Available online at https://ganitjournal.bdmathsociety.org/

GANIT: Journal of Bangladesh Mathematical Society

GANIT *J. Bangladesh Math. Soc.* 44.2 (2024) 039–046
DOI: https://doi.org/10.3329/ganit.v44i2.78531

Rationalized Toeplitz Hankel operators on the space of Torus $L^2(\mathbb{T}^n)$

Ruchika Verma (Batra)*

Ramjas College, University of Delhi, Delhi-110007, India.

ABSTRACT

In this paper we introduce the class of Rationalized Toeplitz Hankel operators on the space $L^2(T^n), T$ being the unit circle in complex plane and $L^2(T^n)$ is the space of Lebesgue square integrable functions on T^n. We also introduce the Rationalized Toeplitz Hankel matrix of level n and give the characterization of Rationalized Toeplitz Hankel Operator .

Received: August 15, 2024 **Accepted:** September 30, 2024 **Published Online:** December 30, 2024

Keywords: Rationalized Toeplitz Hankel operators; Slant Toeplitz operators; slant hankel operators; toeplitz; hankel operators
AMS Subject Classifications 2024: Primary 47B35; Secondary 47B38.

1 Introduction

Toeplitz operators, Hankel operators, Slant Toeplitz operators have vast literature and have been studied extensively in the last few decades .This class of operators have many applications in wavelet theory and dynamical systems. In 1911, O.Toeplitz introduced the notion of Toeplitz operators and subsequently many researchers have worked on these operators on different spaces . A parallel study of Hankel operators have been a subject of investigation for many scholars. Later in 1995, Slant Toeplitz operators were introduced. The first systematic study of Slant Toeplitz operators was introduced by Ho [8] where he studied many algebraic and spectral properties of these operators. In [3] the authors introduced Slant Hankel operators on the space $L^2(\mathbb{T})$ and also generalize the notion of Slant Toeplitz operators [1]. After that a lot of work has been done on this class of operators and their generalizations on different spaces. Recently many researchers (see [6,7,12]) have studied Slant Hankel Operators and Slant Toeplitz operators on the Lebesgue space of the Torus $\mathbb{T}^n$; so it is still an interesting area for mathematicians.

Motivated by all these , in 2022 [4], the author has introduced the generalization of all such kind of Toeplitz, Hankel, Slant Toeplitz and Slant Hankel operators on $L^2(\mathbb{T})$ as the Rationalized Toeplitz Hankel operators on $L^2(\mathbb{T})$. Many algebraic and spectral properties can be seen in [4,5]. All these extend the scope of study of Rationalized Toeplitz Hankel operators on the Lebesgue space $L^2(\mathbb{T}^n)$ of the Torus $\mathbb{T}^n$. We begin with the following preliminaries.

*Corresponding author, Ruchika Verma (Batra),*E-mail address:* ruchikaverma@ramjas.du.ac.in

If $\mathbb{T}$ is the unit circle in the complex plane $\mathbb{C}$, then $\mathbb{T}^n$ is the cartesian product of n copies of $\mathbb{T}$ which is a subset of $\mathbb{C}^n$. Let $d\mu$ be the normalized Haar measure on $\mathbb{T}^n$. Henceforth , we denote vectors in $\mathbb{C}^n$ as $z = (z_1, z_2, ..., z_n)$ and $z^t = z_1^{t_1} z_2^{t_2} ... z_n^{t_n}$ for $t = (t_1, t_2, ..., t_n) \in \mathbb{Z}^n$,

The space $L^2(\mathbb{T}^n, d\mu)$ or $L^2(\mathbb{T}^n)$ is the space of all complex valued square integrable functions on $\mathbb{T}^n$ with respect to the measure $d\mu$. That is

$$L^2(\mathbb{T}^n) = \left\{ f \middle| f : \mathbb{T}^n \to \mathbb{C} \text{ such that with } \int_{\mathbb{T}^n} |f|^2 < \infty \right\}$$

It is known that $L^2(\mathbb{T}^n)$ is a Hilbert space with the inner product defined as follows:
For $f, g \in L^2(\mathbb{T}^n)$,

$$\langle f, g \rangle = \int_{\mathbb{T}^n} f(z)\overline{g}(z) d\mu(z).$$

We see that $e_t(z) = z^t$ .That is, $e_{t_1 t_2 .... t_n}(z_1 .... z_n) = z_1^{t_1} z_2^{t_2} ... z_n^{t_n}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$ and , therefore , in terms of using the Fourier series we define

$$L^2(\mathbb{T}^n) = \left\{ f = f(z) \text{ such that } f(z) = \sum_{t \in \mathbb{Z}^n} f_t z^t \text{ with } \sum_{t \in \mathbb{Z}^n} |f_t|^2 < \infty \right\}.$$

We denote $L^\infty(\mathbb{T}^n)$ as the space of all essentially bounded measurable functions on $\mathbb{T}^n$.
That is, $L^\infty(\mathbb{T}^n)$ is the subspace consisting of all $\phi \in L^2(\mathbb{T}^n)$ such that
i) $\phi f \in L^2(\mathbb{T}^n) \ \ \forall f \in L^2(\mathbb{T}^n)$,

ii) there exists some $c > 0$ such that $\|\phi f\| \leq c\|f\| \ \ \forall f \in L^2(\mathbb{T}^n)$.

In this paper, we say that $t = (t_1, t_2, ..., t_n) \in \mathbb{Z}^n$ is even if each t_i is even, otherwise it is said to be odd. Also the set

$$F = \left\{ (t_1, t_2, ..., t_n) \in \mathbb{Z}^n \text{ such that each } t_i = 1, 2, ..., k_1 - 1 \right\}$$

Let $\epsilon_j = (x_1, x_2, ..., x_n)$ for $j \in [1, n] \cap \mathbb{Z}$ where $x_i = \delta_{ij}$
For $j \in (1, n] \cap \mathbb{Z}$ arbitrarily fixed ,in view of $t_j, t_{j+1}, ..., t_n \in \mathbb{Z}$, let

$$Q_{[t_j, t_{j+1}, ..., t_n]} = \left\{ e_{(t_j, t_{j+1}, ..., t_n)} : t_i \in \mathbb{Z}, 1 \leq i < j \right\}$$

$Q_{[t_j, t_{j+1}, ..., t_n]}$ is an orthonormal basis of $L^2(\mathbb{T}^{j-1})$.

For $1 < j \leq n$ and $t_i \in \mathbb{Z}$ with $j \leq i \leq n$, we use the notation $H_{j-1}(t_j, \ldots, t_n)$ for the space $L^2(\mathbb{T}^{j-1})$. $B(L^2(\mathbb{T}^n))$ is the space of all bounded linear operators on $L^2(\mathbb{T}^n)$.

2 Rationalized Toeplitz Hankel operators on $L^2(\mathbb{T}^n)$

In [4] ,the author had introduced the Rationalized Toeplitz Hankel operator R_ϕ of order (k_1, k_2) on the space $L^2(\mathbb{T})$ as

$$R_\phi f = W_{k_1} M_\phi W_{k_2} f \qquad \forall f \in L^2$$

where k_1 and k_2 are non-zero integers and for $k \neq 0$, W_k is defined as

$$W_k(z^i) = \begin{cases} z^{i/k} & \text{if i is divisible by k} \\ 0 & \text{otherwise .} \end{cases}$$

It is proved in [4] that if k_1 and k_2 are not co-primes then a bounded linear operator R is a Rationalized Toeplitz Hankel operator of order (k_1, k_2) on $L^2(\mathbb{T})$ if and only if $M_{z^{k_2}} R = R M_{z^{k_1}}$. A similar version is there in [4] if k_1 and k_2 are not co-primes. Also with the help of Multiplication operator M_ϕ on $L^2(\mathbb{T})$ the author has defined $R_\phi = W_{k_1} M_\phi W_{k_2}^*$, Therefore, we will be taking k and k_2 as co-primes in this paper. So we begin our study with the following definition.

Definition 1. *For a nonzero integer k, define an operator S_k as*

$$S_k(z^t) = \begin{cases} z^{t/k} & \text{if } t \text{ is a multiple of } k \\ 0 & \text{otherwise .} \end{cases}$$

That is,

$$S_k(z_1^{t_1} z_2^{t_2} ... z_n^{t_n}) = \begin{cases} z_1^{\frac{t_1}{k}} z_2^{\frac{t_2}{k}} \dots z_n^{\frac{t_n}{k}} & \text{if } t_1, t_2, ..., t_n \text{ are all multiples of } k \\ 0 & \text{otherwise .} \end{cases}$$

.

Then , S_k is a bounded linear operator on $L^2(\mathbb{T}^n)$. If S_k^* is the adjoint of S_k ,then,

$$\begin{aligned} \langle S_k^*(z_1^{t_1} z_2^{t_2} ... z_n^{t_n}), z_1^{m_1} z_2^{m_2} ... z_n^{m_n} \rangle &= \langle z_1^{t_1} z_2^{t_2} ... z_n^{t_n} S_k, (z_1^{m_1} z_2^{m_2} ... z_n^{m_n}) \rangle. \\ &= \begin{cases} 1 & \text{if } m_i = kt_i \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

Hence,

$$S_k^*(z_1^{t_1} z_2^{t_2} ... z_n^{t_n}) = z_1^{k_1 t_1} z_2^{k_2 t_2} ... z_n^{k_n t_n}$$

for each $(t_1 \dots t_n) \in \mathbb{Z}^n$ or $S_k^*(z^t) = z^{kt}, \forall t \in \mathbb{Z}^n$.

Definition 2. *Let k_1 and k_2 be non-zero integers. For $\phi \in L(\mathbb{T}^n)$, the Rationalized Toeplitz Hankel operator R_ϕ of order (k_1, k_2), induced by ϕ ,is an operator $L^2(\mathbb{T}^n)$ defined as*

$$R_\phi f = S_{k_1} M_\phi S_{k_2}^* f \qquad \forall \in L^2(\mathbb{T}^n)$$

where M_ϕ is the Multiplication operator induced by ϕ on $L^2(\mathbb{T}^n)$ and S_k is the operator as defined above.

R_ϕ is a bounded linear operator on $L^2(\mathbb{T}^n)$. If R_ϕ^* is the adjoint of the Rationalized Toeplitz Hankel operator on $L^2(\mathbb{T}^n)$ then , for $\phi(z) = \sum \phi_t z^t$ where $z = z_1 \dots z_n \in \mathbb{Z}^n$,
we have

$$\begin{aligned} R_\phi(z^m) = R_\phi(z_1^{m_1} z_2^{m_2} ... z_n^{m_n}) &= S_{k_1} M_\phi S_{k_2}^* (z_1^{m_1} z_2^{m_2} ... z_n^{m_n}) \\ &= S_{k_1} M_\phi (z_1^{k_1 m_1} z_2^{k_2 m_2} ... z_n^{k_n m_n}) \\ &= \sum_{t \in \mathbb{Z}^n} \phi_{t_1 - k_2 m_1, t_2 - k_2 m_2, ..., t_n - k_2 m_n} S_{k_1}(z_1^{t_1} z_2^{t_2} ... z_n^{t_n}) \\ &= \sum_{t \in \mathbb{Z}^n} \phi_{k_1 t_1 - k_2 m_1, k_2 t_2 - k_2 m_2, ..., k_1 t_n - k_2 m_n} (z_1^{t_1} z_2^{t_2} ... z_n^{t_n}). \end{aligned}$$

Thus, we have for $m = m_1, ..., m_n$ and $i = (i_1, i_2, ..., i_n)$,

$$\langle R_\phi^* z^m, z^i \rangle = \langle z^m, R_\phi z^i \rangle$$

$$= \left\langle z^m, \sum_{(t_1,...,t_n)\in\mathbb{Z}^n} \phi_{k_1t_1-k_2i_1,k_1t_2-k_2i_2,...,k_1t_n-k_2i_n}(z_1^{t_1}z_2^{t_2}...z_n^{t_n})\right\rangle$$
$$= \left\langle \sum_{t_1..t_n\in\mathbb{Z}^n} \bar{\phi}_{k_1m_1-k_2t_1,k_1m_2-k_2t_2,...,k_1m_n-k_2t_n}(z_1^{t_1}z_2^{t_2}...z_n^{t_n}), z_1^{i_1}z_2^{i_2}...z_n^{i_n}\right\rangle .$$

This is true $\forall\ i = (i_1, i_2, ..., i_n) \in \mathbb{Z}^n$. Thus, the adjoint R_ϕ^* of R_ϕ is defined as

$$R_\phi^*(z_1^{m_1}z_2^{m_2}...z_n^{m_n}) = \sum_{t_1...t_n\in\mathbb{Z}^n} \bar{\phi}_{k_1m_1-k_2t_1k_1m_2-k_2t_2...k_1m_n-k_2t_n}(z_1^{t_1}z_2^{t_2}...z_n^{t_n}).$$

Thus, from here we can have the following two trivial and straightforward results

1) The map $\phi \to R_\phi$ from $L^\infty(\mathbb{T}^n)$ to $B(L^2(\mathbb{T}^n))$ is linear and one-to-one.

2) $M_{z^{k_2t}}S_{k_1} = S_{k_1}M_{z^{k_1k_2t}}$.

We are now in a position to get the main result of this class of operators.

Theorem 1. *A bounded linear operator R on $B(L^2(\mathbb{T}^n))$ is a Rationalized Toeplitz Hankel operator on $L^2(\mathbb{T}^n)$ iff*

$$M_{z^{k_2t}}R = RM_{z^{k_1t}}.$$

Proof: Let $R = R_\phi$ be a Rationalized Toeplitz Hankel operator on $L^2(\mathbb{T}^n)$. Then $R = S_{k_1}M_\phi S_{k_2}^*$. So consider

$$\begin{aligned} M_{z^{k_2t}}R_\phi &= M_{z^{k_2t}}S_{k_1}M_\phi S_{k_2}^* \\ &= S_{k_1}M_\phi M_{z^{k_1k_2t}}S_{k_2}^* \\ &= R_\phi M_{z^{k_1t}}. \end{aligned}$$

Thus, R_ϕ satisfies the equation $M_{z^{k_2t}}R = RM_{z^{k_1t}}$.
Conversely, suppose that a bounded linear operator R on $L^2(\phi)$ satisfies the operator equation

$$M_{z^{k_2t}}X = XM_{z^{k_1t}}$$

and let $f(z) = \sum_{t\in\mathbb{Z}^n} f_t z^t$, then

$$\begin{aligned} Rf(z^{k_1}) &= R\left(\sum_{t\in\mathbb{Z}^n} f_t z^{k_1t}\right) \\ &= \sum_{t\in\mathbb{Z}^n} f_t M_{z^{k_2t}}R(1) \\ &= f(z^{k_2})R(1) \end{aligned}$$

In a similar way, for $t \in \mathbb{Z}^n$, we have

$$R(z^t f(z^{k_1})) = f(z^{k_2})Rz^t$$

As pointed in [4,7] and [12] it can be proved that $Rz^t \in L^\infty(\mathbb{T}^n)$ for every $t \in \mathbb{Z}^n$. So if we define for $t \in \mathbb{Z}^n$, $\phi_t(z) = \bar{z}^{k_2t}Rz^t(z^{k_1})$, then each ϕ_t is bounded and, if

$$\phi(z) = \sum_{t\in\mathbb{Z}^n} \phi_t(z),$$

we get $\phi \in L^{\infty}(\mathbb{T}^n)$. Thus, for any function $g \in L^2(\mathbb{T}^n)$, we can write

$$g(z) = g_o(z^{k_1}) + \sum_{o \neq t \in \mathbb{Z}^n} \bar{z}^{k_2 t} g_t(z^{k_1}).$$

So it can be proved that $S_{k_1} M_\phi S^*_{k_2} g = Rg$ and ,correspondingly,R is a Rationalized Toeplitz Hankel Operator on $L^2(\mathbb{T}^n)$ of order (k_1, k_2).

3 Rationalized Toeplitz Hankel matrix of level n

In [5], the author had defined Rationalized Toeplitz Hankel matrix of order (k_1, k_2) and gave a characterization of the Rationalized Toeplitz Hankel operator on the space $L^2(\mathbb{T})$ in terms of its matrix. Motivated by this, here we are ready to give the following notion.

Definition 3. *Let $\{a_t\}_{t \in \mathbb{Z}^n}$ be a sequence of scalars. The Rationalized Toeplitz Hankel matrix of order (k_1, k_2) of level 1 is the matrix of the form*

$$\left(\begin{array}{cccccc|cccccc}
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & a_{-k_1,t_2,\ldots,t_n} & a_{-k_1,-k_2,t_2\ldots,t_n} & a_{k_1,-2k_2,t_2,\ldots,t_n} & . & . & . \\
. & . & . & . & . & . & a_{0,t_2,\ldots,t_n} & a_{-k_2,t_2\ldots,t_n} & a_{-2k_2,t_2,\ldots,t_n} & . & . & . \\
. & . & . & . & . & . & a_{k_1,t_2,\ldots,t_n} & a_{k_1,-k_2,t_2\ldots,t_n} & a_{k_1,-2k_2,t_2,\ldots,t_n} & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & .
\end{array}\right)$$

We denote this matrix by $R^{(1)}_{t_2,\ldots,t_n}$.

A block matrix is said to be Rationalized Toeplitz Hankel matrix of order (k_1, k_2) of level 2 if it is represented by

$$R^{(2)}_{t_3,\ldots,t_n} = \left(\begin{array}{cccccc|cccccc}
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & R^{(1)}_{-k_1,t_3,\ldots,t_n} & R^{(1)}_{-k_1,-k_2,t_3\ldots,t_n} & R^{(1)}_{k_1,-2k_2,t_3,\ldots,t_n} & . & . & . \\
. & . & . & . & . & . & R^{(1)}_{0,t_3,\ldots,t_n} & R^{(1)}_{-k_2,t_3\ldots,t_n} & R^{(1)}_{-2k_2,t_3,\ldots,t_n} & . & . & . \\
. & . & . & . & . & . & R^{(1)}_{k_1,t_3,\ldots,t_n} & R^{(1)}_{k_1,-k_2,t_3\ldots,t_n} & R^{(1)}_{k_1,-2k_2,t_3,\ldots,t_n} & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & .
\end{array}\right)$$

and ,in general ,the Rationalized Toeplitz Hankel matrix of order (k_1, k_2) of level n, denoted by $R^{(n)}$, is defined as the block matrix whose expression is

$$R^{(n)} = \left(\begin{array}{cccccc|ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & R^{(n-1)}_{-k_1} & R^{(n-1)}_{-k_1,-k_2,t_3\ldots,t_n} & R^{(n-1)}_{k_1,-2k_2} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & R^{(n-1)}_{0} & R^{(n-1)}_{-k_2} & R^{(n-1)}_{-2k_2} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & R^{(n-1)}_{k_1} & R^{(n-1)}_{k_1,-k_2} & R^{(n-1)}_{k_1,-2k_2} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right).$$

We observe the following from our definition:

1.If $k_1 = k_2 = 1$ then it becomes a Laurent matrix of level n.

2.If $k_1 = -2$ and $k_2 = 1$ then it becomes a Slant Hankel matrix of level n.

3.If $k_1 = 2$ and $k_2 = 1$ then it becomes a Slant Toeplitz matrix of level n.

The pattern above continues for general values of k_1 and k_2 .

As proved in [4] and [5], now we give the characterization of Rationalized Toeplitz Hankel operator R_ϕ on the space $L^2(\mathbb{T}^n)$ in terms of the Rationalized Toeplitz Hankel matrix.

Theorem 2. *A bounded linear operator R on $L^2(\mathbb{T}^n)$ can be represented by Rationalized Toeplitz Hankel matrix of order (k_1, k_2) of level n if and only if its matrix satisfies*

$$\langle Re_t, e'_t\rangle = \langle Re_{t+k_1\epsilon_j}, e_{t'+t_2\epsilon_j}\rangle.$$

.

Proof: As $\{e_t\}_{t\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$,we have

$$Q_{[t_j,t_{j+1},\ldots,t_n]} = \left\{e_{(t_j,\ldots,t_n)} : t_i \in \mathbb{Z}, 1 \le i < j\right\}$$

is an orthonormal basis of $L^2(\mathbb{T}^{j-1})$. So, let $\{a_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Z}^n}$ be scalars and

$$Re_\beta = \sum_{\alpha\in\mathbb{Z}^n} a_{\alpha,\beta}e_\alpha \qquad \forall \beta \in \mathbb{Z}^n.$$

So if

$$\langle Re_t, e_{t'}\rangle = \langle Re_{t+k_1\epsilon_j}, e_{t'+k_2\epsilon_j}\rangle$$

for $t = (t_2, ..., t_n), t' = (t'_2, ..., t'_n) \in \mathbb{Z}^n$ and $1 \le j \le n$. We fix $t = (t_2, ...t_n), t' = (t'_2, ...t'_n)$ and vary t_1, t'_1 . Hence,we get

$$\begin{aligned}
&\langle Re_t(z), e_{t'}(z)\rangle = \langle Re_{t+k_1\epsilon_1}, e_{t'+k_2\epsilon_1}\rangle \\
&\implies \sum a_{\alpha,t+k_1\epsilon_1}\langle e_\alpha(z), e_{t'+k_2\epsilon_1}\rangle = \sum a_{\alpha,t}\langle e_\alpha(z), e'_t(z)\rangle \\
&\implies a_{t'+k_2\epsilon_1,t+k_1\epsilon_1} = a_{t',t}.
\end{aligned}$$

So, $R : Q_{1,(t_2,\ldots,t_n)} \to Q_{1,(t'_2,\ldots,t'_n)}$ can be represented as Rationalized Toeplitz Hankel matrix of level 1 as $R^{(1)}_{(t_2,t_3,\ldots t_n)}$.
Now vary t_2, t'_2 and fix $(t_3,\ldots,t_n)$ and $(t'_3,\ldots,t'_n)$ .Then we get

$$\langle Re_{t+k_1\epsilon_2}, e_{t'+k_2\epsilon_2}\rangle = \langle Re_t, e_{t'}\rangle.$$

We get

$$R^{(1)}_{(t'_2+k_2,t'_3,\ldots,t'_n)(t_2+k,t_3,\ldots,t_n)} = R^{(1)}_{(t'_2,\ldots,t'_n)(t_2,\ldots,t_n)}.$$

So,$R : Q_{2,(t_3,\ldots,t_n)} \to Q_{2,(t'_2,\ldots,t'_n)}$ can be represented as Rationalized Toeplitz Hankel matrix of level 2 as $R^{(2)}_{(t_3,\ldots,t_n)}$.
We continue in this way and can see that the n- Torus $L^2(\mathbb{T}^n)$ can be expressed as

$$L^2(\mathbb{T}^n) = \oplus_{t_i\in\mathbb{Z}} Q_{n-1}(t_i).$$

So ,in the Rationalized Toeplitz Hankel Matrix of level n, $(t'_n, t_n)^{th}$ entry is $R^{n-1}_{t'_n,t_n}$ and

$$R^{n-1}_{t'_n,t_n} = R^{n-1}_{t'_n+k_2,t_n+k_1}.$$

Hence ,$\langle Re_{t+k_1\epsilon_n}, e_{t'+k_2\epsilon_n}\rangle = \langle Re_t, e_{t'}\rangle$. Thus, we can say $R^{(n)} = R$ is the Rationalized Toeplitz Hankel operator of level n. We have our main characterization as follows:

Theorem 3. *A bounded linear operator R on $L^2(\mathbb{T}^n)$ is a Rationalized Toeplitz Hankel operator of order (k_1,k_2) iff it can be represented by Rationalized Toeplitz Hankel matrix of order (k_1,k_2) of level n.*

Proof : For $\phi \in L^\infty(\mathbb{T}^n)$,let $R^{(n)}_\phi$ is a Rationalized Toeplitz Hankel operator of order (k_1,k_2) of level n. Therefore,

$$\begin{aligned}
\langle R^{(n)}_\phi e_{t+k_1\epsilon_n}, e_{t'+k_2\epsilon_n}\rangle &= \langle S_{k_1} M_\phi S_{k_2} e_{t+k_1\epsilon_n}, e_{t'+k_2\epsilon_n}\rangle \\
&= \langle \phi(z) z^{k_2 t+k_1k_2\epsilon_n}, z^{k_1t'+k_1k_2\epsilon_n}\rangle \\
&= \sum_{r\in\mathbb{Z}^n} a_r \langle z^{r+k_2t+k_1k_2\epsilon_n}, z^{k_1t'+k_1k_2\epsilon_n}\rangle \\
&= \sum_{r\in\mathbb{Z}^n} a_r \langle z^{r+k_2t}, z^{k_1t'}\rangle \\
&= \langle \phi(z) z^{k_2t}, (S_{k_1})^* z^{t'}\rangle \\
&= \langle S_{k_1} M_\phi S^*_{k_2} e_t(z), e'_t(z)\rangle.
\end{aligned}$$

Thus, the matrix of $R^{(n)}_\phi$ is a Rationalized Toeplitz Hankel matrix of level n.

Conversely, let $(\alpha_{t,t'})_{t,t'\in\mathbb{Z}^n}$ be the matrix of level n . Then for $1 \le j \le n$,

$$\alpha_{t,t'} = \langle Re_t, e_{t'}\rangle = (\alpha_{t+k\epsilon_j,t'+k_2\epsilon_j}) = \langle Re_{t+k_1\epsilon_j}, e_{t'+k_2\epsilon_j}\rangle.$$

Now,

$$\begin{aligned}
\langle S_{k_1} M S^*_{k_2} e_t, e_{t'}\rangle &= \langle M e_{k_2t}, S^*_{k_1} e_{t'}\rangle \\
&= \langle M e_{k_2t}, e_{k_1t'}\rangle.
\end{aligned}$$

Also,

$$\begin{aligned}\langle M_{z^{k_2 t}} Re_t, e_{t'} \rangle &= \langle Re_t, M_{z^{k_2 t}} e_{t'} \rangle \\ &= \langle Re_t, M_{z^{k_2 t}} e_{t'} \rangle \\ &= \langle Re_t, e_{t' - k_2 \epsilon_j} \rangle \\ &= \langle Re_t + k_1, \epsilon_j, e_{t'} \rangle \\ &= \langle RM_{z^{k_1 t}} e_t, e_{t'} \rangle\end{aligned}$$

This is true $\forall t, t^{'} \in \mathbb{Z}^n$. Hence $M_{z^{k_2 t}} R = RM_{z^{k_1 t}}$, $\forall t \in \mathbb{Z}^n$.

Thus, $M_{z^{k_2}} R = RM_{z^{k_1}}$ and hence it is a Rationalized Toeplitz Hankel operator on $L^2(\mathbb{T}^n)$. This completes the proof.

References

[1] S.C. Arora ,R. Batra,On generalised slant Toeplitz operators ,*Indian J. Math* ,45(2), 121-134 (2003).

[2] S.C. Arora ,R. Batra,Generalised slant Toeplitz operators on H^2,*Math.Nachr.*, 278(4), 347-355 (2005).

[3] S.C. Arora ,R. Batra, M.P.Singh,Slant Hankel Operators,*Archivum Mathematicum*,42,125-133 (2006).

[4] R. Batra(Verma),Rationalization of Toeplitz Hankel operators , *Gulf Journal of Mathematics*,12(2),86-96 (2022),

[5] R. Batra(Verma), Properties of Rationalized Toeplitz Hankel operators, *Jordan Journal of Mathematics and Statistics*,16,67-78 (2023),

[6] G.Datt,S.K. Pandey,Slant Toeplitz operators on the Lebesgue space of n-dimensional Torus, *JHokkaido Mathematics Journal*,49(3),363-389,(2020).

[7] Munmun Hazarika and S.Marik, Toeplitz and Slant Toeplitz operators on the polydisk, *Arab Journal of Mathematics*,27(1) , 73-93 (2021),

[8] M.C.Ho ,Properties of Slant Toeplitz operators, *Indiana University Mathematics Journal*,45,843-862 (1996),

[9] L.Kong,Y.Lu, Commuting Toeplitz operators on the Hardy space of the polydisk, *Acta.Math.Sin(Eng.Sev)*,31(2),695-702(2015).

[10] W.Rudin,Function theory in Polydiscs, *W A Benjsmin* ,Vol.41,(1969).

[11] W.Rudin, Function theory in the unit ball of $\mathbb{C}$, *Springer science and Business Media*,Vol.241,(2012).

[12] M.P.Singh and O.N.Singh, k^{th} order Slant Hankel operators on the polydisk,*math.FA.*,arXiv:2022.09366vi,Feb,(2017).

[13] S.Sun,D.Zheng,Toeplitz operators on the polydisk, *Proc.Am.Math.Soc.*,124,3351-3356 (1996).

[14] Ruchika Verma, ,Rationalized Toeplitz Hankel operators via operator equations, *Annals of Mathematics and Computer Science*,vol.7,25-31 (2022).

[15] Ruchika Verma,Weighted rationalized Toeplitz Hankel operator, *Annals of Mathematics and Computer Science*,vol.13, 12-24,(2023).

[16] K.Zhu,Operator Theory in Function spaces, *American Mathematical Society*,138,(2007).