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A Characterization of Jordan Automorphisms on Jordan Ideals of Prime Gamma Rings

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ABSTRACT

We study and develop the concepts of homomorphism and anti-homomorphism to derive some important results in the theory of gamma rings. This article attempts to analyze some of the results of Ali et al. [1] in case of classical rings for extending those in the context of gamma rings. We establish a number of results related to automorphism, anti-automorphism and Jordan automorphism on Jordan ideal of prime gamma rings to obtain a new characterizing result. If M is a 2-torsion free prime gamma ring fulfilling a suitable condition, J is a non-zero Jordan ideal as well as a subring of M and $\varphi: M \to M$ is a Jordan automorphism, then we prove that φ is an automorphism or φ is an anti-automorphism.

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1. Introduction

We begin by recalling the concept and some properties of gamma ring. If M, Γ are two abelian groups under addition and $(a, \alpha, b) \to a\alpha b$ is a map of $M \times \Gamma \times M \to M$ such that $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + +a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$, and $(a\alpha b)\beta c = a\alpha(b\beta c)$ $\forall\, a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then M is called a *gamma ring* (shortly: a Γ-*ring*). Nobusawa introduced the concept of gamma ring in [5] and Barnes developed various properties of gamma rings in [2]. M is called 2-*torsion free* $\Leftrightarrow 2x = 0$ forces $x = 0\ \forall\, x \in M$. Here M is called (i) *prime* $\Leftrightarrow a\Gamma M\Gamma b = 0$ (with $a, b \in M$) forces $a = 0$ or $b = 0$; and (ii) *commutative* $\Leftrightarrow x\gamma y = y\gamma x\ \forall\, x, y \in M$ and $\gamma \in \Gamma$. If $a, b \in M$ and $\alpha \in \Gamma$, then $[a, b]_\alpha = a\alpha b - b\alpha a$ is called the *commutator* (sometimes called the *Lie product*) of a, b with respect to α. *As a result*, M is commutative $\Leftrightarrow [a, b]_\alpha = 0\ \forall\, a, b \in M$ and $\alpha \in \Gamma$. If $a, b \in M$ and $\alpha \in \Gamma$, then $(a \circ b)_\alpha = a\alpha b + b\alpha a$ is called the *Jordan product* of a, b with respect to α.

Consider J as an additive subgroup of M. J is called (i) a *left* (or, *right*) *ideal* $\Leftrightarrow M\Gamma J \subset J$ (or, $J\Gamma M \subset J$); (ii) an *ideal* $\Leftrightarrow m\gamma u \in J$ and $u\gamma m \in J\ \forall\, m \in M, \gamma \in \Gamma$ and $u \in J$; and (iii) a *Jordan ideal* $\Leftrightarrow (u \circ m)_\gamma \in J$ (that is, iff $u\gamma m + m\gamma u \in J$) $\forall\, u \in J, m \in M$ and $\gamma \in \Gamma$. $x \in M$ is called a *nilpotent element* $\Leftrightarrow \forall\, \gamma \in \Gamma\ \exists\, n \in \mathbb{Z}^+$ (depends on γ) such that $(x\gamma)^n x = (x\gamma)(x\gamma) \ldots (x\gamma)x = 0$. J is called (i) a *nil ideal* $\Leftrightarrow$ each element of J is nilpotent; and (ii) a *nilpotent ideal* $\Leftrightarrow \exists\, n \in \mathbb{Z}^+$ for which $(J\Gamma)^n J = (J\Gamma)(J\Gamma) \ldots (J\Gamma)J = 0$. Every nilpotent ideal of a Γ-ring is *thus* a nil ideal.

For Γ-rings M and N, an additive map $\varphi: M \to N$ is called (i) a *homomorphism* $\Leftrightarrow \varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)\ \forall\, a, b \in M$ and $\alpha \in \Gamma$; (ii) an *anti-homomorphism* $\Leftrightarrow \varphi(a\alpha b) = \varphi(b)\alpha\varphi(a)\ \forall\, a, b \in M$ and $\alpha \in \Gamma$; and (iii) a *Jordan homomorphism* $\Leftrightarrow$

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$\varphi(a\alpha a) = \varphi(a)\alpha\varphi(a)\ \forall\ a \in M$ and $\alpha \in \Gamma$. From the very definition, *it follows that* every homomorphism and anti-homomorphism of Γ-rings is a Jordan homomorphism of the same, but the converse is not always true.

We know that a bijective homomorphism of a Γ-ring onto another Γ-ring is called an isomorphism, and an isomorphism of a Γ-ring onto itself is called an automorphism. A bijective additive map $\varphi: M \to M$ is thus called an *automorphism*, an *anti-automorphism* and a *Jordan automorphism* $\Leftrightarrow$ $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$, $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$ and $\varphi(a\alpha a) = \varphi(a)\alpha\varphi(a)\ \forall\ a, b \in M$ and $\alpha \in \Gamma$, respectively.

Oukhtite et al. [6] developed the concept of homomorphism and anti-homomorphism on σ-Lie ideals in the theory of classical rings. Afterwards, Dey and Paul [3] worked with the notions of homomorphism and anti-homomorphism on a non-zero ideal in the theory of gamma rings. Later on, Paul and Chakraborty [7] also worked on the development of the concepts of homomorphism and anti-homomorphism to derive some more important results in the theory of Γ-rings.

We are inspired to perform this attempt from the work of Ali et al. [1] in the theory of classical rings in order to generalize extensively some of their results in the context of gamma rings. If M is a 2-torsion free prime Γ-ring fulfilling a suitable condition, J is a non-zero Jordan ideal as well as a subring of M and $\varphi: M \to M$ is a Jordan automorphism, then we prove that φ is an automorphism or φ is an anti-automorphism.

2. Main Results

We start this section with the following well-known *identities on commutators*.

(i) $[a, b]_\alpha + [b, a]_\alpha = 0$,
(ii) $[a + b, c]_\alpha = [a, c]_\alpha + [b, c]_\alpha$,
(iii) $[a, b + c]_\alpha = [a, b]_\alpha + [a, c]_\alpha$,
(iv) $[a, b]_{\alpha+\beta} = [a, b]_\alpha + [a, b]_\beta$,
(v) $[a\beta b, c]_\alpha = a\beta[b, c]_\alpha + [a, c]_\alpha\beta b + a\beta(c\alpha b) - a\alpha(c\beta b)$, and
(vi) $[a, b\beta c]_\alpha = b\beta[a, c]_\alpha + [a, b]_\alpha\beta c + b\alpha(a\beta c) - b\beta(a\alpha c)$.

We need to consider throughout this section hereafter that a Γ-ring M fulfills the condition

(*) $\quad a\alpha b\beta c = a\beta b\alpha c\ \forall\ a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

If M fulfills the condition (*), then the last two commutator identities become (respectively)

(V) $[a\beta b, c]_\alpha = a\beta[b, c]_\alpha + [a, c]_\alpha\beta b$, and
(VI) $[a, b\beta c]_\alpha = b\beta[a, c]_\alpha + [a, b]_\alpha\beta c$.

Unless otherwise stated, M will represent a prime Γ-ring and J will represent a non-zero Jordan ideal of M (throughout this section hereafter)

Lemma 2.1 *Let M fulfills the condition* (*). *Then* $2[M, M]_\Gamma\Gamma J \subseteq J$ *and* $2J\Gamma[M, M]_\Gamma \subseteq J$.

Proof. Let $a, b \in M$ and $c \in J$.
For any $\alpha, \beta, \gamma \in \Gamma$, we get $(c \circ [a, b]_\alpha)_\beta - ((c \circ a)_\alpha \circ b)_\beta + ((c \circ b)_\alpha \circ a)_\beta \in J$.
It gives $c\beta(a\alpha b - b\alpha a) + (a\alpha b - b\alpha a)\beta c - ((c\alpha a + a\alpha c) \circ b)_\beta + ((c\alpha b + b\alpha c) \circ a)_\beta \in J$.
This implies that $c\beta a\alpha b - c\beta b\alpha a + a\alpha b\beta c - b\alpha a\beta c - c\alpha a\beta b - a\alpha c\beta b$
$-b\beta c\alpha a - b\beta a\alpha c + c\alpha b\beta a + b\alpha c\beta a + a\beta c\alpha b + a\beta b\alpha c \in J$.
Using (*), we obtain $2(a\alpha b - b\alpha a)\beta c \in J$.
It gives that $2[a, b]_\alpha\beta c \in J\ \forall\ a, b \in M$, $c \in J$ and $\alpha, \beta \in \Gamma$.
That is, $2[M, M]_\Gamma\Gamma J \subseteq J$.
Similarly, $\forall\ a, b \in M$, $c \in J$ and $\alpha, \beta \in \Gamma$,

$$2c\beta(a\alpha b - b\alpha a) = ((c \circ b)_\alpha \circ a)_\beta - (c \circ [a, b]_\alpha)_\beta - ((c \circ a)_\alpha \circ b)_\beta \in J .$$

This gives $2J\Gamma[M, M]_\Gamma \subseteq J$. □

Lemma 2.2 *If* $a \in M$ *and* $a\Gamma J = (0)$ [*or* $J\Gamma a = (0)$], *then* $a = 0$.

Proof. For any $x \in M$, $c \in J$ and $\alpha \in \Gamma$, we get $(c \circ x)_\alpha \in J$.

According to hypothesis, $a\beta((c\circ x)_\alpha)=0\ \forall\ x\in M$, $c\in J$ and $\alpha,\beta\in\Gamma$.
Now we get $0=a\beta(c\alpha x+x\alpha c)=a\beta c\alpha x+a\beta x\alpha c=a\beta x\alpha c$, since $a\beta c=0$.
That is, $a\beta M\alpha J=0$.
Since $J\neq 0$ and M is prime, the above equation yields that $a=0$.
If $J\Gamma a=(0)$, in view of the similar arguments with necessary variations, we also obtain that $a=0$. □

Lemma 2.3 *Let M be 2-torsion free fulfilling the condition (*) such that $a\alpha J\beta b=(0)\ \forall\ a,b\in M$ and $\alpha,\beta\in\Gamma$. Then $a=0$ or $b=0$.*

Proof. From Lemma 2.1, we get $2[M,M]_\Gamma\Gamma J\subseteq J$.
Therefore, $\forall\ u,v\in M$, $c\in J$ and $\gamma,\delta\in\Gamma$, we get $2[u,v]_\gamma\delta c\in J$.
By hypothesis, we get $2a\alpha[u,v]_\gamma\delta c\beta b=0$.
The 2-torsion freeness of M then gives

$$a\alpha[u,v]_\gamma\delta c\beta b=0. \quad (1)$$

Now substituting $v\nu a$ (with $\nu\in\Gamma$) for v in (1), we obtain

$$0=a\alpha[u,v\nu a]_\gamma\delta c\beta b=a\alpha v\nu[u,a]_\gamma\delta c\beta b+a\alpha[u,v]_\gamma\nu a\delta c\beta b.$$

Since $a\alpha J\beta b=(0)$, we get $a\alpha[u,v]_\gamma\nu a\delta c\beta b=0$.
Hence we obtain $a\alpha v\nu[u,a]_\gamma\delta c\beta b=0\ \forall\ u,v\in M$, $c\in J$ and $\alpha,\beta,\gamma,\delta,\nu\in\Gamma$.
This implies that $a\alpha M\nu[u,a]_\gamma\delta c\beta b=0$.
By the primeness of M, we find that $a=0$ or $[u,a]_\gamma\delta c\beta b=0$.
If $[u,a]_\gamma\delta c\beta b=0\ \forall\ u\in M$, $c\in J$ and $\beta,\gamma,\delta\in\Gamma$, it gives $u\gamma a\delta c\beta b-a\gamma u\delta c\beta b=0$.
Since $a\delta c\beta b=0$, we get $a\gamma u\delta c\beta b=0\ \forall\ u\in M$, $c\in J$ and $\beta,\gamma,\delta\in\Gamma$.
This implies that $a\gamma M\delta c\beta b=0$.
Since M is prime, we get $a=0$ or $c\beta b=0$.
If $c\beta b=0\ \forall\ c\in J$ and $\beta\in\Gamma$, then by Lemma 2.2, $b=0$. □

Lemma 2.4 *Let M be 2-torsion free fulfilling the condition (*). If J is commutative, then $J\subseteq Z(M)$.*

Proof. From Lemma 2.1, we get $2[M,M]_\Gamma\Gamma J\subseteq J$.
For any $u,v\in M$, $c,d\in J$ and $\alpha,\beta,\gamma\in\Gamma$, we thus get $[2[u,v]_\alpha\beta c,d]_\gamma=0$.
Hence $[2[u,v]_\alpha,d]_\gamma\beta c+2[u,v]_\alpha\beta[c,d]_\gamma=0$.
Since J is a commutative Jordan ideal, $2[u,v]_\alpha\beta[c,d]_\gamma=0$.
Using this fact in the above equation, we obtain $2[[u,v]_\alpha,d]_\gamma\beta c=0$.
The 2-torsion freeness of M then gives $[[u,v]_\alpha,d]_\gamma\beta c=0\ \forall\ u,v\in M$, $c,d\in J$ and $\alpha,\beta,\gamma\in\Gamma$.
By Lemma 2.2, $[[u,v]_\alpha,d]_\gamma=0\ \forall\ u,v\in M$, $d\in J$ and $\alpha,\gamma\in\Gamma$.
Now putting $u\beta v$ (with $\beta\in\Gamma$) for v in (1), we get $[[u,u\beta v]_\alpha,d]_\gamma=0$.
This implies that $[u\beta[u,v]_\alpha,d]_\gamma+[[u,u]_\alpha\beta v,d]_\gamma=0$.
It becomes $0=[u\beta[u,v]_\alpha,d]_\gamma=u\beta[[u,v]_\alpha,d]_\gamma+[u,d]_\gamma\beta[u,v]_\alpha=[u,d]_\gamma\beta[u,v]_\alpha$
$\forall\ u,v\in M$, $d\in J$ and $\alpha,\beta,\gamma\in\Gamma$.
Further replacing v by $v\delta d$ (with $\delta\in\Gamma$), the last equation becomes

$$0=[u,d]_\gamma\beta[u,v\delta d]_\alpha=[u,d]_\gamma\beta v\delta[u,d]_\alpha+[u,d]_\gamma\beta[u,v]_\alpha\delta d=[u,d]_\gamma\beta v\delta[u,d]_\alpha.$$

This implies that $[u,d]_\gamma\beta M\delta[u,d]_\alpha=0\ \forall\ u\in M$, $d\in J$ and $\alpha,\beta,\gamma,\delta\in\Gamma$.
The primeness of M then gives $[u,d]_\gamma=0$ or $[u,d]_\alpha=0$.
In both cases, $d\in Z(M)\ \forall\ d\in J$. Hence $J\subseteq Z(M)$. □

Lemma 2.5 *Let M be 2-torsion free fulfilling the condition (*) and let J be a subring of M* as well. *If $\varphi: M\to M$ is an additive map with $\varphi(a\alpha a)=\varphi(a)\alpha\varphi(a)\ \forall\ a\in J$ and $\alpha\in\Gamma$, then $\forall\ a,b,c\in J$ and $\alpha,\beta\in\Gamma$,*
(i) $\varphi(a\alpha b+b\alpha a)=\varphi(a)\alpha\varphi(b)+\varphi(b)\alpha\varphi(a)$;
(ii) $\varphi(a\alpha b\beta a)=\varphi(a)\alpha\varphi(b)\beta\varphi(a)$;
(iii) $\varphi(a\alpha b\beta c+c\alpha b\beta a)=\varphi(a)\alpha\varphi(b)\beta\varphi(c)+\varphi(c)\alpha\varphi(b)\beta\varphi(a)$.

Proof. (i) For any $a,b\in J$ and $\alpha\in\Gamma$, we get $a\alpha b+b\alpha a\in J$, as J is a Jordan ideal and a subring of M.
By assumption, we get $\varphi(a\alpha a)=\varphi(a)\alpha\varphi(a)\ \forall\ a\in J$ and $\alpha\in\Gamma$.
Substitute a by $a+b$ to obtain $\varphi((a+b)\alpha(a+b))=\varphi(a+b)\alpha\varphi(a+b)$.

It gives $\varphi(a\alpha a + a\alpha b + b\alpha a + b\alpha b) = (\varphi(a) + \varphi(b))\alpha(\varphi(a) + \varphi(b))$.
This yields

$$\varphi(a\alpha a) + \varphi(a\alpha b + b\alpha a) + \varphi(b\alpha b) = \varphi(a)\alpha\varphi(a) + \varphi(a)\alpha\varphi(b) + \varphi(b)\alpha\varphi(a) + \varphi(b)\alpha\varphi(b).$$

It then gives

$$\varphi(a)\alpha\varphi(a) + \varphi(a\alpha b + b\alpha a) + \varphi(b)\alpha\varphi(b) = \varphi(a)\alpha\varphi(a) + \varphi(a)\alpha\varphi(b) + \varphi(b)\alpha\varphi(a) + \varphi(b)\alpha\varphi(b).$$

Thus we get

$$(1) \qquad \varphi(a\alpha b + b\alpha a) = \varphi(a)\alpha\varphi(b) + \varphi(b)\alpha\varphi(a),$$

which proves the claim.

(ii) Substituting b by $a\beta b + b\beta a$ (where $\beta \in \Gamma$) in (1) and using (*), we get

$$(2) \qquad \varphi(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \varphi(a)\alpha\varphi(a\beta b + b\beta a) + \varphi(a\beta b + b\beta a)\alpha\varphi(a)$$
$$= \varphi(a)\alpha\big(\varphi(a)\beta\varphi(b) + \varphi(b)\beta\varphi(a)\big) + (\varphi(a)\beta\varphi(b) + \varphi(b)\beta\varphi(a))\alpha\varphi(a)$$
$$= \varphi(a)\alpha\varphi(a)\beta\varphi(b) + \varphi(a)\alpha\varphi(b)\beta\varphi(a) + \varphi(a)\beta\varphi(b)\alpha\varphi(a) + \varphi(b)\beta\varphi(a)\alpha\varphi(a).$$

On the other hand,

$$(3) \qquad \varphi(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \varphi(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$$
$$= \varphi(a\alpha a\beta b + b\beta a\alpha a) + \varphi(a\alpha b\beta a) + \varphi(a\beta b\alpha a)$$
$$= \varphi(a\alpha a)\beta\varphi(b) + \varphi(b)\beta\varphi(a\alpha a) + 2\varphi(a\alpha b\beta a)$$
$$= \varphi(a)\alpha\varphi(a)\beta\varphi(b) + \varphi(b)\beta\varphi(a)\alpha\varphi(a) + 2\varphi(a\alpha b\beta a).$$

Comparing (2) and (3), and using (*), we obtain $\varphi(a\alpha b\beta a) = \varphi(a)\alpha\varphi(b)\beta\varphi(a)$.

(iii) Putting $a + c$ for a (where $c \in J$) in (ii), we get

$$(4) \qquad \varphi\big((a + c)\alpha b\beta(a + c)\big) = \varphi(a + c)\alpha\varphi(b)\beta\varphi(a + c)$$
$$= (\varphi(a) + \varphi(c))\alpha\varphi(b)\beta(\varphi(a) + \varphi(c))$$
$$= \varphi(a)\alpha\varphi(b)\beta\varphi(a) + \varphi(a)\alpha\varphi(b)\beta\varphi(c) + \varphi(c)\alpha\varphi(b)\beta\varphi(a) + \varphi(c)\alpha\varphi(b)\beta\varphi(c).$$

On the other hand,

$$(5) \qquad \varphi\big((a + c)\alpha b\beta(a + c)\big) = \varphi(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$$
$$= \varphi(a\alpha b\beta c + c\alpha b\beta a) + \varphi(a\alpha b\beta a) + \varphi(c\alpha b\beta c)$$
$$= \varphi(a\alpha b\beta c + c\alpha b\beta a) + \varphi(a)\alpha\varphi(b)\beta\varphi(a) + \varphi(c)\alpha\varphi(b)\beta\varphi(c).$$

Comparing (4) and (5), we obtain $\varphi(a\alpha b\beta c + c\alpha b\beta a) = \varphi(a)\alpha\varphi(b)\beta\varphi(c) + \varphi(c)\alpha\varphi(b)\beta\varphi(a)$. □

Definition 2.1 For convenience we now define $\Phi_\alpha(a, b) = \varphi(a\alpha b) - \varphi(a)\alpha\varphi(b)$.

Lemma 2.6 *For any $a, b, c \in J$ and $\alpha, \beta \in \Gamma$, the following are true.*
(i) $\Phi_\alpha(a, b) + \Phi_\alpha(b, a) = 0$;
(ii) $\Phi_\alpha(a + b, c) = \Phi_\alpha(a, c) + \Phi_\alpha(b, c)$;
(iii) $\Phi_\alpha(a, b + c) = \Phi_\alpha(a, b) + \Phi_\alpha(a, c)$;
(iv) $\Phi_{\alpha+\beta}(a, b) = \Phi_\alpha(a, b) + \Phi_\beta(a, b)$.

Proof. Obvious. □

Definition 2.2 For convenience we next define $\Psi_\alpha(a, b) = \varphi(a\alpha b) - \varphi(b)\alpha\varphi(a)$.

Lemma 2.7 *For any $a, b, c \in J$ and $\alpha, \beta \in \Gamma$, the following results hold.*
(i) $\Psi_\alpha(a, b) + \Psi_\alpha(b, a) = 0$;
(ii) $\Psi_\alpha(a + b, c) = \Psi_\alpha(a, c) + \Psi_\alpha(b, c)$;
(iii) $\Psi_\alpha(a, b + c) = \Psi_\alpha(a, b) + \Psi_\alpha(a, c)$;
(iv) $\Psi_{\alpha+\beta}(a, b) = \Psi_\alpha(a, b) + \Psi_\beta(a, b)$.

Proof. Clear. □

Lemma 2.8 *For any $a, b \in J$ and $\alpha, \beta \in \Gamma$, the following results are true.*
(i) $\Phi_\alpha(a, b)\beta\Psi_\alpha(a, b) = 0$;
(ii) $\Psi_\alpha(a, b)\beta\Phi_\alpha(a, b) = 0$;
(iii) $[\varphi(a), \varphi(b)]_\alpha = \Psi_\alpha(a, b) - \Phi_\alpha(a, b)$;
(iv) $\varphi([a, b]_\alpha) = \Psi_\alpha(a, b) + \Phi_\alpha(a, b)$.

Proof. (i) $\Phi_\alpha(a,b)\beta\Psi_\alpha(a,b) = (\varphi(a\alpha b) - \varphi(a)\alpha\varphi(b))\beta(\varphi(a\alpha b) - \varphi(b)\alpha\varphi(a))$

$$= \varphi(a\alpha b)\beta\varphi(a\alpha b) - \varphi(a\alpha b)\beta\varphi(b)\alpha\varphi(a) - \varphi(a)\alpha\varphi(b)\beta\varphi(a\alpha b) + \varphi(a)\alpha\varphi(b)\beta\varphi(b)\alpha\varphi(a)$$
$$= \varphi(a\alpha b\beta a\alpha b) - \varphi(a\alpha b\beta b\alpha a) - \varphi(a\alpha b\beta a\alpha b) + \varphi(a\alpha b\beta b\alpha a) = 0.$$

(ii) Similar as (i).

(iii) $\Psi_\alpha(a,b) - \Phi_\alpha(a,b) = \varphi(a\alpha b) - \varphi(b)\alpha\varphi(a) - \varphi(a\alpha b) + \varphi(a)\alpha\varphi(b)$

$$= \varphi(a)\alpha\varphi(b) - \varphi(b)\alpha\varphi(a) = [\varphi(a),\varphi(b)]_\alpha.$$

(iv) $\Psi_\alpha(a,b) + \Phi_\alpha(a,b) = \varphi(a\alpha b) - \varphi(b)\alpha\varphi(a) + \varphi(a\alpha b) - \varphi(a)\alpha\varphi(b)$

$$= \varphi(a\alpha b) - \varphi(b\alpha a) + \varphi(a\alpha b) - \varphi(a\alpha b)$$
$$= \varphi(a\alpha b - b\alpha a + a\alpha b - a\alpha b) = \varphi(a\alpha b - b\alpha a) = \varphi([a,b]_\alpha). \ \square$$

Lemma 2.9 *For any $a,b,m \in J$ and $\alpha,\beta \in \Gamma$, the following results happen.*
(i) $\varphi([a,b]_\alpha\beta m) = \varphi(m)\beta\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)$;
(ii) $\varphi(m\beta[a,b]_\alpha) = \Phi_\alpha(a,b)\beta\varphi(m) + \varphi(m)\beta\Psi_\alpha(a,b)$.

Proof. (i) We get $\varphi(m)\beta\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)$

$$= \varphi(m)\beta\big(\varphi(a\alpha b) - \varphi(a)\alpha\varphi(b)\big) + (\varphi(a\alpha b) - \varphi(b)\alpha\varphi(a))\beta\varphi(m)$$
$$= \varphi(m)\beta\varphi(a\alpha b) - \varphi(m)\beta\varphi(a)\alpha\varphi(b) + \varphi(a\alpha b)\beta\varphi(m) - \varphi(b)\alpha\varphi(a)\beta\varphi(m)$$
$$= \varphi(m\beta(a\alpha b)) - \varphi(m\beta a\alpha b) + \varphi((a\alpha b)\beta m) - \varphi(b\alpha a\beta m)$$
$$= \varphi(m\beta a\alpha b - m\beta a\alpha b + a\alpha b\beta m - b\alpha a\beta m)$$
$$= \varphi(a\alpha b\beta m - b\alpha a\beta m) = \varphi([a,b]_\alpha\beta m).$$

(ii) Similar as (i). □

Lemma 2.10 *If $\varphi: M \to M$ is a Jordan homomorphism on a Γ-ring M fulfilling the condition* (*)*, then $\forall\ a,b,m \in J$ and $\alpha,\beta,\gamma \in \Gamma$, $\Phi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) = 0$.*

Proof. Lemma 2.9(i) yields us that

$$(1)\qquad \varphi(m)\beta\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m) = \varphi([a,b]_\alpha\beta m).$$

Multiplying both sides of (1) by $\Phi_\alpha(a,b)$ from the left, we obtain

$$\Phi_\alpha(a,b)\gamma\varphi(m)\beta\Phi_\alpha(a,b) + \Phi_\alpha(a,b)\gamma\Psi_\alpha(a,b)\beta\varphi(m) = \Phi_\alpha(a,b)\gamma\varphi([a,b]_\alpha\beta m).$$

Hence, by using Lemma 2.8(i), we get

$$(2)\qquad \Phi_\alpha(a,b)\gamma\varphi(m)\beta\Phi_\alpha(a,b) = \Phi_\alpha(a,b)\gamma\varphi([a,b]_\alpha\beta m).$$

Again multiplying both sides of (1) by $\Psi_\alpha(a,b)$ to the right, we get

$$\varphi(m)\beta\Phi_\alpha(a,b)\gamma\Psi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b) = \varphi([a,b]_\alpha\beta m)\gamma\Psi_\alpha(a,b).$$

Using Lemma 2.8(i) again, we then get

$$(3)\qquad \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b) = \varphi([a,b]_\alpha\beta m)\gamma\Psi_\alpha(a,b).$$

Also, from Lemma 2.9(ii), we get

$$(4)\qquad \varphi(m\gamma[a,b]_\alpha) = \Phi_\alpha(a,b)\gamma\varphi(m) + \varphi(m)\gamma\Psi_\alpha(a,b).$$

Replacing m by $[a,b]_\alpha\beta m$ in (4), we get

$$(5)\qquad \varphi([a,b]_\alpha\beta m\gamma[a,b]_\alpha) = \Phi_\alpha(a,b)\gamma\varphi([a,b]_\alpha\beta m) + \varphi([a,b]_\alpha\beta m)\gamma\Psi_\alpha(a,b).$$

By adding (2) and (3), we obtain

$$(6)\qquad \Phi_\alpha(a,b)\gamma\varphi(m)\beta\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b)$$
$$= \Phi_\alpha(a,b)\gamma\varphi([a,b]_\alpha\beta m) + \varphi([a,b]_\alpha\beta m)\gamma\Psi_\alpha(a,b).$$

Combing (5) and (6), we get

$$(7)\qquad \varphi([a,b]_\alpha\beta m\gamma[a,b]_\alpha) = \Phi_\alpha(a,b)\gamma\varphi(m)\beta\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b).$$

On the other hand, applying Lemma 2.5(ii) and 2.8(iv), we get

$$\varphi([a,b]_\alpha\beta m\gamma[a,b]_\alpha) = \varphi([a,b]_\alpha)\beta\varphi(m)\gamma\varphi([a,b]_\alpha)$$
$$= \big(\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\big)\beta\varphi(m)\gamma(\Phi_\alpha(a,b) + \Psi_\alpha(a,b))$$
$$= \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b)$$
$$+\Psi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b).$$

By (7), we thus get

$$\Phi_\alpha(a,b)\gamma\varphi(m)\beta\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b)$$
$$= \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b)$$
$$+\Psi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b).$$

Using (*), we then obtain

$$\Phi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b)$$
$$= \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b)$$
$$+\Psi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b).$$

Hence, by cancelling the similar terms from both sides, we get

$$\Phi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) = 0. \ \square$$

Lemma 2.11 ([4, Lemma 3.10]) *If M is 2-torsion free and $a,b \in M$ such that $a\alpha m\beta b + b\alpha m\beta a = 0\ \forall\ m \in M$ and $\alpha,\beta \in \Gamma$, then $a = 0$ or $b = 0$.*

Proof. For any $m \in M$ and $\alpha,\beta \in \Gamma$, we get

$$(1) \qquad a\alpha m\beta b + b\alpha m\beta a = 0.$$

Replace m by $n\gamma a\delta t$ with $n,t \in M$ and $\gamma,\delta \in \Gamma$, and we get

$$(2) \qquad a\alpha n\gamma a\delta t\beta b + b\alpha n\gamma a\delta t\beta a = 0.$$

But, by (1), we get $a\delta t\beta b = -b\delta t\beta a$ and $b\alpha n\gamma a = -a\alpha n\gamma b$.
Substituting these in (2), we get $-a\alpha n\gamma b\delta t\beta a - a\alpha n\gamma b\delta t\beta a = 0$.
This implies us that $2a\alpha n\gamma b\delta t\beta a = 0$.
By the 2-torsion freeness of M, it then gives $a\alpha n\gamma b\delta t\beta a = 0$.
Thus we get $a\alpha M\gamma(b\delta t\beta a) = 0$.
Since M is prime, we get $a = 0$ or $b\delta t\beta a = 0$.
If $b\delta t\beta a = 0$, then we get $b\delta M\beta a = 0$.
Again, by the primeness of M, we obtain $b = 0$ or $a = 0$. □

We are now ready to establish our main results as follows.

Theorem 2.1 *Let M fulfils the condition (*) and let J be a subring of M as well. If $\varphi: M \to M$ is an automorphism fulfilling $\varphi(a\alpha a) = \varphi(a)\alpha\varphi(a)\ \forall\ a \in J$ and $\alpha \in \Gamma$, then $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$ or $\varphi(a\alpha b) = \varphi(b)\alpha\varphi(a)\ \forall\ a,b \in J$ and $\alpha \in \Gamma$.*

Proof. First we assume that J is commutative. By Lemma 2.4, we then get $J \subseteq Z(M)$.
Applying this fact to Lemma 2.5(i), $\forall\ a,b \in J$ and $\alpha \in \Gamma$, we obtain

$$(1) \qquad 2\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b) + \varphi(b)\alpha\varphi(a).$$

Again since J is a commutative Jordan ideal and a subring of M, we find that

$$(2) \qquad [a,b]_\alpha = 0\ \forall\ a,b \in J \text{ and } \alpha \in \Gamma.$$

Hence $\varphi([a,b]_\alpha) = \varphi(0) = 0$.
This implies that $[\varphi(a),\varphi(b)]_\alpha = 0\ \forall\ a,b \in J$ and $\alpha \in \Gamma$.
We thus get $\varphi(a)\alpha\varphi(b) = \varphi(b)\alpha\varphi(a)$.
Substituting this result in (1), we get $2\varphi(a\alpha b) = 2\varphi(a)\alpha\varphi(b)$.
The 2-torsion freeness of M then gives $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$.
Likewise, we also get $2\varphi(a\alpha b) = 2\varphi(b)\alpha\varphi(a)$, and hence $\varphi(a\alpha b) = \varphi(b)\alpha\varphi(a)$.

Now onward we shall assume that $J \nsubseteq Z(M)$.
Since we know from [7, Corollary 3.1] that M has no non-zero nilpotent ideal, so J has a non-zero ideal (by [8, Theorem 3.4]).
Therefore, by Lemma 2.3, J is a prime Γ-ring.
Now $\varphi|_J: J \to M$ is a Jordan homomorphism, and so, by Lemma 2.10, we get

$$(3) \qquad \Phi_\alpha(a,b)\beta\varphi(m)\gamma\Psi_\alpha(a,b) + \Psi_\alpha(a,b)\beta\varphi(m)\gamma\Phi_\alpha(a,b) = 0$$
$$\forall\ a,b,m \in J \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

For any $a,b,c \in \varphi(J)$ and $\alpha,\beta,\gamma \in \Gamma$, it is clear that

$$\varphi^{-1}(a\alpha b\beta c + c\alpha b\beta a) = \varphi^{-1}(a)\alpha\varphi^{-1}(b)\beta\varphi^{-1}(c) + \varphi^{-1}(c)\alpha\varphi^{-1}(b)\beta\varphi^{-1}(a).$$

Now (3) can be treated by means of φ^{-1}, it is then found that

$$\varphi^{-1}(\Phi_\alpha(a,b))\beta\varphi^{-1}(\varphi(m))\gamma\varphi^{-1}(\Psi_\alpha(a,b)) + \varphi^{-1}(\Psi_\alpha(a,b))\beta\varphi^{-1}(\varphi(m))\gamma\varphi^{-1}(\Phi_\alpha(a,b)) = 0$$
$$\forall\ a,b,m \in J \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

It gives us $\varphi^{-1}(\Phi_\alpha(a,b))\beta m\gamma\varphi^{-1}(\Psi_\alpha(a,b)) + \varphi^{-1}(\Psi_\alpha(a,b))\beta m\gamma\varphi^{-1}(\Phi_\alpha(a,b)) = 0\ \forall\ a,b,m \in J$ and $\alpha,\beta,\gamma \in \Gamma$.
In view of Lemma 2.11, we obtain $\varphi^{-1}(\Phi_\alpha(a,b)) = 0$ or $\varphi^{-1}\big(\Psi_\alpha(a,b)\big) = 0$.
If $\varphi^{-1}(\Phi_\alpha(a,b)) = 0\ \forall\ a,b \in J$ and $\alpha \in \Gamma$, then $\Phi_\alpha(a,b) = 0\ \forall\ a,b \in J$ and $\alpha \in \Gamma$.

On the other hand, if $\varphi^{-1}\left(\Psi_\alpha(a,b)\right)=0\ \forall\ a,b\in J$ and $\alpha\in\Gamma$, then we find that $\Psi_\alpha(a,b)=0\ \forall\ a,b\in J$ and $\alpha\in\Gamma$.
We thus get $\Phi_\alpha(a,b)=0$ or $\Psi_\alpha(a,b)=0\ \forall\ a,b\in J$ and $\alpha\in\Gamma$.
That is, φ is an automorphism or φ is an anti-automorphism. □

Conflict of Interests

The authors declare no conflict of interest.

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