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Analytic Mapping Associated with Generalized Modular Equation

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ABSTRACT

There is a relation between hypergeometric function and conformal mapping. Based on this relationship, we study the map

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

which is associated with the generalized modular equation

$$\frac{{}_2F_1(s, 1-s; 1; 1-\beta)}{{}_2F_1(s, 1-s; 1; \beta)} = p\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

in the Ramanujan's theories of signatures $\frac{1}{s} = 2$, 3, and 4. It is proved that the inverse mapping ρ of ζ can be analytically extended as a single-valued function on the upper half-plane $\mathbb{H}$ only for $\frac{1}{s} = 2,\ 3$, and 4. For the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$ associated with the generalized modular equation, we also investigate various properties of the mapping $f : \mathbb{H} \to G\backslash\mathbb{H}$ which constructs an infinite cover of the thrice punctured Riemann sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

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1 Introduction

For $\alpha,\ \beta \in (0, 1)$, $s \in (0, \frac{1}{2}]$, the map

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

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is associated with the generalized modular equation given by

$$\frac{{}_2F_1(s,1-s;1;1-\beta)}{{}_2F_1(s,1-s;1;\beta)} = p\,\frac{{}_2F_1(s,1-s;1;1-\alpha)}{{}_2F_1(s,1-s;1;\alpha)}, \tag{1.1}$$

where $p \in \mathbb{N} \setminus \{1\}$. The notation ${}_2F_1$ denotes the Gaussian hypergeometric function defined as

$${}_2F_1(a,b;c;\alpha) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \alpha^n, \quad |\alpha| < 1,$$

where $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots,$ and $(a)_n$ denotes the Pochhammer symbol given by

$$(a)_n = \begin{cases} 1 & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1) & \text{if } n \geq 1. \end{cases}$$

The integer p is called the degree or order and $\frac{1}{s}$ is called the signature of the equation (1.1). The function ${}_2F_1$ can be extended to the slit plane $\mathbb{C} \setminus [1, +\infty)$ by Euler's integral representation formula (see [6, Chapter II] and [26, Chapter XIV]).

The great mathematician Srinivasa Ramanujan investigated the generalized modular equation and provided many remarkable formulas and identities (see [8, 9, 10, 11]). However, he did not provide proof of those results. The identities given by Ramanujan were published in his unpublished notebooks without original proofs (see, e.g., [21, 22]). Ramanujan mainly investigated the generalized modular equation in the theories of signatures 2, 3, 4, and 6. There were no organized and developed theories associated with the generalized modular equation in the theories of signatures 2, 3, 4, and 6 before the 1980s. Later many mathematicians studied Ramanujan's theories and tried to prove the results provided by Ramanujan. For example, Borwein and Borwein (see [14]), Berndt (see [10, 12]), Berndt et al. (see [13]) proved many results given by Ramanujan and organized the theories related to Ramanujan's modular equation for $\frac{1}{s} = 2,\ 3,\ 4,$ and 6. In their proofs, they used hypergeometric functions and the nontrivial identities for Jacobi's theta functions in addition to several new ideas. Also, Anderson et al. (see [3, 5]) have studied Ramanujan's theories of modular equations from other perspectives. Alam and Sugawa (see [2]) provided a geometric method to prove Ramanujan's modular equations arising from the generalized modular equation. In [1], Alam studied the Hecke groups associated with the generalized modular equation in the theories of signatures $\frac{1}{s} = 2,\ 3,$ and 4.

There is a relation between the hypergeometric function ${}_2F_1$ and conformal mapping. It is a known fact that the function

$$\zeta(\alpha) = i\,\frac{{}_2F_1(a,b;a+b+1-c;1-\alpha)}{{}_2F_1(a,b;c;\alpha)}$$

maps the upper half-plane onto a hyperbolic triangle with angles $(1-c)\pi$, $(c-a-b)\pi$ and $(b-a)\pi$ at $\zeta(0)$, $\zeta(1)$ and $\zeta(\infty)$, respectively (see [20, pp. 206, 207]). Based on this relationship, we study the map

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s,1-s;1;1-\alpha)}{{}_2F_1(s,1-s;1;\alpha)}, s \in \big(0, \frac{1}{2}\big],$$

which is related to the generalized modular equation. We prove using the Riemann mapping theorem and the Schwarz reflection principle that one can analytically extend the inverse mapping of ζ to a single-valued function on the upper half-plane only for the theories of signatures $\frac{1}{s} = 2,\ 3,$ and 4. We construct the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$ from the mapping ζ. The triangle group G acts properly discontinuosly on the upper half-plane $\mathbb{H}$ and we obtain a quotient surface denoted by $G\backslash\mathbb{H}$. The surface $G\backslash\mathbb{H}$ is the thrice punctured Riemann sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. We study and prove various properties of the map

$$f : \mathbb{H} \to G\backslash\mathbb{H}$$

which makes an infinite cover of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and is associated with the generalized modular equation.

2 Preliminaries

Let us denote the upper half-plane $\{\alpha \in \mathbb{C} : \operatorname{Im} \alpha > 0\}$ by $\mathbb{H}$. The following group of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}$$

is known as unimodular group and is denoted by $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{PSL}_2(\mathbb{R})$ is defined as

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\},$$

where I_2 is the 2×2 identity matrix (see, for example, [24, Chapter VII], [17, Chapter One]). The action of the group $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is as follows:

$$\alpha \mapsto \gamma \cdot \alpha = \frac{a\alpha + b}{c\alpha + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}),\ \alpha \in \mathbb{H}.$$

Let $\partial\mathbb{H}$ denote the boundary of $\mathbb{H}$, then $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Vertical lines and semicircles that are orthogonal to the real axis are known as geodesics (see [17]). The group $\mathrm{PSL}_2(\mathbb{R})$ together with the transformation $\gamma(\alpha) = -\bar{\alpha}$ construct the group $\mathrm{Isom}(\mathbb{H})$ of isometries of $\mathbb{H}$ (see [16, p. 107]), i.e.,

$$\mathrm{Isom}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R}) \cup \gamma\, \mathrm{PSL}_2(\mathbb{R})$$

and the group of analytic automorphisms of $\mathbb{H}$, denoted by $\mathrm{Aut}\, \mathbb{H}$, is the group $\mathrm{PSL}_2(\mathbb{R})$.

Let the internal angles of a triangle be $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$, then the triangle is

1. Euclidean if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1$,
2. spherical if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1$,
3. hyperbolic if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$.

In our study, we are interested in the hyperbolic triangle. We will denote the hyperbolic triangle by Ω. If the angles of two hyperbolic triangles are the same, then they are congruent. The area of Ω depends on the angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$ and is given by the following theorem known as Gauss-Bonnet theorem.

Theorem 1 ([17, p. 13]). *Let Ω be a hyperbolic triangle with angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$, then*

$$\mathrm{Area}(\Omega) = \pi\Big(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\Big).$$

The upper half-plane $\mathbb{H}$ can be tessellated by the successive reflection of the hyperbolic triangle Ω about its sides. If K is the group generated by the three reflections of Ω about its sides, then it is the discrete subgroup of Isom($\mathbb{H}$). Let G be the subgroup of K such that G has only orientation-preserving isometries. Then,

$$G = K \cap \mathrm{PSL}_2(\mathbb{R}).$$

The group G is called the triangle group with signature (m_1, m_2, m_3). One can also represent the triangle group G as

$$\big\langle A, B \,|\, A^{m_1} = B^{m_2} = (AB)^{m_3} = 1 \big\rangle,$$

where A, B, and AB represents the rotations, respectively, by $\frac{2\pi}{m_1}$, $\frac{2\pi}{m_2}$, and $\frac{2\pi}{m_3}$ about the vertices of Ω.

Let P be a subset of $\mathbb{H}$ and let G be a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. If the following conditions are satisfied, then P is called a fundamental domain for G (see [25, p. 15]):

1. all points of P are G-inequivalent,
2. the subset P is open and connected,
3. if $x \in \mathbb{H}$ and y is a point of the closure of P, then x is G-equivalent to y.

When the subgroup G is a triangle group, then the fundamental domain for G is given by the hyperbolic triangle Ω together with its reflection about one of its sides. Note that one can construct a fundamental domain for a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ in different ways.

Now, we discuss the construction of a quotient Riemann surface. Let $\mathcal{S}$ be a topological space and let G be a topological group. If $\alpha \in \mathcal{S}$ and $g \in G$, then the map

$$\alpha \mapsto g \cdot \alpha$$

is a homeomorphism of $\mathcal{S}$ onto itself. The G-orbit of $x \in \mathcal{S}$ is given by

$$Gx = \big\{g \cdot x : g \in G\big\}.$$

Consider a compact subset R of $\mathcal{S}$, then the action of G on $\mathcal{S}$ is called properly discontinuous if and only if

$$g(R) \cap R = \emptyset,$$

except for the identity element of G (see [7, p. 94]). The quotient space $G\backslash\mathcal{S}$ is the set of all G-orbits of points on $\mathcal{S}$. Let the mapping

$$f : \mathcal{S} \to G\backslash\mathcal{S}$$

is defined by

$$f(\alpha) = G\alpha.$$

Then the mapping f is known as the canonical projection. The topology defined by the mapping f is known as the quotient topology (see [25]). If D is a subset of $G\backslash\mathcal{S}$, then D is open in $G\backslash\mathcal{S}$ when $f^{-1}(D)$ is open in $\mathcal{S}$. If G acts properly discontinuously on $\mathcal{S}$, then $G\backslash\mathcal{S}$ is a Riemann surface according to the following theorem.

Theorem 2 ([7, Theorem 6.2.1])**.** *Let G be a group of linear fractional transformations and let $\mathcal{S}$ be a simply connected open subset of the Riemann sphere $\widehat{\mathbb{C}}$. Assume that $\mathcal{S}$ is invariant under G. Then $G\backslash\mathcal{S}$ is a Riemann surface if the action of G on $\mathcal{S}$ is properly discontinuous.*

If P is a fundamental domain for G, then the boundary points of P is G-equivalent. As a result, the action of G on P identifies the sides of P and we get an oriented surface $G\backslash P$. The surfaces $G\backslash P$ and $G\backslash\mathbb{H}$ are homeomorphic to each other by the following theorem.

Theorem 3 ([7, Theorem 9.2.4])**.** *Assume that G is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ and P is the fundamental domain for G. Then, P is locally finite if and only if the map $\sigma : G\backslash P \to G\backslash\mathbb{H}$ is a homeomorphism.*

Now, consider the triangle group G with signature (m_1, m_2, m_3). The action of G on $\mathbb{H}$ is properly discontinuous and we obtain the quotient Riemann surface $G\backslash\mathbb{H}$ (see, for example, [7, 17]). In [1] and [2], a special type of triangle group known as the Hecke group is investigated to study modular equations. For example, if the signature of G is (∞, ∞, ∞), then $G\backslash\mathbb{H}$ is a thrice punctured Riemann surface as depicted in Figure 2.1.

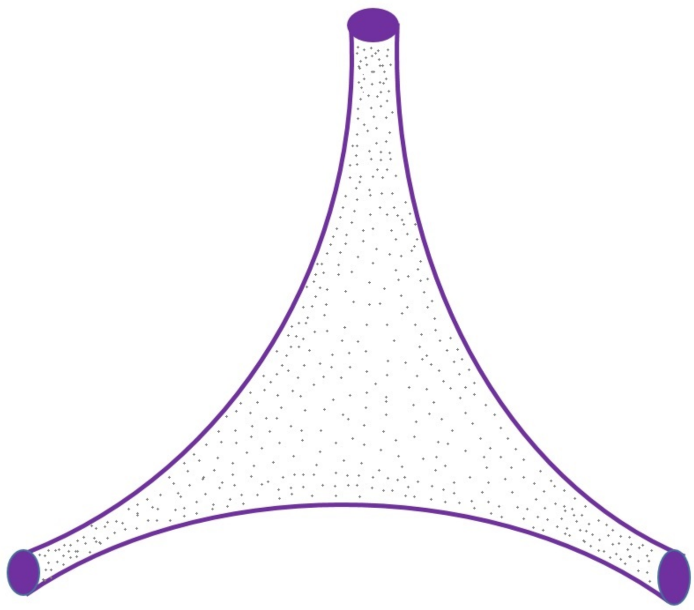

Figure 2.1: Thrice punctured Riemann surface associated with the triangle group $G = (\infty, \infty, \infty)$.

Consider the following hypergeometric differential equation

$$\alpha(1-\alpha)\frac{d^2 z}{d\alpha^2} + \{c - (a+b+1)\alpha\}\frac{dz}{d\alpha} - abz = 0. \tag{2.1}$$

The equation (2.1) has two linearly independent solutions

$$z_1 = {}_2F_1(a, b; c; \alpha)$$

and

$$z_2 = {}_2F_1(a, b; a+b+1-c; 1-\alpha).$$

If

$$\zeta(\alpha) = i\,\frac{{}_2F_1(a, b; a+b+1-c; 1-\alpha)}{{}_2F_1(a, b; c; \alpha)},$$

then ζ maps $\mathbb{H}$ conformally onto a hyperbolic triangle Ω (see Figure 2.2). At the vertices $\zeta(0)$, $\zeta(1)$ and $\zeta(\infty)$, the interior angles of Ω are $(1-c)\pi$, $(c-a-b)\pi$ and $(b-a)\pi$, respectively (see [20, Chapter V]).

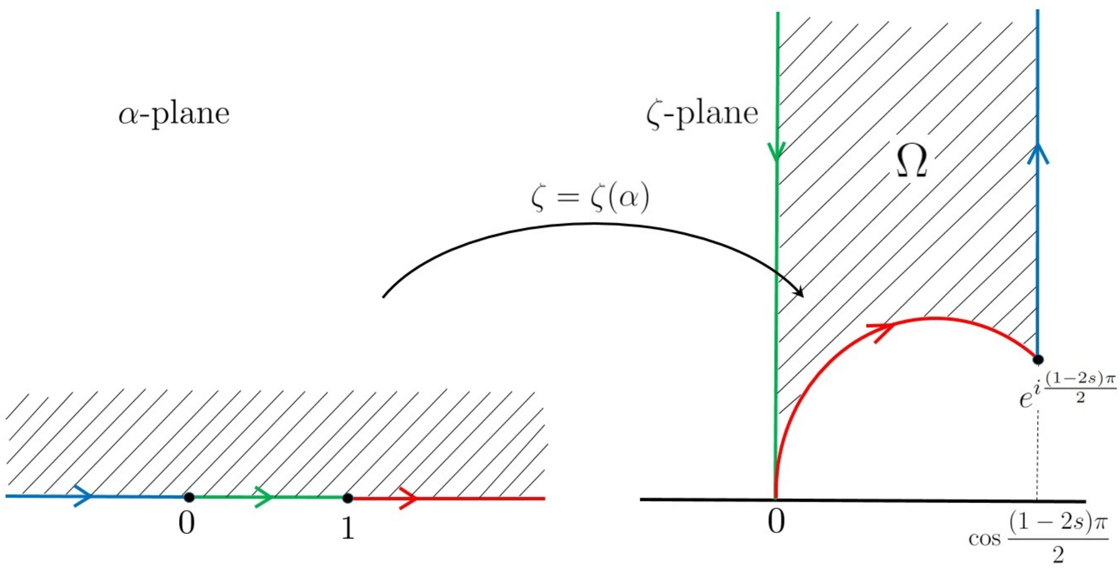

Figure 2.2: The mapping ζ maps the upper half α-plane to Ω on ζ-plane.

For $s \in (0, \frac{1}{2}]$, let

$$a = s, \quad b = 1-s \quad \text{and} \quad c = 1.$$

Then

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}. \tag{2.2}$$

The following lemma describes the above facts.

Lemma 1 ([4, Lemma 4.1])**.** *For* $s \in (0, \frac{1}{2}]$, *if*

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)},$$

then ζ *maps* $\mathbb{H} = \{\alpha \in \mathbb{C} : \operatorname{Im}\alpha > 0\}$ *onto the following hyperbolic triangle*

$$\Omega = \left\{\zeta \in \mathbb{H} : 0 < Re\,\zeta < \cos\frac{(1-2s)\pi}{2}, \left|2\zeta\cos\frac{(1-2s)\pi}{2} - 1\right| > 1\right\} \tag{2.3}$$

in the ζ*-plane. At the vertices*

$$\zeta(1) = 0, \quad \zeta(0) = \infty \quad \textit{and} \quad \zeta(\infty) = e^{i\frac{(1-2s)\pi}{2}},$$

the interior angles of Ω *are* 0, 0, *and* $(1-2s)\pi$, *respectively.*

The following theorem is known as the Riemann mapping theorem which will be used to prove Theorem 5.

Theorem 4 ([18, Theorem 4.0.1])**.** *If V is a simply connected open subset of $\mathbb{C}$ such that V is not entire $\mathbb{C}$, then there exists a biholomorphic mapping*

$$f : \mathbb{D} \to V,$$

where $\mathbb{D} = \{\alpha \in \mathbb{C} : |\alpha| < 1\}$.

3 Main Results

Theorem 5. *Consider the mapping*

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}, \tag{3.1}$$

where $s \in (0, \frac{1}{2}]$, $\alpha \in \mathbb{H}$. Let

$$\rho : \Omega \to \mathbb{H}$$

be the inverse mapping of ζ, where Ω is the hyperbolic triangle defined in (2.3). Then the mapping ρ can be analytically extended to a single-valued function on $\mathbb{H}$ only for Ramanujan's theories of signatures $\frac{1}{s} = 2,\ 3,$ and 4.

Proof. Let

$$\theta_1 = \frac{\pi}{m_1}, \quad \theta_2 = \frac{\pi}{m_2} \quad \text{and} \quad \theta_3 = \frac{\pi}{m_3}$$

be the internal angles of a hyperbolic triangle Ω, then Ω can be continued across its sides as a single-valued function if and only if $m_j > 1$ and $m_j \in \mathbb{N} \cup \{\infty\}$ (see [23, p. 416]). Therefore,

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1. \tag{3.2}$$

By Lemma 1, the mapping

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

transforms $\mathbb{H}$ to the hyperbolic triangle Ω. The angles of Ω are

$$\theta_1 = \frac{\pi}{m_1} = 0, \quad \theta_2 = \frac{\pi}{m_2} = 0 \quad \text{and} \quad \theta_3 = \frac{\pi}{m_3} = (1-2s)\pi$$

at

$$\zeta(0) = \infty, \quad \zeta(1) = 0 \quad \text{and} \quad \zeta(\infty) = e^{i\frac{\theta_3}{2}},$$

respectively. Therefore, we have

$$m_1 = \infty, \quad m_2 = \infty \quad \text{and} \quad m_3 = \frac{1}{1-2s}.$$

Thus, the condition (3.2) becomes

$$\frac{1}{m_3} < 1,$$

which implies that it depends only on the point

$$\zeta(\infty) = e^{i\frac{\theta_3}{2}}.$$

As $m_3 = \frac{1}{1-2s}$, we conclude that m_3 is an integer for $s = \frac{1}{3}$, $\frac{1}{4}$ and $m_3 = \infty$ for $s = \frac{1}{2}$. Since Ω is a simply connected domain that is not the whole $\mathbb{C}$ and $\mathbb{D} = \{\alpha \in \mathbb{C} : |\alpha| < 1\}$ is conformally equivalent to $\mathbb{H}$, the mapping $\zeta : \mathbb{H} \to \Omega$ is biholomorphic by the Riemann Mapping Theorem (Theorem 4). As a result, the inverse mapping $\rho : \Omega \to \mathbb{H}$ of ζ is holomorphic and one can analytically extend $\rho(\zeta)$ to a single-valued function on $\mathbb{H}$ for $s = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ by applying repeatedly the Schwarz reflection principle. □

From Theorem 5, we have seen that the map ρ can be extended analytically to the whole $\mathbb{H}$ as a single-valued function for $s = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Since the hyperbolic triangle Ω has angles 0, 0, and $(1-2s)\pi$ at the vertices ∞, 0, and $e^{i\frac{\theta_3}{2}}$, respectively, we can construct the triangle group $(\frac{1}{1-2s}, \infty, \infty)$ from the triangle Ω. Note that we use the notation $(\frac{1}{1-2s}, \infty, \infty)$ instead of $(\infty, \infty, \frac{1}{1-2s})$. Let G denote the triangle group $(\frac{1}{1-2s}, \infty, \infty)$, then G is the triangle group associated with the generalized modular equation

$$\frac{{}_2F_1(s, 1-s; 1; 1-\beta)}{{}_2F_1(s, 1-s; 1; \beta)} = p\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

for $s = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. If G acts on $\mathbb{H}$, then, by Theorem 2, we obtain the quotient surface $G\backslash\mathbb{H}$ which is $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and we have the following natural projection map (canonical map)

$$f : \mathbb{H} \to G\backslash\mathbb{H}.$$

Theorem 6. *If G is the triangle group associated with the generalized modular equation*

$$\frac{{}_2F_1(s, 1-s; 1; 1-\beta)}{{}_2F_1(s, 1-s; 1; \beta)} = p\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

in the Ramanujan's theories of signatures 2, 3, and 4, then the mapping

$$f : \mathbb{H} \to G\backslash\mathbb{H}$$

constructs an infinite cover of the thrice punctured Riemann sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

Proof. We have the function

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

maps the upper half α-plane to the hyperbolic triangle Ω with angles 0, 0 and $(1-2s)\pi$ at

$$\zeta(0) = \infty, \quad \zeta(1) = 0 \quad \text{and} \quad \zeta(\infty) = e^{i\frac{\theta_3}{2}},$$

respectively, in the upper half ζ-plane.

If we reflect the hyperbolic triangle Ω about the geodesic joining 0 and ∞, then we obtain the hyperbolic triangle Ω' with vertices at 0, $e^{-i\frac{\theta_3}{2}}$, and ∞ (see Figure 3.1). The triangle Ω represents the upper half-plane

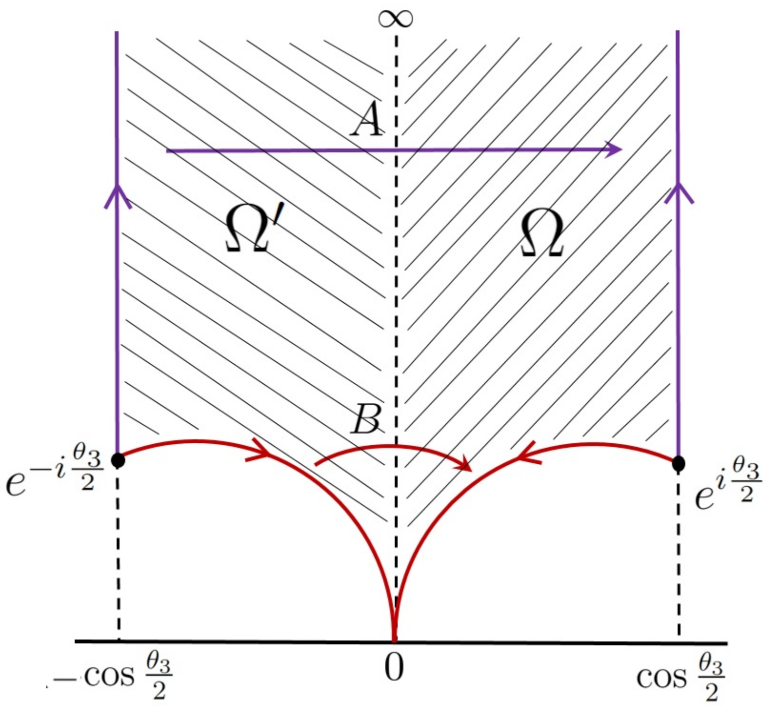

Figure 3.1: Fundamental domain for the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$.

and the triangle Ω' represents the lower half-plane. If we identify the geodesic joining $e^{-i\frac{\theta_3}{2}}$ and ∞ with the geodesic joining $e^{i\frac{\theta_3}{2}}$ and ∞ by

$$A = \begin{pmatrix} 1 & 2\cos\frac{\theta_3}{2} \\ 0 & 1 \end{pmatrix},$$

and identify the geodesic joining 0 and $e^{-i\frac{\theta_3}{2}}$ with the geodesic joining 0 and $e^{i\frac{\theta_3}{2}}$ by

$$B = \begin{pmatrix} 1 & 0 \\ 2\cos\frac{\theta_3}{2} & 1 \end{pmatrix},$$

then we obtain a Riemann surface which has branch points at 0, $e^{i\frac{\theta_3}{2}}$ and ∞. Note that one can reflect Ω about any side of Ω. Therefore, the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$ is generated by

$$A = \begin{pmatrix} 1 & 2\cos\frac{\theta_3}{2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2\cos\frac{\theta_3}{2} & 1 \end{pmatrix}.$$

The hyperbolic triangle Ω and one copy of its reflection about any side of it construct the fundamental domain for the triangle group G. If the triangle group G acts on the upper half-plane, then we obtain the quotient surface $G\backslash\mathbb{H}$ which is $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

The triangle group G acts properly discontinuously on the upper half-plane as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. As a result, for each $\alpha \in \mathbb{H}$, one can find a neighborhood U of α such that U does not contain any other element of the orbit of α. Therefore, we can construct the fundamental domain for G which contains exactly one representative from the orbit of each $\alpha \in \mathbb{H}$. Let P denote the hyperbolic polygon which is the fundamental domain for the triangle group G constructed by joining the hyperbolic triangles Ω and Ω'. We have an infinite number of copies of the hyperbolic polygon P on $\mathbb{H}$.

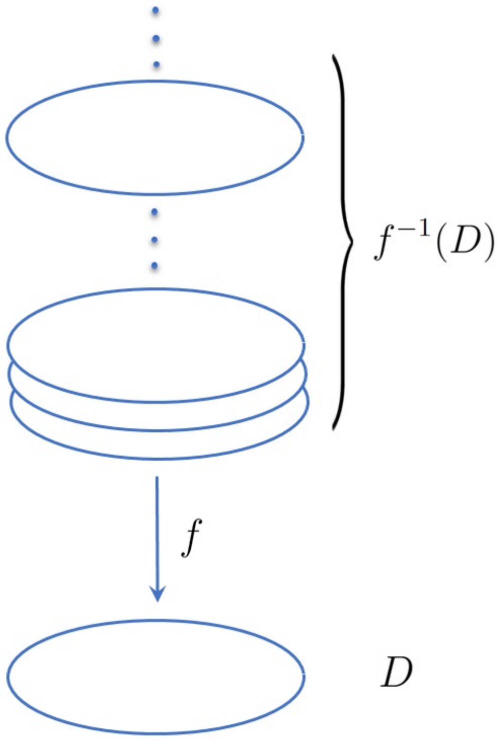

Figure 3.2: An illustration of the mapping $f : \mathbb{H} \to G\backslash\mathbb{H}$.

Let X denote the surface $G\backslash\mathbb{H} = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. For the mapping

$$f : \mathbb{H} \to X,$$

consider a point $x \in X$. If we choose a small disc $D = D(x)$ such that $x \in D \subset X$, then

$$f^{-1}(D) = \bigcup_{i \in I} \tilde{D}_i,$$

where $\tilde{D}_i$ for $i \in I$ are disjoint open discs of $\mathbb{H}$ and each mapping

$$f|_{\tilde{D}_i} : \tilde{D}_i \to D$$

is a homeomorphism for all $i \in I$ (see [15, p. 24]).

The open discs $\tilde{D}_i \subset \mathbb{H}$ are known as sheets. Since the disc D is connected, the discs $\tilde{D}_i$ can be uniquely determined up to homeomorphism. As we have an infinite number of copies of the hyperbolic polygon P in the upper half-plane $\mathbb{H}$, the number of sheets $\tilde{D}_i$ are infinite. Therefore, we conclude that the mapping f makes an infinite cover of $X = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. □

Lemma 2. *For the triangle group* $G = (\frac{1}{1-2s}, \infty, \infty)$*, the mapping*

$$f : \mathbb{H} \to G\backslash\mathbb{H}$$

is locally $\frac{1}{1-2s}$ *to 1 at the point* $e^{i\frac{(1-2s)\pi}{2}}$*.*

Proof. We have seen in the proof of Theorem 6 that the generators of the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$ are

$$A = \begin{pmatrix} 1 & 2\cos\frac{\theta_3}{2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2\cos\frac{\theta_3}{2} & 1 \end{pmatrix},$$

where $\theta_3 = (1-2s)\pi$. The fundamental domain for G is shown in Figure 3.1. Now, we modify the fundamental domain as follows. If we reflect the hyperbolic triangle Ω about the geodesic joining $e^{i\frac{\theta_3}{2}}$ and ∞, then we obtain the hyperbolic triangle Ω' whose vertices are at $2\cos\frac{\theta_3}{2}$, $e^{i\frac{\theta_3}{2}}$, and ∞.

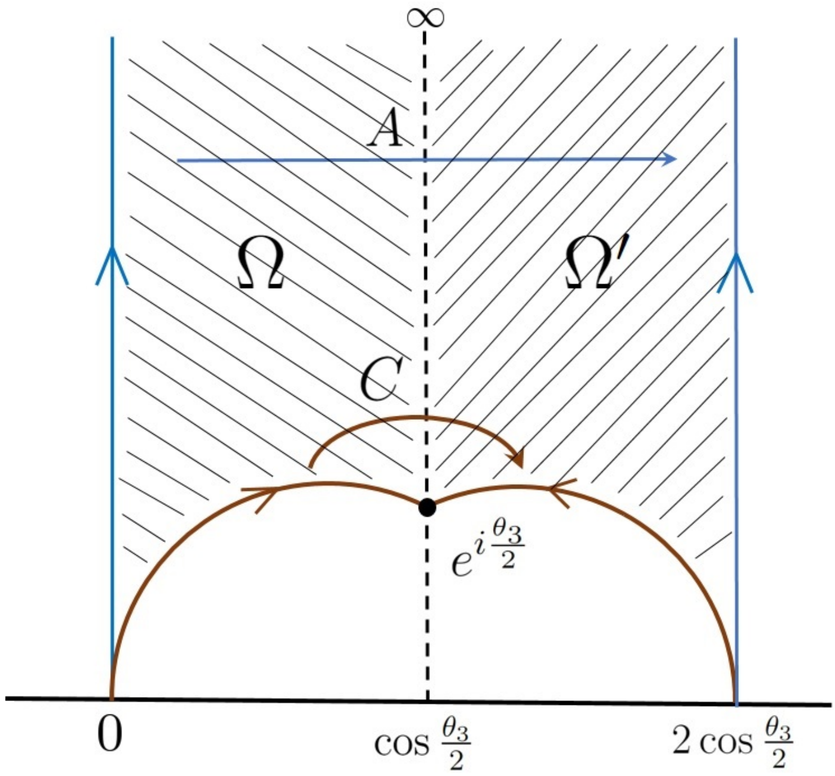

Figure 3.3: The modified fundamental domain for the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$.

The triangle Ω represents the upper half-plane and the triangle Ω' represents the lower half-plane. If the geodesic side between 0 and ∞ is identified with the geodesic side between $2\cos\frac{\theta_3}{2}$ and ∞, then the side-pairing transformation is

$$A = \begin{pmatrix} 1 & 2\cos\frac{\theta_3}{2} \\ 0 & 1 \end{pmatrix}$$

and if the geodesic side between 0 and $e^{i\frac{\theta_3}{2}}$ is identified with the geodesic side between $2\cos\frac{\theta_3}{2}$ and $e^{i\frac{\theta_3}{2}}$, then the side-pairing transformation is

$$C = -AB^{-1} = \begin{pmatrix} 4\cos^2\frac{\theta_3}{2} - 1 & -2\cos\frac{\theta_3}{2} \\ 2\cos\frac{\theta_3}{2} & -1 \end{pmatrix}.$$

At the point $e^{i\frac{(1-2s)\pi}{2}}$, $\frac{1}{1-2s}$ copies of the fundamental domain meet, i.e., $\frac{1}{1-2s}$ branches meet at the point $e^{i\frac{(1-2s)\pi}{2}}$. Therefore, we conclude that the mapping

$$f : \mathbb{H} \to G\backslash\mathbb{H}$$

is locally $\frac{1}{1-2s}$ to 1 at the point $e^{i\frac{(1-2s)\pi}{2}}$. □

Lemma 3. *The mapping*

$$f : \mathbb{H} \to G\backslash\mathbb{H},$$

where G is the triangle group $(\frac{1}{1-2s}, \infty, \infty)$, has branch points at 0, 1, *and* ∞.

Proof. We have seen that the vertices of the hyperbolic triangle Ω are at ∞, 0, and $e^{i\frac{\theta_3}{2}}$ and Ω is the half fundamental domain for the triangle group $G = (\frac{1}{1-2s}, \infty, \infty)$. When the triangle group G acts on $\mathbb{H}$, we obtain the quotient surface $G\backslash\mathbb{H}$. Since $\rho : \Omega \to \mathbb{H}$ is the inverse map of

$$\zeta(\alpha) = i\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

and

$$\zeta(0) = \infty, \quad \zeta(1) = 0 \quad \text{and} \quad \zeta(\infty) = e^{i\frac{\theta_3}{2}},$$

we have

$$\rho(\infty) = 0, \quad \rho(0) = 1 \quad \text{and} \quad \rho(e^{i\frac{\theta_3}{2}}) = \infty.$$

Since the mapping f comes from the mapping ρ, so f maps the points ∞, 0, and $e^{i\frac{\theta_3}{2}}$ and all equivalent points of them under the triangle group G to the points 0, 1, and ∞, respectively, which implies that

$$f\Big(\frac{aw+b}{cw+d}\Big) = f(w) \in \{0, 1, \infty\}$$

for $w \in \{\infty, 0, e^{i\frac{\theta_3}{2}}\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. As a result, the mapping

$$f : \mathbb{H} \to G\backslash\mathbb{H}$$

is ramified only over the points 0, 1, and ∞. Thus 0, 1, and ∞ are the only branch points of f. □

Remark 1. *Ramanujan investigated the theories of signatures* $\frac{1}{s} = 2$, 3, 4, *and* 6. *For* $\frac{1}{s} = 2$, 3, 4, *the surface* $G\backslash\mathbb{H}$ *is a modular surface, but for* $\frac{1}{s} = 6$, *the surface* $G\backslash\mathbb{H}$ *is not a modular surface as* $m_3 = \frac{1}{1-2s}$ *is not an integer greater than 1 for* $\frac{1}{s} = 6$ *(see [19, Section 10]).*

Remark 2. *In [13, Section 12], Bruce C. Berndt, S. Bhargava, and Frank G. Garvan provided the following reason why Ramanujan's theories of signatures are restricted to the values 2, 3, 4, and 6. For* $\alpha \in (0,1)$ *and* $s \in (0, \frac{1}{2}]$*, let*

$$x = \frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

and

$$q_s(\alpha) := \exp\Big(-\frac{\pi\alpha}{\sin \pi s}\Big).$$

Then,

$$\begin{aligned} q_s(\alpha) =& \alpha \exp\big(\psi(s) + \psi(1-s) + 2\gamma\big) \times \big(1 + (2s^2 - 2s + 1)\alpha \\ &+ \big(1 - \frac{7}{2}(s - s^2) + \frac{13}{4}(s - s^2)^2\big)\alpha^2 + \cdots\big), \end{aligned}$$

where $\psi(\alpha) = \frac{d(\ln \Gamma(\alpha))}{d\alpha}$*. Let* $h(s) = \exp\big(\psi(s) + \psi(1-s) + 2\gamma\big)$*, then*

$$\begin{aligned} h(\frac{1}{2}) &= \frac{1}{16}, \\ h(\frac{1}{3}) &= \frac{1}{27}, \\ h(\frac{1}{4}) &= \frac{1}{64}, \\ h(\frac{1}{6}) &= \frac{1}{432}, \end{aligned}$$

which implies that $h(s)$ *is a rational number for* $s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ *and* $\frac{1}{6}$*. If* s *takes other values, then* $h(s)$ *is transcendental, e.g., if* $s = \frac{1}{5}$*, then*

$$h(\frac{1}{5}) = (\sqrt{5})^{-5}\Big(\frac{1+\sqrt{5}}{2}\Big)^{-\sqrt{5}}.$$

For this reason, Ramanujan's theories of signatures are restricted to the values 2, 3, 4, and 6.

4 Conclusion

We have studied geometrically the analytic mapping associated with the generalized modular equation

$$\frac{{}_2F_1(s, 1-s; 1; 1-\beta)}{{}_2F_1(s, 1-s; 1; \beta)} = p\, \frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

for $\frac{1}{s} = 2$, 3, and 4. We have shown that the mapping $\rho : \Omega \to \mathbb{H}$, which is the inverse map of

$$\zeta(\alpha) = i\, \frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)},$$

can be analytically extended as a single-valued function on $\mathbb{H}$ for $\frac{1}{s} = 2,\ 3$, and 4. Also, we have proved that the mapping

$$f : \mathbb{H} \to G\backslash\mathbb{H},$$

where f comes from the mapping ρ and G is the triangle group $(\frac{1}{1-2s}, \infty, \infty)$, constructs an infinite cover of the thrice punctured Riemann sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. It has been proved that the mapping f is locally $\frac{1}{1-2s}$ to 1 at the point $e^{i\frac{(1-2s)\pi}{2}}$ and has branch points at 0, 1, and ∞. Though Ramanujan investigated the theories of signatures 2, 3, 4, and 6, we were not able to study the theories of signature 6 geometrically because the corresponding surface is not modular.

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