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Evolution of harvesting in competitive population dynamics with spatial heterogeneity

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ABSTRACT

The study investigates the intricate interactions, particularly the antagonistic dynamics, between two entities inhabiting a nonhomogeneous circumstance subjected to harvesting pressures. We initiate our analysis by constructing a robust mathematical framework utilizing partial differential equations (PDEs) to model the behaviors of the two species. We rigorously demonstrate the substantiality and exclusivity of solutions to the formulated model. In this context, we establish pivotal conditions that facilitate species coexistence and delineate scenarios wherein one species may exert competitive pressures sufficient to drive the other towards extinction. Additionally, we identify conditions that could culminate in the simultaneous extinction of both species. The findings yield a comprehensive relative analysis of two distinct harvesting levels, providing critical insights into their differential impacts on species dynamics. Furthermore, we substantiate our theoretical conclusions through a series of numerical simulations, which serve to validate our model and its implications.

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1 Introduction

In ecological studies, understanding the dynamics of entity competition within spatially heterogeneous conditions is crucial for effective management and conservation strategies. This paper investigates the isolated competition between two entities, each exhibiting distinct diffusion behaviors influenced by their respective carrying capacities. Traditional models often assume uniform diffusion; however, this assumption may not hold in practical scenarios. To address this limitation, we consider a framework where one species engages in carrying capacity-driven diffusion, reflecting its resource limitations, while the other species diffuses regularly. This dual approach allows us to capture the complexities of interspecific interactions in varying environmental conditions. We formulate our model using partial differential equations that account for the intrinsic growth rates of both species and their interactions, leading to a comprehensive understanding of their competitive dynamics.

- Our analysis aims to reveal insights into the conditions under which these species coexist or face endangerment, particularly in response to harvesting pressures and environmental variability.

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Harvesting significantly impacts how two entities interact and compete in different areas. In this framework, competition emerges for identical resources, which are limited by a constrained food supply and restricted living space. Furthermore, predatory species impose significant pressure on prey species as they strive for sustenance. We consider a harvesting rate that is directly proportional to the space-dependent growth rate, mathematically articulated as follows:

$$H_1(x) = \theta r(x), \quad H_2(x) = \tau r(x),$$

where θ and τ denote nonnegative coefficients of proportionality. The assumption that the harvesting rate is proportional to the intrinsic growth rate is grounded in ecological principles. This relationship reflects the notion that the capacity to harvest is inherently linked to the growth potential of the populations involved. As the intrinsic growth rate $r(x)$ varies spatially, it encapsulates the availability of resources and environmental conditions, thereby influencing the extent to which harvesting can occur sustainably. By modeling harvesting in this manner, we aim to capture the dynamic interplay between resource availability and population recovery, ultimately providing a more accurate representation of ecological interactions. The proposed model equations are

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta \left(\dfrac{u(t,x)}{K(x)} \right) + r(x)u(t,x) \left(1 - \dfrac{u(t,x)+v(t,x)}{K(x)} \right) - \theta r(x) u(t,x), \\ \dfrac{\partial v}{\partial t} = \nabla(d(x)\nabla v(t,x)) + r(x)v(t,x) \left(1 - \dfrac{u(t,x)+v(t,x)}{K(x)} \right) - \tau r(x) v(t,x), \\ t > 0, \quad x \in \Omega, \\ \dfrac{\partial (u/K)}{\partial \iota} = \dfrac{\partial v}{\partial \iota} = 0, \quad x \in \ \partial\Omega, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

Let $u(t,x)$ and $v(t,x)$ denote the densities of two competing species, both of which are constrained to non-negative values, with associated migration rates denoted by d_1 and d, respectively. This framework sets the stage for examining the interaction dynamics between the species. It is noteworthy that analogous models have been explored in the literature, specifically in [1, 2, 3]. The research presented in [2] delves into regular (random) diffusion strategies, providing a baseline understanding of how species disperse in unpredictable environments. Conversely, [3] investigates directed diffusion strategies that are intricately linked to harvesting practices, offering insights into how targeted harvesting can influence species movement. These diffusion strategies lay a foundational framework for comprehending species mobility, as elaborated in multiple studies [2, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Each of these references contributes to a more nuanced understanding of the mechanisms governing dispersal and competition, emphasizing the importance of ecological context.

Various scenarios can arise when harvesting is implemented on one or more interacting species, particularly under differing diffusion strategies [2, 3, 13, 14, 15]. In [14], the author investigates a single species with a harvesting function that is both time and space dependent, revealing the complexities involved in managing resources effectively. In contrast, [3] examines the competition between two species under a time-independent harvesting effort, highlighting the implications for coexistence and resource allocation. Furthermore, the study in [15] presents a nonhomogeneous Gilpin-Ayala diffusive equation tailored for a single species subjected to harvesting, where the harvesting function is spatially dependent. This model underscores the significance of spatial variability in resource availability and its impact on population dynamics. This work presents several novel contributions that surpass previous efforts in the literature:

- Integration of both random and directed diffusion strategies within a single model framework, enhancing the understanding of species mobility under varying ecological conditions.
- Exploration of interactions between multiple species subjected to time- and space-dependent harvesting, addressing gaps in existing research.
- Incorporation of spatial heterogeneity in resource availability, providing deeper insights into its effects on competitive dynamics and population stability.
- Emphasis on the ecological context, highlighting the implications of different harvesting strategies for biodiversity conservation and resource management.

For the purposes of this study, we establish the initial conditions $u_0(x) \geq 0$ and $v_0(x) \geq 0$ for $x \in \overline{\Omega}$, where both functions u_0 and v_0 are strictly positive within an open, nonempty subdomain of Ω. In this framework, the function $K(x)$ serves as the carrying capacity, while $r(x)$ denotes the intrinsic growth rate of the two competing species. We postulate that both the carrying capacity $K(x)$ and the intrinsic growth rate $r(x)$ are space dependent, allowing for a stable analysis of population dynamics. The function $K(x)$ is assumed to be continuous and positive across $\overline{\Omega}$, while we stipulate that $r(x) \geq 0$ for all $x \in \overline{\Omega}$, with $r(x) > 0$ in at least one open, nonempty subdomain of Ω. The notation Ω is utilized to denote a bounded region in $\mathbb{R}^n$ (where $n = 1, 2$, or 3), characterized by a smooth boundary $\partial\Omega$ that belongs to the class $C^{2+\alpha}$ for $0 < \alpha < 1$. The unit normal vector at the boundary, denoted by ι, is employed to describe directional properties of the boundary. We implement the Neumann boundary condition, which specifies that no individuals are permitted to cross the boundary of the habitat, thereby ensuring conservation of the population within the defined region. The Laplace operator Δ describes the random movement of a species in space.

To modify the system presented in equation (1.1) such that the harvesting rate is excluded, we can reformulate the first equation of the model as follows:

$$\frac{\partial u(t,x)}{\partial t} = d_1 \Delta \left(\frac{u(t,x)}{K(x)} \right) + f(u(t,x), v(t,x), K(x)),$$

where, $u(t,x)$ represents the population density of the first species, $f(u(t,x), v(t,x), K(x))$ denotes the intrinsic growth dynamics influenced by interspecific interactions and the carrying capacity $K(x)$, and $d_1 \Delta \left(u(t,x)/K(x) \right)$ captures the carrying capacity-driven diffusion of the population. This formulation eliminates any dependency on harvesting, allowing for a focus on natural population dynamics and competitive interactions.

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta \left(\frac{u(t,x)}{K(x)} \right) + r(x) u(t,x) \left(1 - \theta - \frac{u(t,x) + v(t,x)}{K(x)} \right) \\ &= d_1 (1-\theta) \Delta \left(\frac{u(t,x)}{(1-\theta)K(x)} \right) + r(x) u(t,x)(1-\theta) \left(1 - \frac{u(t,x) + v(t,x)}{(1-\theta)K(x)} \right). \end{aligned}$$

Now, introduce the effective diffusion coefficient, effective growth rate, and effective carrying capacity, such as

$$D_\theta = d_1(1-\theta), \; r_\theta(x) = (1-\theta) r(x), \; K_\theta(x) = (1-\theta) K(x).$$

Upon applying a similar approach to the second equation of the system presented in equation (1.1), we ultimately arrive at the following formulation

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_\theta \Delta \left(\dfrac{u}{K_\theta} \right) + r_\theta u \left(1 - \dfrac{u+v}{K_\theta} \right), \\ \dfrac{\partial v}{\partial t} = \nabla (d(x) \nabla v) + r_\tau v \left(1 - \dfrac{u+v}{K_\tau} \right), \\ t > 0, \quad x \in \Omega, \\ \dfrac{\partial (u/K_\theta)}{\partial \iota} = \dfrac{\partial v}{\partial \iota} = 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \; v(0,x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.2}$$

where, $r_\tau(x) = (1-\tau) r(x), \; K_\tau(x) = (1-\tau) K(x)$.

2 Substantiality and exclusivity

We will now separate each equation to explore the substantiality and exclusivity of the coupled system. The subsequent initial-boundary value problem, subject to Neumann boundary conditions, is formulated as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_\theta \Delta \left(\dfrac{u(t,x)}{K_\theta(x)} \right) + r_\theta(x) u(t,x) \left(1 - \dfrac{u(t,x)}{K_\theta(x)} \right), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ \dfrac{\partial (u/K_\theta)}{\partial \iota} = 0, \; x \in \partial\Omega. \end{cases} \tag{2.1}$$

The following results also discussed in [2, 16, 17, 18, 19].

Lemma 1. *[2, 16, 17, 18, 19] Assuming that the parameters are strictly positive throughout the closure of the domain $\overline{\Omega}$, and that the initial condition for equation (2.1) is represented by a nonnegative continuity function $u_0(x) \in C(\Omega)$, where $u_0(x) \geq 0$ in Ω and $u_0(x) > 0$ within some nonempty, bounded open subset $\Omega_1 \subset \Omega$, it follows that there exists a unique positive solution to the problem described by equation (2.1).*

Lemma 2. *[2, 16, 17, 18, 19] Examine the scenario articulated by equation (2.1). It can be affirmed that there exists a function $K_\theta(x) > 0$, which serves as a distinct stable solution to this equation. Moreover, the initial condition that satisfies $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$, the solution $u(t,x)$ adheres to the following condition*

$$\lim_{t\to\infty} u(t,x) = K_\theta(x)$$

consistently for all $x \in \overline{\Omega}$.

Now, let us consider the subsequent initial-boundary value predicament characterized by Neumann boundary conditions for the density $v(t,x)$.

$$\begin{cases} \dfrac{\partial v}{\partial t} = \nabla(d(x)\nabla v) + r_\tau v\left(1 - \dfrac{v}{K_\tau}\right), \\ v(0,x) = v_0(x), \quad x \in \Omega, \\ \dfrac{\partial v}{\partial \iota} = 0, \;\; x \in \;\partial\Omega. \end{cases} \tag{2.2}$$

Lemma 3. *[2, 16, 17, 18, 19] Assume the initial condition associated with equation (2.2) to be a continuous, non-negative function $v_0(x) \in C(\Omega)$ such that $v_0(x) \geq 0$ throughout Ω and $v_0(x) > 0$ within a certain open, bounded, and non-empty subset $\Omega_1 \subset \Omega$. Under these circumstances, it can be asserted that there exists a unique positive solution to the problem delineated by equation (2.2).*

Lemma 4. *[2, 16, 17, 18, 19] Considering the problem delineated by equation (2.2), we can assert the existence of a function $v^*(x) > 0$, which represents a distinct equilibrium solution to (2.2). The initial condition satisfying $v_0(x) \geq 0$ and $v_0(x) \not\equiv 0$, the solution $v(t,x)$ fulfills the condition*

$$\lim_{t\to\infty} v(t,x) = v^*(x)$$

consistently for all $x \in \overline{\Omega}$.

The concluding result establishes both the substantiality and exclusivity of solutions for the coupled system defined by equations (1.1) and (1.2).

Theorem 1. *[2, 18] Consider the functions $K_\theta(x)$ and $K_\tau(x)$, both of which are positive, indicating that $\theta, \tau \in [0,1)$. Additionally, let $r(x) > 0$ for $x \in \overline{\Omega}$. For any continuous functions $u_0(x)$ and $v_0(x)$ in $C(\Omega)$, the equations (1.1) and (1.2) yield a unique solution represented as (u,v). Furthermore, if the initial functions u_0 and v_0 are both non-negative and nontrivial, it can be concluded that $u(t,x) > 0$ and $v(t,x) > 0$ for every $t > 0$.*

Theorem 2. *[2, 19] Assuming $0 \leq \tau < 1 \leq \theta$ and given initial conditions $u_0, v_0 \geq 0$, the system (1.1) possesses a unique, positive, time-dependent, and nontrivial solution.*

Proof. The proof is similar with [[2] Theorem 2, [19] Theorem 2.5]. □

Theorem 3. *[2, 19] Assuming $0 \leq \theta < 1 \leq \tau$ and given initial conditions $u_0, v_0 \geq 0$, the system (1.1) possesses a unique, positive, time-dependent, and nontrivial solution.*

Proof. The proof is similar with [[2] Theorem 3, [19] Theorem 2.5]. □

3 Preliminary analysis

Let $\widetilde{u}(x) = K_\theta(x)$ denote the stationary solution associated with the following boundary value problem, specifically when the entity v is absent in equation (1.2). The problem is formulated as follows

$$D_\theta \Delta \left(\frac{\widetilde{u}}{K_\theta}\right) + r_\theta \widetilde{u}\left(1 - \frac{\widetilde{u}}{K_\theta}\right) = 0, \quad x \in \Omega, \quad \frac{\partial\left(\widetilde{u}/K_\theta\right)}{\partial \iota} = 0, \quad x \in \partial\Omega. \tag{3.1}$$

Analogously, the function $\widetilde{v}(x) = v^*(x)$ represents the solution to another boundary value problem, which arises when the entity u is absent in equation (1.2). This problem is expressed as

$$\nabla\left(d(x)\nabla\widetilde{v}\right) + r_\tau \widetilde{v}\left(1 - \frac{\widetilde{v}}{K_\tau}\right) = 0, \quad x \in \Omega, \quad \frac{\partial \widetilde{v}}{\partial \iota} = 0, \quad x \in \partial\Omega. \tag{3.2}$$

Lemma 5. *[18] Assuming $0 \leq \theta < 1$, it follows that the function $\widetilde{u}(x) = K_\theta(x)$ serves as the unique positive solution to the boundary value problem delineated by equation (3.1).*

Proof. The solution $\widetilde{u}(x)$ denotes the steady-state solution corresponding to the equation presented in (2.1). It can be inferred from Lemma 2 that $\widetilde{u}(x) = K_\theta(x)$ holds true under the condition $0 \leq \theta < 1$. □

Lemma 6. *[2, 18] Assuming that $v^*(x)$ constitutes a positive solution to the boundary value problem described in equation (3.2), it follows that if $K_\tau(x)$ is not identically constant, certain implications arise. Thus,*

$$\int_\Omega r K_\tau \left(1 - \frac{v^*}{K_\tau}\right) dx > 0. \tag{3.3}$$

4 Stability evaluation of equilibrium positions

In examining the outcomes of competition between two competing entities, it becomes imperative to perform a stability analysis of the semi-trivial steady states, namely $(K_\theta(x), 0)$ and $(0, v^*(x))$. This analysis should also encompass the trivial solution $(0,0)$ as well as the nontrivial stationary solution (u_s, v_s), which denotes the condition of coexistence.

4.1 Scenario of growth rates outrun harvesting levels

This section is organized around the scenario in which the intrinsic growth rate outpaces the harvesting rate, specifically for parameters $\theta, \tau \in [0,1)$. Given that $H_1(x) = \theta r(x)$ and $H_2(x) = \tau r(x)$, it follows that if $\theta, \tau \in [0,1)$, then it is evident that $0 \leq H_1(x) < r(x)$ and $0 \leq H_2(x) < r(x)$. In this section, we will explore two distinct cases: the first case in which $\theta \leq \tau$, and the second case where $\theta \geq \tau$.

Lemma 7. *Consider $\theta, \tau \in [0,1)$. This condition ensures that $K_\theta(x), K_\tau(x), r_\theta(x) > 0$ and $r_\tau(x) > 0$ are positive across $\overline{\Omega}$. As a result, the solution $(0,0)$ for the system (1.2) is identified as an unstable repelling stationary solution.*

Proof. Assume $w(t,x) = \frac{u(t,x)}{K_\theta(x)}$ with the preliminary condition specified as $w(0,x) = \frac{u(0,x)}{K_\theta(x)} = w_0(x)$. Consequently, the linearized equation (1.2) is considered in the vicinity of this solution.

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{D_\theta}{K_\theta(x)}\Delta w + r_\theta w, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial t} = \nabla(d(x)\nabla v) + r_\tau v, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial w}{\partial \iota} = \dfrac{\partial v}{\partial \iota} = 0, \quad x \in \partial\Omega, \\ w(0,x) = w_0(x),\ v(0,x) = v_0(x), \quad x \in \Omega. \end{cases} \tag{4.1}$$

The eigenvalue equations relevant to this discussion are presented below:

$$\begin{cases} \xi\zeta = \dfrac{D_\theta}{K_\theta(x)}\Delta\zeta + r_\theta\zeta, \ x \in \Omega, \\ \pi\varphi = \nabla(d(x)\nabla\varphi) + r_\tau\varphi, \ x \in \Omega, \\ \dfrac{\partial\zeta}{\partial\iota} = \dfrac{\partial\varphi}{\partial\iota} = 0, \ \ x \in \ \partial\Omega. \end{cases} \tag{4.2}$$

Employing the characterization of eigenvalues as described in [16], we derive the principal eigenvalue by selecting the eigenfunction $\zeta = 1$.

$$\xi_1 \geq \frac{1}{|\Omega|}\int_\Omega r_\theta \, dx = \frac{1}{|\Omega|}\int_\Omega (1-\theta)r \, dx > 0, \ \ \theta \in [0,1).$$

Similarly, by applying the characterization of eigenvalues as outlined in [16], we determine the principal eigenvalue by selecting the eigenfunction $\varphi = 1$.

$$\pi_1 \geq \frac{1}{|\Omega|}\int_\Omega (1-\tau)r \, dx > 0, \ \ \tau \in [0,1).$$

Consequently, the stationary point $(0,0)$ is unstable. We will now demonstrate that the solution $(0,0)$ acts as a repeller. The proof follows the same approach as that presented in [[17], Theorem 5]. ☐

The subsequent case illustrates the results of the rivalry when the growth rate outruns the harvesting level for $0 \leq \theta \leq \tau < 1$.

4.1.1 Scenario $\theta \leq \tau$

In the subsequent lemma, we establish that the steady state $(K_\theta, 0)$ exhibits instability whenever $\theta \leq \tau$. It is noteworthy that the proof presented here bears a resemblance to that found in [3].

Lemma 8. *Assume that θ and τ are elements of the interval $[0,1)$ with the condition that $\theta \leq \tau$. Under these circumstances, it follows that the steady state $(K_\theta, 0)$ is deemed unstable to the equation presented in (1.2).*

Proof. The case where $\theta = \tau$ has been thoroughly investigated in [20], particularly in the context of entities that exhibit a carrying capacity. In light of this, our discourse will now shift to the scenario in which $\theta < \tau$. To facilitate our analysis, we will undertake a linearization of the second function derived from (1.2), specifically in the neighborhood of the steady state represented by $(K_\theta(x), 0)$.

$$\begin{aligned} &\frac{\partial v}{\partial t} = \nabla(d(x)\nabla v) + r_\tau v\left(1 - \frac{K_\theta}{K_\tau}\right), \ \ x \in \Omega, \\ &v(0,x) = v_0(x), \quad x \in \Omega, \\ &\frac{\partial v}{\partial \iota} = 0, \ \ x \in \partial\Omega. \end{aligned}$$

The associated eigenvalue can be articulated in the following manner:

$$\pi\zeta = \nabla(d(x)\nabla\zeta) + r_\tau\zeta\left(1 - \frac{K_\theta}{K_\tau}\right), \ \ x \in \Omega, \ \ \frac{\partial\zeta}{\partial\iota} = 0, \ \ x \in \partial\Omega. \tag{4.3}$$

As detailed in [16], the principal eigenvalue associated with this equation is comprehensively characterized.

$$\pi_1 = \sup_{\zeta \neq 0, \zeta \in W^{1,2}} \left\{ \frac{-\int_\Omega d(x)|\nabla(\zeta)|^2 \, dx + \int_\Omega r_\tau \zeta^2 \left(1 - \dfrac{K_\theta}{K_\tau}\right) \, dx}{\int_\Omega \zeta^2 \, dx} \right\}. \tag{4.4}$$

Let $\zeta = 1$ and substitute the value in (4.4), we obtain

$$\pi_1 \geq \frac{\int_\Omega r_\tau \left(\frac{K_\tau - K_\theta}{K_\tau}\right) dx}{\int_\Omega 1\, dx} > 0, \quad \text{whenever } \; K_\tau > K_\theta.$$

Therefore, the steady state $(K_\theta, 0)$ is unstable of the system (1.2). □

In the subsequent lemma, we establish that the steady state $(0, v^*)$ is rendered unstable whenever the condition $\theta \leq \tau$ holds.

Lemma 9. *Consider the scenario where we assume θ and τ reside in the interval $[0,1)$, with the condition that $\theta \leq \tau$. Under these circumstances, it follows that the steady state $(0, v^*)$ is classified as unstable within the framework of the system governed by equation (1.2).*

Proof. In the context where $\theta = \tau$, as discussed in [20], we now turn our attention to the case where $\theta < \tau$. Let us define $w(t,x) = \frac{u(t,x)}{K_\theta(x)}$, with the initial condition specified by $w(0,x) = \frac{u(0,x)}{K_\theta(x)} = w_0(x)$. We proceed by linearizing the first equation of (1.2) in the vicinity of the steady state $(0, v^*)$ in the following manner:

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \frac{D_\theta}{K_\theta}\Delta w + r_\theta w \left(1 - \frac{v^*}{K_\theta}\right), \quad t > 0,\ x \in \Omega,\\
&w(0,x) = w_0(x), \quad x \in \Omega,\\
&\frac{\partial w}{\partial \iota} = 0, \quad x \in \partial\Omega.
\end{aligned}$$

The associated eigenvalue can be articulated as follows:

$$\pi K_\theta \zeta = D_\theta \Delta \zeta + r_\theta K_\theta \zeta \left(1 - \frac{v^*}{K_\theta}\right), \quad x \in \Omega, \quad \frac{\partial \zeta}{\partial \iota} = 0, \quad x \in \partial\Omega. \tag{4.5}$$

The principal eigenvalue associated with this problem is delineated in [16], where it is discussed in the context of the underlying mathematical framework.

$$\pi_1 = \sup_{\zeta \neq 0, \zeta \in W^{1,2}} \left\{ \frac{-D_\theta \int_\Omega |\nabla(\zeta)|^2 dx + \int_\Omega r_\theta \zeta^2 \left(K_\theta - v^*\right)\, dx}{\int_\Omega K_\theta \zeta^2\, dx} \right\}. \tag{4.6}$$

Let $\zeta = 1$ and substitute the value in (4.6),

$$\pi_1 \geq \frac{\int_\Omega r_\theta \left(K_\theta - v^*\right) dx}{\int_\Omega K_\theta\, dx}. \tag{4.7}$$

The sign of the fraction present on the right-hand side is dictated exclusively by the behavior of the numerator, which retains its positivity contingent upon the following conditions:

$$\int_\Omega r_\theta K_\theta dx > \int_\Omega r_\theta v^* dx. \tag{4.8}$$

In [18], it has been rigorously demonstrated that, in the absence of harvesting, the coexistence of species is rendered infeasible, and the equilibrium point $(0, v^*)$ is identified as an unstable solution. Thus, we can conclusively assert that the proof is complete. □

In the ensuing theorem, we rigorously demonstrate that the equilibrium point (u_s, v_s) possesses global stability for the dynamical system delineated by (1.2), provided that the condition $\theta \leq \tau$ holds. This significant result is substantiated through the meticulous application of Lemma 7, Lemma 8, and Lemma 9.

Theorem 4. *Considering $\theta, \tau \in [0, 1)$ such that the inequality $\theta \leq \tau$ is satisfied. Within this framework, it can be asserted that the equilibrium point (u_s, v_s) associated with both the system delineated by (1.1) and that characterized by (1.2) is rigorously established to exhibit global stability.*

Proof. According to the theorem of monotone dynamical systems [[21], Theorem A], if the equilibrium points $(0, 0)$, $(K_\theta, 0)$, and $(0, v^*)$ are unstable, which demonstrated in Lemma 7, Lemma 8, and Lemma 9 respectively, then the coexistence state (u_s, v_s) is globally stable. □

The next case demonstrates the result of the consequences of competition, specifically when the growth function outruns the harvesting level for all $0 \leq \tau \leq \theta < 1$, leading to a scenario where the population density increases without bound, thereby affecting the stability of the ecosystem and potentially resulting in the depletion of resources.

4.1.2 Scenario $\theta \geq \tau$

The section encompasses symmetrical lemmata that have been established in Subsection 4.1.1, and consequently, we shall omit the proofs while instead referring to the corresponding lemmata presented therein.

Lemma 10. *Considering $\theta \geq \tau$, where $\theta, \tau \in [0, 1)$, we conclude that the equilibrium point $(0, v^*)$ of the system described by (1.2) is deemed unstable.*

Proof. Given the similarities in structure and reasoning, the proof is similar to that of Lemma 8, thereby reinforcing the established conclusions. □

Lemma 11. *Assuming that $\theta \geq \tau$, with both θ and τ belonging to the interval $[0, 1)$, we deduce that the steady state $(K_\theta, 0)$ of the equation (1.2) is inherently characterized as unstable, reflecting the underlying dynamics of the system.*

Proof. The proof unfolds in a manner analogous to that of Lemma 9, thereby illustrating the consistency of the methodologies employed in this context. □

Theorem 5. *Supposing that $\theta \geq \tau$, where both θ and τ reside within the interval $[0, 1)$, we conclude that the coexistence equilibrium of the equation described by (1.2) is globally stable, indicating a robust resilience of the system's dynamics.*

Proof. The proof is similar to Theorem 4. □

4.2 Harvesting level of one entity surpasses the intrinsic growth rate

Here, we explore the outcomes concerning the interactions of two competitive entities under the condition that the harvesting level within the equation described by (1.1) outruns their respective intrinsic growth rates. This scenario gives rise to two distinct possibilities: either $H_1(x) \geq r(x)$ or $H_2(x) \geq r(x)$, corresponding to the conditions $0 \leq \tau < 1 \leq \theta$ or $0 \leq \theta < 1 \leq \tau$, respectively.

Initially, we illustrate the results on the effects of competition when one harvesting function surpasses the corresponding intrinsic growth function in the specific case where $0 \leq \tau < 1 \leq \theta$.

4.2.1 Scenario $\tau < 1 \leq \theta$

Lemma 12. *Assuming that $0 \leq \tau < 1 \leq \theta$, it follows that there is no coexistence steady state (u_s, v_s) for the problems (1.1) and (1.2).*

Proof. Assuming the existence of a coexistence solution $(u_s(x), v_s(x))$, where $u_s, v_s \geq 0$ for all $x \in \Omega$, it follows that this solution must satisfy the subsequent system of equations, which delineates the interactions and dependencies between the species in the given ecological framework.

$$\begin{cases} D_\theta \Delta\left(\frac{u_s}{K}\right) + ru_s\left(1-\theta-\frac{u_s+v_s}{K}\right) = 0, \\ \nabla(d(x)\nabla v_s) + rv_s\left(1-\tau-\frac{u_s+v_s}{K}\right) = 0, \\ x \in \Omega, \\ \frac{\partial(u_s/K)}{\partial\iota} = \frac{\partial v_s}{\partial\iota} = 0, \quad x \in \partial\Omega. \end{cases} \tag{4.9}$$

By integrating the first equation of the problem over the domain Ω, while duly accounting for the relevant boundary conditions, we arrive at the following expression:

$$\int_\Omega ru_s\left(1-\theta-\frac{u_s+v_s}{K}\right)dx = 0.$$

For all $x \in \Omega$, the integrand is non-positive whenever $\theta \geq 1$. This is particularly true under the condition that $u_s \not\equiv 0$, a fact assured by our assumption that u_s serves as a coexistence solution. Now, if we assume $\theta = 1$, we observe that since $u_s \not\equiv 0$, the integrand remains non-positive, except in the case where $u_s + v_s \equiv 0$. However, this situation is not feasible for a non-negative coexistence solution, resulting in a contradiction. Next, let us examine the scenario where $\theta > 1$. In this case, if we posit that $u_s + v_s \equiv K(1-\theta)$, the problem represented by (4.9) transforms into

$$\begin{cases} D_\theta \Delta\left(\frac{u_s}{K}\right) + ru_s\left(1-\theta-\frac{K(1-\theta)}{K}\right) = 0, \\ \nabla(d(x)\nabla v_s) + rv_s\left(1-\tau-\frac{K(1-\theta)}{K}\right) = 0, \\ x \in \Omega, \\ \frac{\partial(u_s/K)}{\partial\iota} = \frac{\partial v_s}{\partial\iota} = 0, \quad x \in \partial\Omega, \end{cases}$$

$$\Rightarrow \begin{cases} D_\theta \Delta\left(\frac{u_s}{K}\right) = 0, \quad x \in \Omega, \\ \frac{\partial(u_s/K)}{\partial\iota} = 0, \quad x \in \partial\Omega, \\ \nabla(d(x)\nabla v_s) + rv_s(\theta-\tau) = 0, \quad x \in \Omega, \\ \frac{\partial v_s}{\partial\iota} = 0, \quad x \in \partial\Omega. \end{cases}$$

As a consequence, the maximum principle [22] implies that the solution must take the form $a_s \equiv$ const on $\overline{\Omega}$. Furthermore, by integrating the second problem while applying the appropriate boundary conditions, we arrive at the following result:

$$\int_\Omega rv_s(\theta-\tau)\,dx = 0.$$

This situation is untenable unless v_s is trivial, which would imply that $u_s = K(1-\theta)$. Such a conclusion contradicts our assumption that the pair (u_s, v_s) is nontrivial. Therefore, we conclude that there cannot exist a coexistence equilibrium (u_s, v_s), thereby proving the lemma. □

Lemma 13. *Considering the case where $\theta \geq 1$, it follows that $(0, v^*)$ is the sole nontrivial equilibrium solution to the equations (1.1) and (1.2).*

Proof. The proof is similar with [2]. □

Lemma 14. *Let* $0 \leq \tau < 1 \leq \theta$*; under these conditions, while the trivial solution* $(0,0)$ *of the model defined by (1.1) and (1.2) is characterized as unstable, it simultaneously fails to meet the criteria necessary to be classified as a repeller [21].*

Proof. Initially, we assume that $\theta > 1$ and let $w = \frac{u}{K_\theta}$; consequently, we consider the linearized system defined by (1.2) in the vicinity of the trivial equilibrium point.

$$\begin{cases} \dfrac{\partial w}{\partial t} = D_\theta w + r_\theta w, \\ \dfrac{\partial v}{\partial t} = \nabla(d(x)\nabla v) + r_\tau v, \\ t > 0, \quad x \in \Omega, \\ \dfrac{\partial w}{\partial \iota} = \dfrac{\partial v}{\partial \iota} = 0, \quad x \in \ \partial\Omega, \\ w(0,x) = w_0(x), \ v(0,x) = v_0(x), \quad x \in \Omega. \end{cases}$$

The eigenvalue associated with this system are expressed as follows:

$$\begin{cases} \xi\zeta = D_\theta \Delta\zeta + r_\theta \zeta, \\ \pi\varphi = \nabla(d(x)\nabla\varphi) + r_\tau \varphi, \\ x \in \Omega, \\ \dfrac{\partial \zeta}{\partial \iota} = \dfrac{\partial \varphi}{\partial \iota} = 0, \quad x \in \ \partial\Omega. \end{cases} \tag{4.10}$$

Let ζ_1 and φ_1 denote two eigenfunctions, which can be selected to be positive, along with their corresponding principal eigenvalues ξ_1 and π_1 from the eigenvalue defined by (4.10) [16]. By integrating (4.10) under the given boundary constraints, we obtain

$$\xi_1 = \frac{\int_\Omega r_\theta \zeta_1 \, dx}{\int_\Omega \zeta_1 \, dx},$$

this entails that

$$\xi_1 = \frac{\int_\Omega (1-\theta) r \zeta_1 \, dx}{\int_\Omega \zeta_1 \, dx} < 0, \quad \theta > 1. \tag{4.11}$$

Thus,

$$\pi_1 = \frac{\int_\Omega r_\tau \varphi_1 \, dx}{\int_\Omega \varphi_1 \, dx},$$

this suggests that

$$\pi_1 = \frac{\int_\Omega (1-\tau) r \varphi_1 \, dx}{\int_\Omega \varphi_1 \, dx} > 0, \quad \tau < 1, \tag{4.12}$$

respectively, the trivial equilibrium point $(0,0)$ is deemed unstable. Considering the first equation from (1.2), it is important to note that when $\theta > 1$, the parameters are negative. According to Lemma 2, the space independent solutions (w,v) remain positive as long as $w_0 \neq 0$ or $v_0 \neq 0$. We will now establish the following inequality for the case where $\theta > 1$ and recall that $K_\theta = (1-\theta)K$,

$$1 - \frac{w+v}{K_\theta} = 1 + \frac{w+v}{\mid K_\theta \mid} \geq 1.$$

Let $r_1 = 1-\theta$, and multiplying by r_1 whenever $\theta > 1$, where $r_1 = 1-\theta$, we obtain

$$r_1 \left(1 - \frac{w+v}{K_\theta}\right) = r_1 \left(1 + \frac{w+v}{\mid K_\theta \mid}\right) \leq r_1$$

$$\Rightarrow r_1\left(1-\frac{w+v}{K_\theta}\right)\le r_1.$$

Thus, we derive from the first problem in (1.2) that

$$\begin{aligned}\frac{\partial w}{\partial t}&=D_\theta\Delta w+r_\theta w\left(1-\frac{w+v}{K_\theta}\right)\le D_\theta\Delta w+r_\theta w,\\ \Rightarrow\frac{\partial w}{\partial t}&\le D_\theta\Delta w+r_\theta w.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial w}{\partial t}&\le D_\theta\Delta w+r_\theta w,\\ \frac{\partial v}{\partial t}&\ge\nabla(d(x)\Delta v)+r_\tau v\left(1-\frac{w+v}{K_\tau}\right).\end{aligned}$$

By integrating and applying the boundary condition, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega w\,dx&\le\int_\Omega r_\theta w\,dx,\\ \frac{d}{dt}\int_\Omega v\,dx&\ge\int_\Omega r_\tau v\left(1-\frac{w+v}{K_\tau}\right)\,dx.\end{aligned}$$

Now, examine the positive numbers $0<\varrho\le\inf_{x\in\Omega} r_\tau\left(1-\frac{2\delta}{K_\tau}\right)$ also $0<\delta\le\inf_{x\in\Omega}\left(\frac{K_\tau}{4}\right)$ (see [[17] Theorem 5, [18] Theorem 9]) such that for initial conditions satisfying $w_0+v_0<\delta,\ w_0\ne 0,\ v_0\ne 0,\ w_0\ge 0$, also $v_0\ge 0$, we obtain

$$\frac{d}{dt}\int_\Omega v\,dx>\int_\Omega r_\tau v\left(1-\frac{2\delta}{K_\tau}\right)\,dx.$$

Finally, we obtain

$$\frac{d}{dt}\int_\Omega v\,dx>\rho\int_\Omega v\,dx.$$

By applying the Grönwall inequality [23], we obtain

$$\int_\Omega v\,dx\ge e^{\varrho t}\int_\Omega v(0,x)\,dx,\quad t>0.$$

It is important to note that ϱ is positive, which implies that the integral on the right side increases exponentially. Now, let us consider the first problem

$$\frac{d}{dt}\int_\Omega w\,dx\le\int_\Omega r_\theta w\,dx.$$

When $r_\theta<0$ for $\theta>1$, there can be found a real number $\epsilon=\sup_{x\in\Omega} r_\theta<0$, for all $\theta>1$ (see [18] Theorem 9, [19] Theorem 3.4), such that $r_\theta<-|\epsilon|<0$, which leads to

$$\begin{aligned}\frac{d}{dt}\int_\Omega w\,dx&\le\int_\Omega r_\theta w\,dx<-\mid\epsilon\mid\int_\Omega w\,dx,\\ \Rightarrow\frac{d}{dt}\int_\Omega w\,dx&<-\mid\epsilon\mid\int_\Omega w\,dx.\end{aligned}$$

We can now apply the Grönwall inequality [23] to derive

$$\int_\Omega w\, dx \le e^{-|\epsilon| t} \int_\Omega w(0,x)\, dx.$$

On the right side, there exists an exponential term that converges to zero over time; consequently, the equilibrium point $(0,0)$ is repelling with respect to v and enticing with respect to w, thus failing to satisfy the definition provided in [21]. Furthermore, considering the case when $\theta = 1$, the instability of $(0,0)$ arises from the inequality (4.12), leading to the transformation of the first equation in (1.1) into

$$\begin{cases} \dfrac{\partial w}{\partial t} = d_1 \Delta w + rw\left(1 - 1 - \dfrac{w+v}{K}\right), \\ \dfrac{\partial v}{\partial t} = \nabla(d(x)\nabla v) + rv\left(1 - \tau - \dfrac{w+v}{K}\right), \\ t > 0, \quad x \in \Omega, \\ \dfrac{\partial w}{\partial \iota} = \dfrac{\partial v}{\partial \iota} = 0, \quad x \in \partial\Omega, \\ w(0,x) = w_0(x),\ v(0,x) = v_0(x), \quad x \in \Omega, \end{cases}$$

which implies

$$\begin{cases} \dfrac{\partial w}{\partial t} = d_1 \Delta w - rw\left(\dfrac{w+v}{K}\right), \\ \dfrac{\partial v}{\partial t} = \nabla(d(x)\nabla v) + rv\left(1 - \tau - \dfrac{w+v}{K}\right). \end{cases} \tag{4.13}$$

The subsequent part of the proof follows the same methods described earlier, and we shall forgo the detailed proof for $\theta = 1$. Thus, for $0 \le \tau < 1 \le \theta$, the trivial equilibrium $(0,0)$ exhibits instability but does not function as a repeller. □

Theorem 6. *The case where $0 \le \tau < 1 \le \theta$. Under these conditions, the equilibrium point $(0, v^*(x))$ of the equation defined by equations (1.1) and (1.2) is globally asymptotically stable.*

Proof. The only remaining non-negative steady state is given by the solution $(0, v^*(x))$ for the case $0 \le \tau < 1 \le \theta$, which is proved in Lemma 13. Therefore, according to the existence-uniqueness theorem for parabolic paired systems [24, Theorem 10.5.3], we can conclude that the time-dependent solution of (1.1) converges to the unique semi-trivial steady state $(0, v^*)$ for any initial condition from $\mathbf{S}_\varrho$. This completes the proof. □

4.2.2 Scenario $\theta < 1 \le \tau$

The section contains lemmas that are symmetrical and have been proven in Subsection 4.2.1. Therefore, we will omit the proofs and refer to the corresponding lemmas from Subsection 4.2.1. We will investigate the case when the harvesting rate H_2 is greater than or equal to the intrinsic growth rate $r(x)$ for all $x \in \Omega$.

Lemma 15. *Assuming that $0 \le \theta < 1 \le \tau$, we can conclude that the model defined by equations (1.1) does not admit any coexistence, thereby highlighting the implications of these parameter constraints on the dynamics of the system.*

Proof. The proof follows a similar approach to that of Lemma 12. □

Lemma 16. *Under the assumption that $0 \le \theta < 1 \le \tau$, it can be established that the only nontrivial stationary solution for the model described by equations (1.1) is $(K_\theta, 0)$.*

Proof. The approach taken in the proof closely resembles that of Lemma 13, incorporating the insights from Lemma 5. □

Lemma 17. *Given the conditions $0 \leq \theta < 1$ and $1 \leq \tau$, we conclude that the trivial solution $(0,0)$ of the model described by equations (1.1) is unstable. Nevertheless, it does not meet the criteria for being classified as a repeller, as outlined in Theorem [21].*

Proof. The proof resembles that of Lemma 14, as established by the Theorem in [21]. □

Theorem 7. *The case where $0 \leq \theta < 1$ and $1 \leq \tau$. In this scenario, the semi-trivial equilibrium $(K_\theta, 0)$ of the model given by equations (1.1) is globally asymptotically stable.*

Proof. The proof is similar to Theorem 6. □

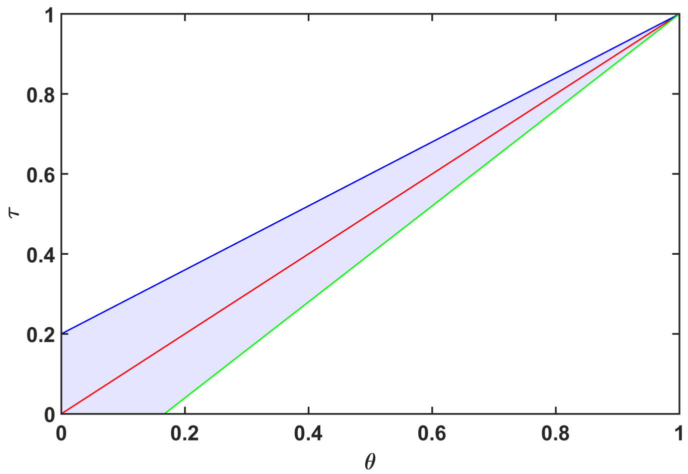


Figure 4.1: Illustration of the harvesting rate zones that ensure the global asymptotic stability of coexistence solutions.

Figure 4.1, illustrates relationship between the harvesting coefficients θ and τ. The shaded region represents the feasible parameter space for the coefficients. The blue line indicates the boundary where $\tau = 0.2$, while the red and green lines show the boundaries for other specific values of τ. The area below the blue line corresponds to harvesting rates where both species can potentially coexist, while the region above the lines signifies conditions that may lead to the extinction of one or both species.

4.3 Since the harvesting level outruns the intrinsic growth rate

The situation where both harvesting rates are greater than the intrinsic growth rates, particularly when $\theta, \tau \geq 1$.

4.3.1 Scenario $\theta, \tau \geq 1$

In the forthcoming theorem, we will elucidate the global asymptotic stability of the trivial equilibrium $(0,0)$ by employing Lemma 14 from Subsection 4.2.1, although one may also consider Lemma 17 from Subsection 4.2.2 as an alternative reference.

Theorem 8. *Assuming that $\theta, \tau \geq 1$, we conclude that the solution $(0,0)$ of the model described by equation (1.1) exhibits global asymptotic stability.*

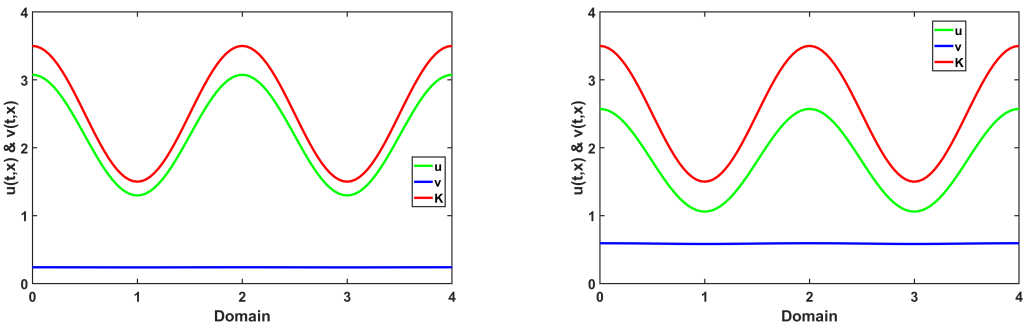
Proof. The argument presented in the proof of Lemma 14 (or Lemma 17) is explicitly applicable in this context, demonstrating convergence to the state. Thus, we conclude that the proof is complete. □

5 Empirical outcomes

For the analysis, taking the parameters θ and τ to be greater than zero, specifically $\theta, \tau > 0$. Additionally, we assume that the diffusion coefficients are constant, setting $d_1 = 1$ and $d(x) \equiv 1$ for simplicity.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\left(\dfrac{u}{K}\right) + ru\left(1 - \theta - \dfrac{u+v}{K}\right), \\ \dfrac{\partial v}{\partial t} = \Delta v + rv\left(1 - \tau - \dfrac{u+v}{K}\right), \\ t > 0, \quad x \in \Omega, \\ \dfrac{\partial(u/K)}{\partial \iota} = \dfrac{\partial v}{\partial \iota} = 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in \Omega. \end{cases} \tag{5.1}$$

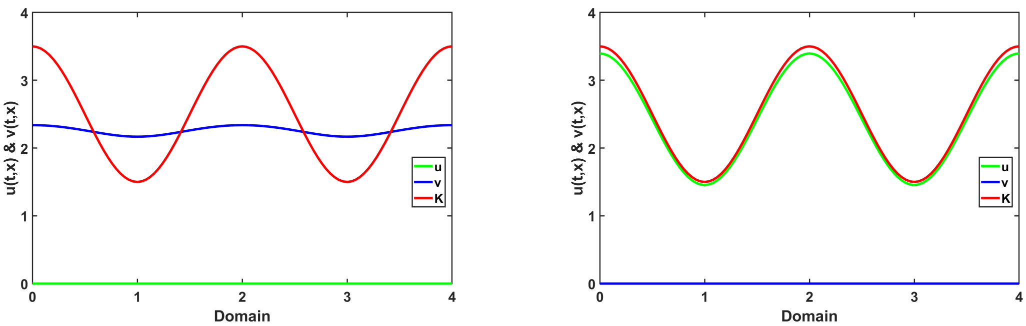
Example 1. *Assume* $K = \cos(\pi x) + 2.5$, $u_0 = 0.6$, $v_0 = 1.9$, *and* $r = 1$ *on* $x \in \Omega = (0, 4)$.



(a) (b)

Figure 5.1: When (a) $\theta = 0.03$, $\tau = 0.04$, and (b) $\theta = 0.04$, $\tau = 0.03$ for $r = 1$, $u_0 = 0.6$, $v_0 = 1.9$, $K = 2.5 + \cos(\pi x)$, on $x \in (0, 4)$ at time $t = 100$, illustrates the solutions of (1.1).

Figure 5.1(a) satisfies Theorem 4, while Figure 5.1(b) satisfies Theorem 5. These figures demonstrate that when $\tau \geq \theta$ *or* $\theta \geq \tau$ *whenever* $\theta, \tau \in [0, 1)$, *both species can coexist in a heterogeneous environment. Conversely, the subsequent figure illustrates a scenario where only one species persists, while the other is endangered due to the non-homogeneous environment.*



(a) (b)

Figure 5.2: When (a) $\theta = 1.1$, $\tau = 0.03$, and (b) $\theta = 0.03$, $\tau = 1.1$ for $r(x) = 1$, $u_0 = 0.6$, $v_0 = 1.9$, $K(x) = 2.5 + \cos(\pi x)$, on $x \in (0, 4)$ at time $t = 100$, illustrates the solutions of (1.1).

Figure 5.2(a) justifies Theorem 6, while Figure 5.2(b) justifies Theorem 7. These figures illustrate that when $0 \leq \tau < 1 \leq \theta$ *or* $0 \leq \theta < 1 \leq \tau$, *only one species can survive in the heterogeneous environment. The subsequent figure delineates the trivial solution, indicating that no species can persist in the environment.*

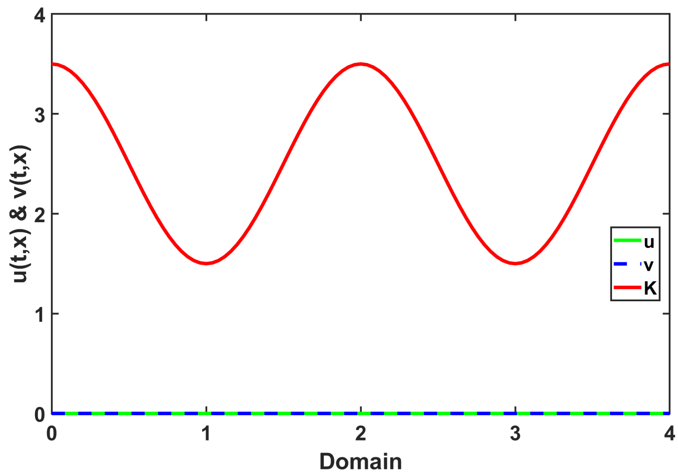
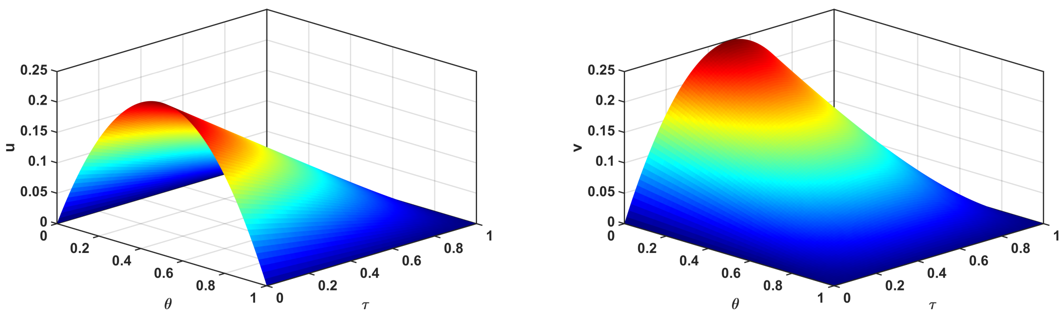


Figure 5.3: When $\theta = \tau = 1.1$ for $r = 1,\ u_0 = 0.6,\ v_0 = 1.9,\ K = 2.5 + \cos(\pi x)$, on $x \in (0, 4)$ at time $t = 100$, illustrates the solutions of (1.1).

Figure 5.3 which support the Theorem 8. This figure demonstrates that when $\theta, \tau \geq 1$ *both species die out from the environment.*

Example 2. *In ecological dynamics, it is common for two competing populations to vie for limited resources such as food and habitat. This competition can significantly influence the persistence or extinction of species. A key aspect of this interaction is the relationship between the harvesting coefficients, denoted as* θ *and* τ. *These coefficients represent the rates at which each population is harvested and can be critical in determining population dynamics. By analyzing the relationship between* θ *and* τ, *we can gain insights into the conditions under which species coexist, thrive, or face extinction. This example illustrates the population densities as functions of the harvesting rates* θ *and* τ. *The figure depicts the conditions under which the species can coexist, as well as the scenarios that may threaten their survival.*



(a) (b)

Figure 5.4: Mean densities of species (a) u, and (b) v as functions of harvesting rates in the ranges $\theta, \tau \in [0, 1]$ for $r(x) \equiv 1,\ u_0 = 0.6,\ v_0 = 1.9,\ K = 2.5 + \cos(\pi x)$, on $x \in (0, 4)$.

The first surface plot 5.4(a) illustrates the relationship between the parameters θ and τ with the output variable u. The surface appears to have a peak or maximum point, suggesting there is a particular combination of θ and τ that maximizes u. The shape suggests that there are regions of increasing u as either θ or τ increases, followed by a decline, indicating that there is an optimal range for these parameters. The second surface plot 5.4(b) shows the relationship between the same parameters θ and τ but with the output variable v. In contrast to the first plot, this surface exhibits a different behavior, potentially indicating diminishing returns or a different interaction between θ and τ as they influence v. Both plots collectively provide insight into how these parameters interact to affect the respective outputs u and v, suggesting the existence of optimal parameter combinations for desired outcomes.

The interaction of the harvesting coefficients θ and τ plays a crucial role in determining the stability of the system. By varying these coefficients, we can analyze scenarios that lead to the extinction of one or both populations, or conditions that allow for their coexistence. In the following figure, we determine the threshold of harvesting coefficients, calculating θ while keeping τ fixed. For Figure 5.5, we use the equation (3.3) in the following manner:

$$\int_\Omega rK_\tau\,dx - \int_\Omega rv^*\,dx > 0$$
$$\Rightarrow \frac{\int_\Omega rv^*\,dx}{\int_\Omega rK_\tau\,dx} < 1.$$

Therefore, we express the critical value of θ as:

$$\theta^* := 1 - \frac{\int_\Omega rv^*\,dx}{\int_\Omega rK_\tau\,dx}. \tag{5.2}$$

We use the equation (5.2) to determine the critical value of θ for Figure 5.5.

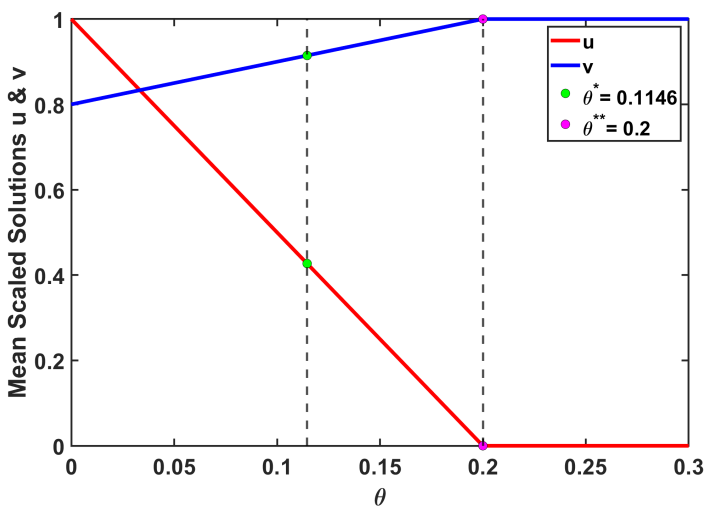


Figure 5.5: Mean solutions of (1.1) for fixed $\tau = 0.02$, $r = 1$, $u_0 = 0.6$, $v_0 = 1.9$, $K = 2.5 + \cos(\pi x)$, on $x \in (0, 4)$ at time $t = 100$.

In Figure 5.5, the entities u and v could represent the population sizes of two interacting entities. For the fixed value of $\tau = 0.02$, the points θ^* (green) and θ^{**} (magenta) indicate critical harvesting thresholds. At these points, interventions might be needed to maintain balance in the ecosystem.

6 Conclusive insights

Two competing species within spatially heterogeneous environments, investigating how variations in habitat and resource distribution influence their interactions. By analyzing these ecological relationships, we aim to understand the mechanisms driving competition and coexistence in diverse landscapes. Coexistence of species is feasible when the harvesting rates do not surpass their intrinsic growth rates, specifically within the parameter range $\theta, \tau \in [0, 1)$. For small values of θ and τ, competitive populations can coexist, as confirmed by both analytical and numerical analyses. We have identified a threshold for the harvesting coefficient, beyond which coexistence is no longer possible. If the harvesting rate exceeds the growth rate of one species, that species will go extinct while the other may persist, provided its harvesting rate remains below its growth rate. Ultimately, when the harvesting rates of both entities surpass their respective growth rates, it leads to a critical threshold that endangers their populations; this imbalance causes the system characterized by equations (1.1) and (1.2) to converge towards the $(0, 0)$ state, indicating a potential collapse of both entities within the ecosystem. These observations indicate that effective management of harvesting rates is crucial for maintaining the balance of competing species in heterogeneous environments. Future research could explore additional factors influencing these dynamics, such as environmental changes or species interactions.

Conflict of interest

The authors declare no conflict of interest.

Data Availability statement

There is no available data regarding this study.

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