I^*}\right), \\ =& \left(1-\frac{S^*}{S}\right)\mu_1\Delta S + \left(1-\frac{V^*}{V}\right)\mu_2\Delta V + \left(1-\frac{I^*}{I}\right)\mu_3\Delta I \\ &+ \epsilon S^*\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right) + \epsilon V^*\left(3-\frac{S^*}{S}-\frac{V}{V^*}-\frac{S}{S^*}\frac{V^*}{V}\right) \\ &+ \beta f(I^*)S^*\left(3-\frac{S^*}{S}-\frac{S}{S^*}\frac{1+I^*}{1+I}-\frac{1+I}{1+I^*}\right) - \beta(S^*+\alpha V^*)\frac{(1-I^*)^2}{(1+I^*)^2(1+I)} \\ &+ \alpha\beta f(I^*)V^*\left(4-\frac{S^*}{S}-\frac{S}{S^*}\frac{V^*}{V}-\frac{1+I}{1+I^*}-\frac{V}{V^*}\frac{1+I^*}{1+I}\right) \\ &- \delta I^*\left(1-\frac{S^*}{S}\right)\left(1-\frac{I}{I^*}\right).\end{aligned}$$

Then, using (5.3)-(5.7), we have

$$\begin{aligned}\frac{\mathrm{d}V_2}{\mathrm{d}t} =& -\mu_1 S^* \int_\Omega \frac{\|\nabla S\|^2}{S^2}\mathrm{d}x - \mu_2 V^* \int_\Omega \frac{\|\nabla V\|^2}{V^2}\mathrm{d}x - \mu_3 I^* \int_\Omega \frac{\|\nabla I\|^2}{I^2}\mathrm{d}x \\ &+ \epsilon S^* \int_\Omega \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right)\mathrm{d}x + \epsilon V^* \int_\Omega \left(3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{S}{S^*}\frac{V^*}{V}\right)\mathrm{d}x \\ &+ \beta f(I^*) S^* \int_\Omega \left(3 - \frac{S^*}{S} - \frac{S}{S^*}\frac{1+I^*}{1+I} - \frac{1+I}{1+I^*}\right)\mathrm{d}x \\ &- \beta(S^* + \alpha V^*) \int_\Omega \frac{(1-I^*)^2}{(1+I^*)^2(1+I)}\mathrm{d}x \\ &+ \alpha\beta f(I^*) V^* \int_\Omega \left(4 - \frac{S^*}{S} - \frac{S}{S^*}\frac{V^*}{V} - \frac{1+I}{1+I^*} - \frac{V}{V^*}\frac{1+I^*}{1+I}\right)\mathrm{d}x \\ &- \delta I^* \int_\Omega \left(1 - \frac{S^*}{S} - \frac{I}{I^*} + \frac{S^*}{S}\frac{I}{I^*}\right)\mathrm{d}x. \end{aligned} \tag{5.9}$$

Since the geometric mean is less than or identical to the arithmetic mean, following this we get

$$2 - \frac{S}{S^*} - \frac{S^*}{S} \le 0,$$

$$3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{S}{S^*}\frac{V^*}{V} \le 0,$$

$$3 - \frac{S^*}{S} - \frac{S}{S^*}\frac{1+I^*}{1+I} - \frac{1+I}{1+I^*} \le 0,$$

$$\text{and } 4 - \frac{S^*}{S} - \frac{S}{S^*}\frac{V^*}{V} - \frac{1+I}{1+I^*} - \frac{V}{V^*}\frac{1+I^*}{1+I} \le 0.$$

If $S, V, I > 0$, then the equation (5.9) demonstrates that, $\frac{\mathrm{d}V_2}{\mathrm{d}t} \le 0$. When, we put $S = S^*, V = V^*$ and $I = I^*$, the inequalities would be equalities, moreover if $(S, V, I) = (S^*, V^*, I^*)$, then $\frac{\mathrm{d}V_2}{\mathrm{d}t} = 0$. Utilizing LaSalle's invariance principle [21], we get $\lim_{t\to\infty}(S, V, I) \to E^*$, it implies that E^* is globally asymptotically stable whenever $\mathcal{R}_0 > 1$, which completes the proof. □

6 Uniform existence

Linearize the model's third equation at E_0, we get

$$\begin{cases} \dfrac{\partial I}{\partial t} = \mu_3 \Delta I + \beta(S_0 + \alpha V_0) I - (\epsilon + \delta) I & \text{in } \mathcal{A}, \\ \dfrac{\partial I}{\partial \eta} = 0 & \text{in } \partial\mathcal{A}. \end{cases} \tag{6.1}$$

We refer the arguments which is in the proof of ([7], Theorem 2.2), ([4], Theorem 2), ([21], Theorem 4.2), ([12], Theorem 2.11), ([22], Theorem 3.4), ([23], Theorem 3.2), ([8], Theorem 4.2); The uniform existence result for the respective system was produced by Y. Yang et al. [24] using the following techniques. If $I(t, x) = e^{\tau t}\hat{\xi}(x)$, yields

$$\begin{cases} \tau\hat{\xi}(x) = \mu_3 \Delta\hat{\xi}(x) + (\beta S_0 + \beta\alpha V_0)\hat{\xi}(x) - (\epsilon + \delta)\hat{\xi}(x), & x \in \Omega, \\ \dfrac{\partial\hat{\xi}(x)}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{6.2}$$

Substituting $\hat{\xi}(x) \equiv 1$ into (6.2). Thus the principal eigenvalue of (6.1)

$$\tau(S_0, V_0) = \beta(S_0 + \alpha V_0) - (\epsilon + \delta) \ = (\epsilon + \delta)(\mathcal{R}_0 - 1).$$

There is a single positive eigen-function $\hat{\xi}(x) \equiv 1$ that corresponds to it. The following lemma can hold according to the above result.

Lemma 3. *The sign of $\tau(S_0, V_0)$ is same as $(\mathcal{R}_0 - 1)$.*

Using similar reasoning from [24], to claim the uniform existence of the model (1.2), we establish the lemma and theorem which are given below

Lemma 4. *If $u(x, t, \xi)$ is the solution of the model (1.2) where $u(\cdot, 0, \xi) = \xi \in \mathcal{X}^+$, then*

(i) since $\xi \in \mathcal{X}^+$, we get $S(x, t, \xi) > 0$ and $V(x, t, \xi) > 0$ in $\mathcal{A}$, moreover

$$\liminf_{t\to\infty} S(x, t) \geq \frac{a}{\epsilon + \gamma + \beta}, \text{ uniformly for } x \in \Omega,$$

as well as

$$\liminf_{t\to\infty} V(x, t) \geq \frac{an}{2(m + \gamma + \beta)(m + \alpha\beta)}, \text{ uniformly for } x \in \Omega,$$

(ii) $I(\cdot, t_0, \xi) \not\equiv 0$ is false for $t_0 \geq 0$. Thus $I(x, t, \xi) > 0$, for each $x \in \Omega,\ t > t_0$.

Proof. It is obvious from the model (1.2) that $S(x, t, \xi) > 0$ as well as $V(x, t, \xi) > 0$. Thus,

$$\frac{\partial S}{\partial t} \geq \mu_1 \Delta S + a - (\beta + \epsilon + \gamma - c)S \quad \text{in } \mathcal{A}.$$

Utilizing ([12], Lemma 1) as well as the comparison principle, yields

$$\liminf_{t\to\infty} S(t, x) \geq \frac{a}{\epsilon + \gamma + \beta}, \text{ uniformly in } x \in \Omega.$$

We get the following inequality for there exists a $t_1 > 0$

$$S(t, x) \geq \frac{1}{2}\frac{a}{\epsilon + \gamma + \beta}, \quad \forall\, t \geq t_1.$$

Analogously, the second equation becomes

$$\begin{cases} \dfrac{\partial V}{\partial t} \geq \mu_2 \Delta V + \dfrac{1}{2}\dfrac{an}{\epsilon + \gamma + \beta} - (\epsilon + \alpha\beta)V & \text{in } \mathcal{A}, \\ \dfrac{\partial V}{\partial n} = 0 & \text{in } \partial\mathcal{A}. \end{cases}$$

Therefore,

$$\liminf_{t\to\infty} V(t, x) \geq \frac{an}{2(\epsilon + \gamma + \beta)(\epsilon + \alpha\beta)}, \text{ uniformly in } x \in \Omega,$$

For the third equation of the model (1.2), we obtain

$$\begin{cases} \dfrac{\partial I}{\partial t} \geq \mu_3 \Delta I - (\epsilon + c)I & \text{in } \mathcal{A}, \\ \dfrac{\partial I}{\partial n} = 0 & \text{in } \partial\mathcal{A}. \end{cases}$$

Hence, this lemma is proved by the Hopf boundary Lemma and the strong maximum principle [25]. □

The following theorem obtains from the above arguments.

Theorem 6. *There exists a constant $n > 0$ as well as $\xi \in \mathbb{X}^+$, whenever $\xi_3(\cdot) \not\equiv 0$, and $\mathcal{R}_0 > 1$. Thus we obtain*

$$\liminf_{t\to\infty} S(t,x) \geq n, \quad \liminf_{t\to\infty} V(t,x) \geq n, \quad \liminf_{t\to\infty} I(t,x) \geq n, \quad \textit{uniformly in } x \in \Omega.$$

Proof. Take into account

$$\mathcal{U}_0 := \{\xi \in \mathbb{X}^+ : \xi_3(\cdot) \neq 0\},$$

as well as

$$\partial\mathcal{U}_0 := \mathbb{X}^+ \setminus \mathcal{U}_0 = \{\xi \in \mathbb{X}^+ : \xi_3(\cdot) = 0\}.$$

By Lemma 4, we obtain $I(x,t,\xi) > 0$, for each $\xi \in \mathcal{U}_0$ in $\mathcal{A}$, then for all $t \geq 0$, we get $\Theta_t\mathcal{U}_0 \subseteq \mathcal{U}_0$.

Let us define $R_\partial := \{\vartheta \in \mathcal{U}_0 : \Theta_t(\vartheta) \in \partial\mathcal{U}_0,\ t \geq 0\}$, moreover $n(\vartheta)$ be the omega limit set of the orbit $\mathcal{O}^+(\vartheta) := \{\Theta_t(\vartheta) : t \geq 0\}$. Take into account by claiming that $n(\xi) = \{(S_0, V_0, 0)\}$, for every $\vartheta \in R_\partial$. If $\xi \in R_\partial$, then $\Theta_t(\xi) \in \partial\mathcal{U}_0$, for each $t \geq 0$. Thus, $I(\cdot,t,\xi) = 0$. We have $\lim_{t\to\infty} S(x,t,\xi) = S_0$ uniformly in $x \in \Omega$ from first equation of the model (1.2). Therefore, $n(\xi) = \{(S_0, V_0, 0)\}$, for every $\xi \in R_\partial$.

According to Lemma (3), if $\mathcal{R}_0 > 1$, we get $\lambda(S_0, V_0) > 0$. For the continuity of $\tau(S_0, V_0)$, there exists a sufficiently small $\nu_0 > 0$, therefore $\tau\left(\frac{S_0-\nu_0}{1+\nu_0}, \frac{V_0-\nu_0}{1+\nu_0}\right) > 0$.

Consider a uniform weak repeller $(S_0, V_0, 0)$ in $\mathcal{U}_0$ by the

$$\limsup_{t\to\infty} |\Theta_t(\xi) - (S_0, V_0, 0)| \geq \nu_0, \quad \text{for each } \xi \in \mathcal{U}_0.$$

Assume there exists $\xi_0 \in \mathcal{U}_0$ by contradiction, thus

$$\limsup_{t\to\infty} |\Theta_t(\xi_0) - (S_0, V_0, 0)| < \nu_0.$$

There persists $t_2 > 0$, then $S(x,t,\xi_0) > S_0 - \nu_0$, $V(x,t,\xi_0) > V_0 - \nu_0$ and $0 < I(x,t,\xi_0) < \nu_0$ whenever $x \in \Omega$ and $t \geq t_2$. Hence, $I(x,t,\xi_0)$ gratifies

$$\begin{cases} \dfrac{\partial I}{\partial t} \geq \nu_3\Delta I + \dfrac{b((S_0-\nu_0)+\alpha(V_0-\nu_0))}{1+\nu_0} I - (\epsilon + c)I, & t \geq t_2 \text{ as well as } x \in \Omega, \\ \dfrac{\partial I}{\partial n} = 0, & t \geq t_2 \text{ as well as } x \in \Omega. \end{cases}$$

By Lemma 3, we obtain $\hat{\xi}$ be the strongly positive eigenfunction such that $\tau\left(\frac{S_0-\nu_0}{1+\nu_0}, \frac{V_0-\nu_0}{1+\nu_0}\right)$. Hence, for each $t > 0$ and $x \in \Omega$, we get $I(x,t,\xi_0) > 0$ which implies $I(x,t,\xi_0) \geq \epsilon\hat{\xi}$ whereas there exists $\epsilon > 0$. Thus, $u(t,x) = \epsilon \exp\left(\tau\left(\frac{S_0-\nu_0}{1+\nu_0}, \frac{V_0-\nu_0}{1+\nu_0}\right)(t-t_2)\right)\hat{\xi}$ is the solution of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} \geq \nu_3\Delta u + \dfrac{b((S_0-\nu_0)+\alpha(V_0-\nu_0))}{1+\nu_0} u - (\epsilon + c)u, & t \geq t_2 \text{ as well as } x \in \partial\Omega, \\ \dfrac{\partial u}{\partial n} = 0, & t \geq t_2 \text{ as well as } x \in \partial\Omega. \end{cases}$$

We may get the following results utilizing the comparison principle

$$I(t,x,\xi_0) \geq \epsilon \exp\left(\tau\left(\frac{S_0-\nu_0}{1+\nu_0}, \frac{V_0-\nu_0}{1+\nu_0}\right)(t-t_2)\right)\hat{\xi}, \quad t \geq t_2 \text{ as well as } x \in \Omega.$$

This leads to the conclusion that $I(x,t,\xi_0)$ is bounded, which is a contradiction. Then for continuous function $\mathcal{L}$ which is defined in $X^+ \to [0,\infty)$ by

$$\mathcal{L}(\xi) = \min_{x\in\Omega} \xi_3(x), \quad \xi \in X^+.$$

It is simple to check that

$$\mathcal{L}^{-1}(0,\infty) \subseteq \mathcal{U}_0.$$

Therefore, for $\mathcal{L}(\xi) > 0$ or $\mathcal{L}(\xi) = 0$ as well as $\xi \in \mathcal{U}_0$, we get $\mathcal{L}(\Theta_t(\xi)) > 0$ whenever $t > 0$. Thus, for the semi-flow $\Theta_t : X^+ \to X^+$ implies $\mathcal{L}$ a generalized distance function which follows that for any forward orbit of Θ_t in R_∂ converges to $\{(S_0, V_0, 0)\}$. Thus, $\{(S_0, V_0, 0)\}$ is separated in X^+ as well as $W^s(S_0, V_0, 0) \cap \mathcal{U}_0 = \varnothing$. Moreover, there is no R_∂ cycle from $\{(S_0, V_0, 0)\}$ to $\{(S_0, V_0, 0)\}$. Utilizing ([26], Theorem 3), we get

$$\min_{\psi \in n(\xi)} \mathcal{L}(\psi) > \varrho, \quad \text{for each } \xi \in \mathcal{U}_0$$

whereas there exists a $\varrho > 0$. Hence,

$$\lim_{t\to\infty} \inf I(\cdot, t, \xi) \geq \varrho, \quad \text{for each } \xi \in \mathcal{U}_0.$$

Then by Lemma 4, this proof is completed. □

7 Numerical examples

The following section contains the numerical examples according to theoretical results, which also justify the analytical results. Assume the following initial values

$$\begin{cases} S^\theta(x) = 100\sin(x) + 500, & \text{in } \Omega, \\ V^\theta(x) = 100\cos(x) + 500, & \text{in } \Omega, \\ I^\theta(x) = 100\sin(0.5x) + 10, & \text{in } \Omega, \end{cases}$$

with zero Neumann boundary conditions

$$\frac{\partial S}{\partial n} = \frac{\partial V}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad \text{on } \partial\mathcal{A}.$$

Example 1. *Assume the parameters*

$$\lambda = 1000, \beta = 0.0005, \alpha = 0.000001, \epsilon = 0.7, \gamma = 0.8, \delta = 0.05;$$

with $\mu_i = 1$, $i = 1, 2, 3$. *The basic reproduction number,* $\mathcal{R}_0 = 0.4445 < 1$, *is obtained using the formula (2.4). Consequently, theorem 4 provides that these parameter values lead to the disease-free steady state results presented in Figure 7.1.*

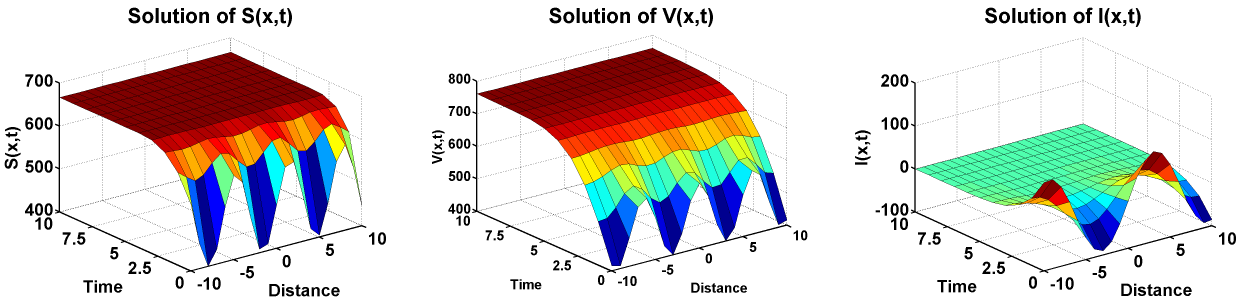

Figure 7.1: The system (1.2) is in a disease free steady state with space and time.

The estimate disease-free steady state $E_0(666.67, 761.90, 0)$ *from* (2.3).

Example 2. *Assume the parameters*

$$\lambda = 1000, \beta = 0.0001, \alpha = 0.000001, \epsilon = 0.1, \gamma = 0.1, \delta = 0.01.$$

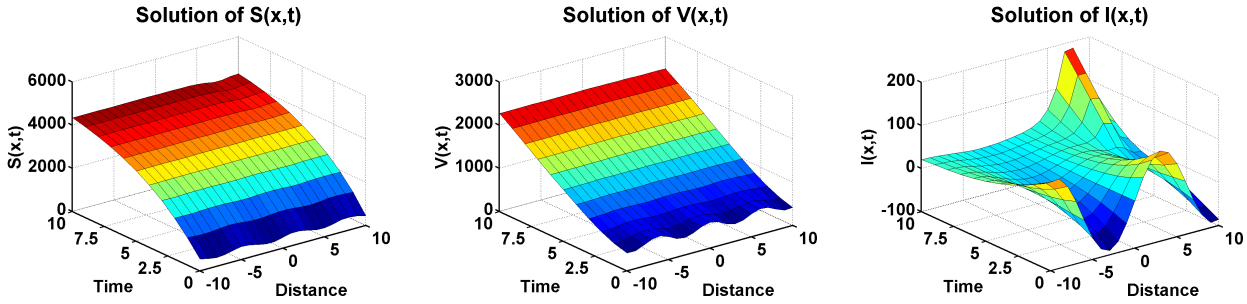

Figure 7.2: The system (1.2) is in a disease free steady state with space and time.

The estimate basic reproduction number is $\mathcal{R}_0 = 22.7273 > 1$ *from the formula* (2.4) *using the above parameters values. This Figure 7.2 depicts the Theorem 5.*

7.1 Bifurcation

In the following section, we have illustrated the model (1.2) behaviour with various values of the parameters.

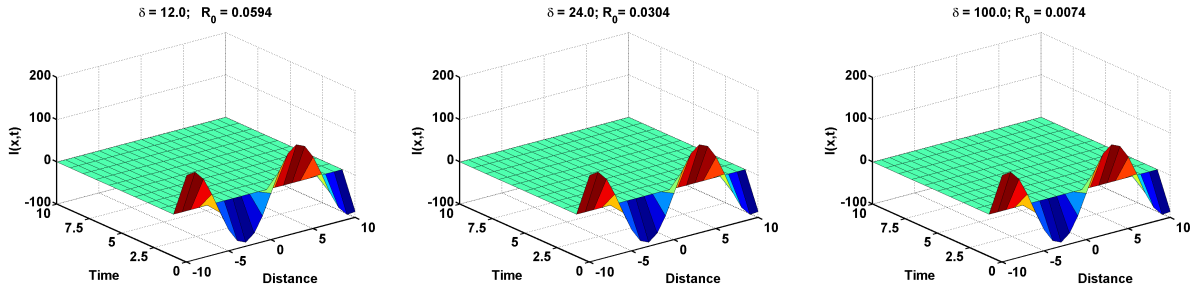

Figure 7.3: Impact of parameter value δ where $\lambda = 1000,\ \beta = 0.0005,\ \alpha = 0.000001,\ m = 0.57,\ \gamma = 0.1$.

We observe from Figures 7.3 that the disease is becoming extinct faster as δ rises. Since δ is greater than 12.0, however, δ has no influence on illness in this parametric arrangement, and, more curiously, there is no disease equilibrium regardless of whether the basic reproduction number is greater than 1.0 or not.

Albeit we considered δ is non-negative in the model (1.2), when the disease occurring environment still prevails, if we consider δ is negative such that $\delta \in [-\epsilon, 0)$. In that case, the following outcomes can occur.

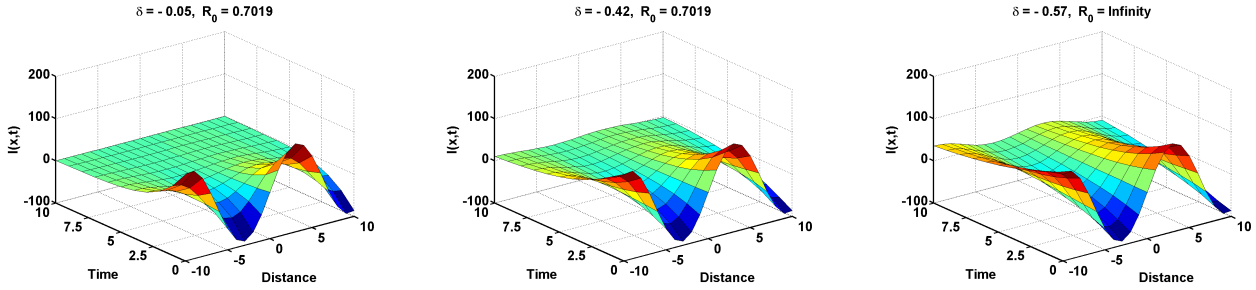

Figure 7.4: Impact of parameter value δ where $\lambda = 1000,\ \beta = 0.0005,\ \alpha = 0.000001,\ \epsilon = 0.57,\ \gamma = 0.8$.

The Figure 7.4 illustrates that when δ is negative and disease occurring environment since $\mathcal{R}_0 < 0.001$, thus disease-free steady state where $\mathcal{R}_0 > 0.001$ disclosures disease steady state. Further, the infection rises in a constant rate since $\mathcal{R}_0$ is undefined when $\delta = -\epsilon$.

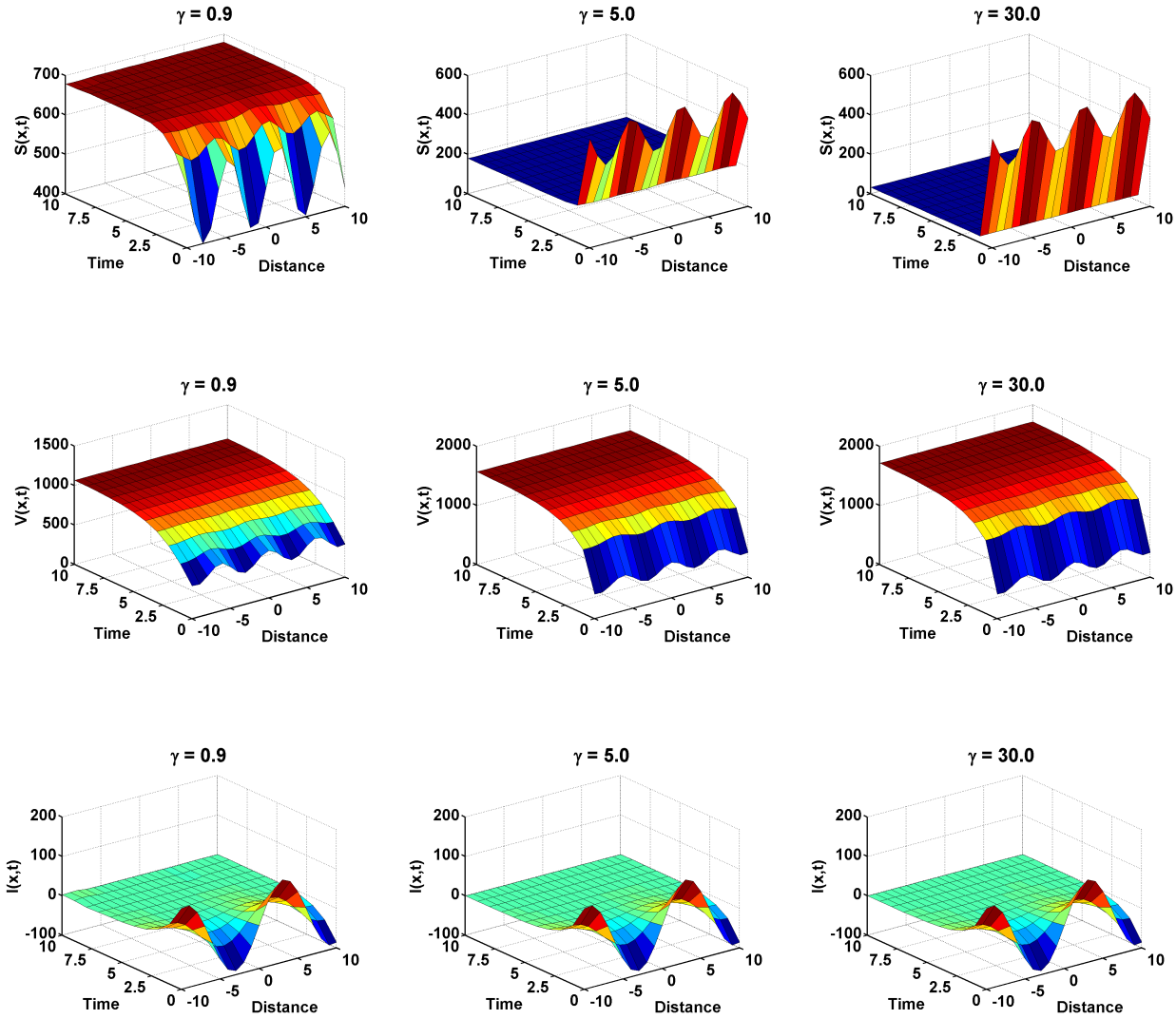

Figure 7.5: Impact of parameter value γ where $\lambda = 1000,\ \beta = 0.0005,\ \alpha = 0.000001,\ \epsilon = 0.57,\ \delta = 0.25$.

The effects of the vaccination coverage parameter γ on susceptible (S) and vaccinated (V) persons are clearly depicted in Figure 7.5. As γ grows larger, the susceptible (S) count converges to a minimum and the vaccinated (V) count rises to a maximum. However, the infectious (I) count is almost the same in each instance.

8 Conclusion

We have demonstrated comprehensive research of disease-free steady state, disease steady-state, basic reproduction number, existence, and the uniqueness of the model solution. We have illustrated local and global stability analysis in the stability analysis part. To assess the system's stability, it is theoretically established that the dynamics are dependent on the basic reproduction number. Further, we have shown the model's

uniform persistence theorem. Such analysis elucidates the stability of disease models at both small and large scales, aiding in the identification of critical control thresholds. Uniform persistence, linked to the threshold $\mathcal{R}_0$, underscores the sustained presence of the disease in the population, emphasizing the necessity for ongoing intervention strategies. Uniqueness existence have ensured the existence of a single, well-defined solution to the model, bolstering the reliability of predictions. Further, positivity boundedness, guided by $\mathcal{R}_0$, ensures realistic and interpretable model outcomes. In this study, bifurcation analysis with $\mathcal{R}_0$ reveals, control of $\mathcal{R}_0$ play effective role in transitions between stable and unstable states. Moreover, analysis with infected and vaccinated rates within compartments quantifies that these parameters have key impact on disease control. Also numerical results in bifurcation indicates disease will rise to pick level along with reinfection when $\mathcal{R}_0 > 1$ and disease burden can be controlled effectively by $\mathcal{R}_0 < 1$. Several numerical examples are used to support all of the theoretical results. It can be an open problem for further study to consider the primary or secondary data fitting with a communicable disease for vaccination and therapy.

Acknowledgments

The research by M. Kamrujjaman was partially supported by the University Grants Commission (UGC), and the University of Dhaka, Bangladesh.

Conflict of interest

The authors declare no conflict of interest.

Data Availability statement

There is no available data regarding this study.

References

[1] A. B. Gumel, S. M. Moghadas, A qualitative study of a vaccination model with non-linear incidence, *App. Math. Comput.* **143** (2003) 409-419.

[2] M. S. Mahmud, M. Kamrujjaman, M. M. I. Y. Adan, ... & M. M. Molla (2022). Vaccine efficacy and SARS-COV-2 control in California and US during the session 2020–2026: A modeling study, Infectious Disease Modelling, 7(1), 62-81.

[3] B. Buonomo, D. Lacitignola, C. Vargar-De-Leon, Qualitative analysis and optimal control of an epidemic model with vaccination and treatment, *Math. Comput. Simulation* **100** (2014) 88-120.

[4] J. D. Murray, Mathematical Biology, I and II, third edition, *Springer, New York*, (2002).

[5] M. Kamrujjaman, M. Shahriar Mahmud & M Shafiqul Islam, Dynamics of a Diffusive Vaccination Model with Therapeutic Impact and Non-linear Incidence in Epidemiology, *Journal of Biological Dynamics*, 1-29, 2020.

[6] M. Shahriar Mahmud, M. Kamrujjaman & M. Shafiqul Islam, A spatially dependent vaccination model with therapeutic impact and non-linear incidence, book chapter: *Recent Trends in Applied Mathematics*, Select Proceedings of AMSE 2019, Springer Nature, Singapore, 2021.

[7] C. Cosner, R. S. Cantrell, Spatial Ecology Via Reaction-Diffusion Equations, John Wiley and Sons Ltd. (2003).

[8] H. L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems: Mathematical Surveys and Monographs **vol. 41**, *Amer. Math. Soc.*, (1995).

[9] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, (1996).

[10] M. Kamrujjaman, M. S. Islam, & M. S. Islam (2022). Lyapunov Mappings and Analysis of a Nonlinear Spatio-temporal Epidemic Model, Dhaka University Journal of Science, 161-170.

[11] S. Ahmed, S. Rahman, & M. Kamrujjaman (2021). Optimal treatment strategies to control acute HIV infection, Infectious Disease Modelling, 6, 1202-1219.

[12] Y. Lou, X.-Q. Zhao, A reaction-diffusion malaria model with incubation period in the vector population, *J. Math. Biol.* **62** (2011), 543-568.

[13] Freedman HI, Zhao X-Q, Global asymptotics in some quasimonotone reaction–diffusion systems with delays, *J Differ Equ*, (1997),137:340–362.

[14] S. Ahmed, M. Kamrujjaman, & S. Rahman (2020), Dynamics of a viral infectiology under treatment, Journal of Applied Analysis and Computation, 10(5), 1800-1822.

[15] Zhao X-Q, Dynamical systems in population biology, Springer, New York, (2003).

[16] J. Hale, Asymptotic Behavior of Dissipative Systems, *American Mathematical Society, Providence*, (1988).

[17] Y. Lou, X. Zhao, A reaction–diffusion malaria model with incubation period in the vector population, *J. Math. Biol.*, **62**, (2011), 543–568.

[18] T. W. Hwang, F. B. Wang, Dynamics of a dengue fever transmission model with crowding effect in human population and spatial variation, *Discrete Contin. Dyn. Syst. Ser. B*, **18**, (2013), 147–161.

[19] H. Amann, Nonhomogeneous linear and quasilinear ellitic and parabolic boundary value problems, in: H.J. Schmeisser, H. Triebel (Eds.), Function Spaces, Differential Operators and Nonlinear Analysis (Friedrichroda, 1992), in: Teubner-Texte zur Mathematik, vol. 133, Teudner, Stuttgart, 1993, 9–126.

[20] R. Martin, H. L. Smith, Abstract functional differential equations and reaction–diffusion systems, *Trans. Amer. Math. Soc.*, **321**, (1990), 1–44.

[21] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, New York, (1981).

[22] W. Wang, X.-Q. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, *SIAM. Appl. Math.* **71** (2011), 147-168.

[23] Z. Xu, Y. Xu, Y. Huang, Stability and traveling waves of a vaccination model with nonlinear incidence, *Comput. and Math. with App.* **75**, (2018) 561-581.

[24] Y. Yang, S. Zhang, Dynamics of a diffusive vaccination model with nonlinear incidence, *Comput. Math. Appl.* **75** (12) (2018) 4355-4360.

[25] M. H. Protter, H. F. Weinberger, Maximum Principles in Differential Equations, *Springer-Verlag, New York*, (1984).

[26] H. L. Smith, X. Zhao, Robust persistence for semi-dynamical systems, *Nonlinear Anal. TMA* **47** (2001), 6169-6179.