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Fractional differential equations and its solutions using the Laplace Variational Iteration Method

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ABSTRACT

In this paper, we discuss fractional differential equations, including the Fokker-Planck equation and fractional diffusion differential equations, which are closely related to chemistry and engineering. To solve these equations, we employ the Laplace Variational Iteration Method (LVIM), which combines the Laplace transform with He's Variational Iteration Method. To demonstrate the efficiency and validity of LVIM, we consider two 1-D Fokker-Planck equations and three fractional diffusion equations in 1-D, 2-D, and 3-D. We solve these equations using LVIM, and the results are presented analytically in tables and graphically using MATLAB for different values of the fractional order and these results are then compared with those obtained by existing methods. The solutions obtain as infinite series, and for certain values of the fractional order, they are found to be similar to the exact results.

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Introduction:

Fractional-order differential equations, in which an unknown function has a fractional-order derivative, have recently received a lot of interest. This interest stems from both the rapid growth of theoretical knowledge and the numerous applications of these differential equations in various scientific and engineering domains and was first mentioned in a

Nomenclature	
α	Fractional order of derivative
f^{α}	Alpha times derivatives of function f
D_x^{α}	Riemann fractional derivative of order α with respect to x
${}_a^C D_x^{\alpha}$	Caputo fractional derivative of order α with respect to x, at the point a
$E_{\alpha}(\xi)$	Mittag-Leffler function

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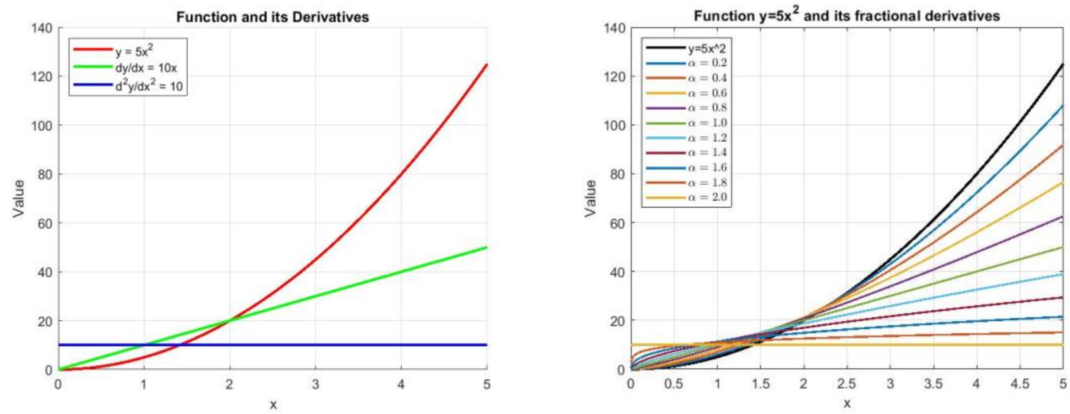
1695 letter from Leibniz to L'Hopital [1, 2]. Several recent studies have employed fractional-order differential models to analyze real-world biological systems. Shah et al. [3] proposed a COVID-19 model using the Caputo derivative, accounting for natural death in all compartments. Naik et al. [4] applied Caputo-Fabrizio and Atangana-Baleanu operators to enzyme-catalyzed processes to capture hereditary behaviors. In 2024, a fractional model using Caputo derivatives was developed for HIV-HCV co-infection [5]. Another study used a Caputo-based optimal control model for RSV transmission, exploring solution properties, stability, bifurcation, and control strategies [6]. Many FDEs do not have exact solutions, therefore mathematicians have been focusing on numerical solutions of these equations. Among them most useful method to find the approximate or numerical solutions are the Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Fractional Difference Method (FDM), Differential Transform Method (DTM), and Homotopy Perturbation Method (HPM). Classical solution techniques like the Laplace transform method, Fractional Green's function method, Mellin transform method, and the method of orthogonal polynomials are also employed [11]. VIM and ADM are particularly noteworthy for their ability to provide both symbolic and numerical solutions to linear and nonlinear differential equations without requiring linearization or discretization [8,9,10,11]. A key advantage of using fractional derivatives over ordinary differential equations is their ability to demonstrate how a curve's slope transitions into a horizontal line, parallel to the x-axis, across varying values of α (α is fractional order i.e. $\alpha \in \mathcal{R}$). For example, if we graph the equation $y = 5x^2$ for both cases i.e. ordinary derivatives and fractional derivatives at $\alpha = 0.2, 0.4, \cdots, 1.0, \cdots, 2.0$ and the function we can easily observe this effect (Shown in the figure 1).



(a) *(b)*

Figure 1: (a) Function and its 1st & 2nd derivatives (b) Function and its fractional derivatives for different values of α

The Brownian motion of particles is modeled by the Fokker-Planck equation, which was first proposed by Fokker and Planck [12]. It is widely used in domains, such as chemistry, biology, astrophysics, economics, nucleation, electron relaxation in gases, optical bi-stability, polymer dynamics, quantum optics, reactive systems, and many more [13].

The Fokker-Planck equation describes the evolution of the probability distribution of a random variable over space and time, making it particularly useful for modeling solute transport. The general form of the Fokker-Planck equation for a concentration field $\phi(x, t)$ in one spatial dimension and at time t is as follows,

$$\frac{\partial^\alpha \phi(x,t)}{\partial t^\alpha} + \left[\frac{\partial}{\partial x} A(x,t) - \frac{\partial^2}{\partial x^2} B(x,t)\right] \phi(x,t) = 0 \tag{1.1}$$

The initial condition $\phi(x,0) = f(x)$, $x \in R$, Here, $A(x,t)$is the drift coefficient and $B(x,t) > 0$ is diffusion coefficient.

The more general FPE is non-linear FPE can be written as follows,

$$\frac{\partial^\alpha \phi(x,t)}{\partial t^\alpha} + \left[\frac{\partial}{\partial x} A(x,t,\phi) - \frac{\partial^2}{\partial x^2} B(x,t,\phi)\right] \phi(x,t) = 0 \tag{1.2}$$

Fractional nonlinear Fokker-Planck-like equations are used to analyze physical situations involving anomalous diffusion, which often includes a combination of nonlinear terms and fractional derivatives [14, 15]. These equations are effective in numerous contexts, such as frequency-dependent damping of materials, viscoelasticity, and diffusion processes.

The distribution of heat or temperature fluctuation within a region over time is described by the heat equation, a significant partial differential equation that Joseph Fourier created in 1822. This formula is noteworthy in several fields of science. It is closely related to 'Brownian motion' through the Fokker-Planck equation and is a standard example of a parabolic partial differential equation in mathematics [16]. Studies of chemical diffusion and other related phenomena use the diffusion equation, an extended version of the heat equation. According to the heat equation, if no additional heat is introduced, the temperature of a heated item submerged in cold water will gradually drop and approach equilibrium [17].

Now, we consider the general structure of fractional heat equations.

$D_t^{\alpha}\Psi(x,y,z,t) = f(x,y,z,t)\frac{\partial^2}{\partial x^2}\Psi(x,y,z,t) + g(x,y,z,t)\frac{\partial^2}{\partial y^2}\Psi(x,y,z,t) + h(x,y,z,t)\frac{\partial^2}{\partial z^2}\Psi(x,y,z,t)\ ; 0 < \alpha \leq 1$

with the initial value $\Psi(x,y,z,0) = H(x,y,z,t), \Psi_t(x,y,z,0) = m(x,y,z).$ (1.3)

This paper is organized as follows: In Section 2, we present some basic definitions and theorems relevant to the study. In Section 3, we explain the Laplace Variational Iteration Method along with its stability and convergence. In Section 4, numerical examples are discussed with graphical representations and numerical tables. Finally, the concluding remarks of this paper with future plan.

Preliminaries:

In this section, we provide the essential definitions and properties of fractional calculus and Laplace transform theory along with the conditions for Picard's T-stability and convergence based on the Banach fixed point theorem, which will be used throughout the paper.

Definitions:

2.1 Grunwald - Letnikov Fractional Derivative [18]

$f^{\alpha}(x) = \lim_{n\to\infty}\left(\frac{1}{h}\right)^{\alpha}\sum_{j=0}^{n}(-1)^j\binom{\alpha}{j}f(x-jh)$, where, $h = \frac{x-a}{n},\ a < x$, h is known as step size.

2.2 Riemann Integral for fractional calculus

$I^{\alpha}(f) = \frac{1}{\Gamma\alpha}\int_a^x (x-u)^{\alpha-1}f(u)du,\ \ a < x.$

2.3 Riemann Liouville's Fractional Derivative

$D_x^{\alpha}f(x) = \frac{d^n}{dx^n}I_x^{n-\alpha}f(x) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f(t)dt$, for $n-1 \leq \alpha \leq n, n \in N, t > 0.$

2.4 Caputo Fractional Derivative

${}_a^C D_x^{\alpha}f(x) = I_x^{n-\alpha}\left\{\frac{d^n}{dx^n}f(x)\right\} = \frac{1}{\Gamma n-\alpha}\int_a^x (x-t)^{n-\alpha-1}\left\{\frac{d^n}{dx^n}.c\right\}dt$, for $n-1 \leq \alpha \leq n, n \in N, t > 0.$

2.5 Relation between Riemann-Liouville and Caputo fractional derivative [19]

${}_0^C D^{\alpha}\psi(x,t) = {}_0^{RL}D^{\alpha}\psi(x,t) - \sum_{k=0}^{n-1}\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)}\psi^k(x,0).$

2.6 Laplace transform of the Caputo fractional derivative [20, 21]

$L\{{}_0^C D_x^{\alpha}f(x)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^k(0).$

2.7 The Mittag-Leffler function $E_{\alpha}(\xi)$ with $\alpha > 0$

$E_{\alpha}(\xi) = \sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(\alpha k+1)}, \alpha > 0,\ \xi \in \mathbb{C}.$

Theorem:

2.1 Let (X,d) be a Banach space and $T: X \to X$ be a self-map of X satisfying $d(Tx,Ty) \leq \beta d(x,Tx) + \gamma\, d(x,y)$, for all $x, y \in X, \beta \geq 0,\ 0 \leq \gamma < 1$. Then T is Picard T – stable [22].

2.2 Let $X = (X,d)$ be a metric space. A mapping $T: X \to X$ is called a contraction on X if there is a nonnegative real number $\gamma < 1$ such that, for all $x, y \in X$, $d(Tx,Ty) \leq \gamma\, d(x,y), 0 \leq \gamma < 1.$

Banach fixed point theorem: Consider a metric space $X = (X,d)$, where $X = \emptyset$. Suppose that X is complete, and let $T: X \to X$ be a contraction on X. Then T has a unique fixed point [22].

In the following paper, the Caputo fractional derivative is preferred due to its effectiveness in modeling real-world problems, handling initial value problems, and its compatibility with the Laplace transform, while also providing a smooth transition to classical models [23, 24].

Laplace Variational Iteration Method (LVIM):

LVIM is a combination of Laplace transform and Variational Iteration Method. To demonstrate the fundamental concept of this method, first, we consider a general fractional nonlinear nonhomogeneous partial differential equation along with the initial conditions of the form:

$\psi_t^{\alpha}(\xi,t) = L\Psi(\xi,t) + N\Psi(\xi,t) + f(\xi,t)$ (3.1)

with the initial values given by $\Psi^n(\xi,0) = h_k(\xi,0), n = 0,1,2,3,\cdots,m-1$

where, α is the order of fractional Caputo derivative and L, N denote the Linear and non-linear differential equation. Using

(2.6) in the left side of the equation (3.1) in order to convert the fractional differential equation into a partial differential equation [13].

$$L[\Psi(\xi,t)] = \frac{1}{s^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{s^{\theta}}L[f(\xi,t)] + L[L\Psi(\xi,t) + N\Psi(\xi,t)] \tag{3.2}$$

Taking inverse Laplace transform in both sides, we have

$$\Psi(\xi,t) = L^{-1}\left[\frac{1}{s^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{s^{\theta}}L[f(\xi,t)]\right] + L^{-1}\big[L[L\Psi(\xi,t) + N\Psi(\xi,t)]\big] \tag{3.3}$$

Differentiating equation (3.3) with respect to t, we get

$$\frac{\partial\Psi(\xi,t)}{\partial t} = \frac{\partial}{\partial t}\left\{L^{-1}\left[\frac{1}{s^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{s^{\theta}}L[f(\xi,t)]\right] + L^{-1}\big[L[L\Psi(\xi,t) + N\Psi(\xi,t)]\big]\right\} \tag{3.4}$$

This is the 1st order partial differential equation. Now constructing the correction functional for solving the following fractional differential equation

$$\Psi_{n+1}(\xi,t) = \Psi_n(\xi,t) + \int_0^t \lambda\left[\frac{\partial\Psi n(\xi,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{s^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{s^{\theta}}L[f(\xi,t)]\right] + L^{-1}\big[L[L\Psi(\xi,\varepsilon) + N\Psi(\xi,\varepsilon)]\big]\right\}\right]d\varepsilon \tag{3.5}$$

where, λ denotes the Lagrange multiplier. To compute its value, Equation (3.5) is reformulated based on the stationary theory [25],

$$\delta\Psi_{n+1}(\xi,t) = \delta\Psi_n(\xi,t) + \delta\int_0^t \lambda\left[\frac{\partial\Psi n(\xi,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{s^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{s^{\theta}}L[f(\xi,t)]\right] + L^{-1}\left[L\big[L\Psi(\xi,\varepsilon) + N\widetilde{\Psi}(\xi,\varepsilon)\big]\right]\right\}\right](d\varepsilon)^{\theta},$$

here, $\widetilde{\Psi}(\xi,\varepsilon)$ is the restricted variation, meaning that, $\widetilde{\Psi}(\xi,\varepsilon) = 0$

So, from variation theory, taking the coefficient of $\delta\Psi$ to zero, we get,

$$\frac{\partial^{\theta}}{\partial\varepsilon^{\theta}}\lambda(\xi,\varepsilon) = 0 \quad \& \; 1 + \lambda|_{\epsilon=t} = 0, \text{ therefore, } \lambda(\xi, t) = -1.$$

And begin with the primary iteration [8]

$$\Psi_0(\xi,t) = \Psi(\xi,0) + t\Psi_t(\xi,0), \tag{3.6}$$

and the exact solution can be found as $\Psi(\xi,t) = \lim_{n\to\infty}\Psi_n(\xi,t)$.

Stability and convergence analysis

In the following, we establish an important result concerning the stability of the LVIM. To verify Picard stability, it is sufficient to show that the iterative operator associated with LVIM satisfies the conditions specified in theorem (2.1).

Theorem 3.1 Let $(X, ||\cdot||)$ be a Banach space and $T: X \to X$ be a self-map of X. Then the LVIM iteration procedure defined by

$$\psi_{n+1}(\xi,t) = T\psi_n(\xi,t) = \Psi_n(\xi,t) + \int_0^t \lambda\left[\frac{\partial\Psi n(\xi,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{S^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{S^{\theta}}L[f(\xi,t)]\right] + L^{-1}\big[L[L\Psi(\xi,\varepsilon) + N\Psi(\xi,\varepsilon)]\big]\right\}\right]d\varepsilon \tag{3.7}$$

is Picard T-stable provided that [26]

i) $||\Psi_n(\xi,0) - \Psi_m(\xi,0)|| \le \delta_0||\Psi_n(\xi,t) - \Psi_m(\xi,t)||$ *for some* $\delta_0 > 0$ *and for any* t *in the domain*;

ii) $\left\|f(\Psi_n(p_0\xi,q_0t) - \Psi_m(p_0\xi,q_0t), \frac{\partial}{\partial\xi}\Psi_n(p_1\xi,q_1t) - \frac{\partial}{\partial\xi}\Psi_m(p_1\xi,q_1t), \ldots\ldots\right\| \le \delta_1||\Psi_n(\xi,t) - \Psi_m(\xi,t)||$ *for some* $\delta_0 > 0$;

iii) $\gamma = \delta_0 + \delta_1\left\|\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\| < 1.$

Proof: Let us consider equation (3.7) as

$$\psi_{n+1}(\xi,t) = \Psi_n(\xi,t) + \int_0^t \lambda R_n(x,\varepsilon)d\varepsilon$$

where $\boldsymbol{R_n}$ represents the entire integrand involving derivatives, inverse Laplace, linear and nonlinear terms in the functional.

That is, $\mathbf{R_n} = \frac{\partial\Psi n(\xi,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{S^{\theta}}\sum_{n=0}^{m-1} S^{\theta-1-n}\,\Psi^{n}(\xi,0) + \frac{1}{S^{\theta}}L[f(\xi,t)]\right] + L^{-1}\big[L[L\Psi(\xi,\varepsilon) + N\Psi(\xi,\varepsilon)]\big]\right\}$

Now,

$$T\Psi_n(\xi,t) - T\Psi_m(\xi,t) = \psi_{n+1}(\xi,t) - \psi_{m+1}(\xi,t) = \Psi_n(\xi,t) - \Psi_m(\xi,t) + \int_0^t \lambda[R_n(x,\varepsilon) - R_m(x,\varepsilon)]d\varepsilon$$

Taking Norm in both sides, we get

$$\|\psi_{n+1}(\xi,t)-\psi_{m+1}(\xi,t)\| \le \|\Psi_n(\xi,t)-\Psi_m(\xi,t)\| + \left\|\int_0^t \lambda[\mathrm{R_n}(x,\varepsilon)-\mathrm{R_m}(x,\varepsilon)]d\varepsilon\right\| \quad (3.8)$$

Let all the operator and function are satisfy a Lipschitz-type condition and the Laplace transform and its inverse are linear and bounded operators, then we assume

$$|\mathbf{R_n}-\mathbf{R_m}| \le \boldsymbol{\delta_1}|\boldsymbol{\Psi_n}-\Psi_m|$$

So the integral becomes:

$$\left|\int_0^t \lambda(\mathrm{R_n}-\mathrm{R_m})d\varepsilon\right| \le \boldsymbol{\delta_1}\int_0^t |\lambda||\boldsymbol{\Psi_n}-\Psi_m|d\varepsilon$$

here, Lagrange multiplier is defined by [27],$\lambda = \frac{(-1)^\theta}{\Gamma\theta}(t-\varepsilon)^{\theta-1}$, so, $\int_0^t|\lambda|d\varepsilon = \frac{t^\theta}{\Gamma(\theta+1)}$

Taking Norm in both sides, we get,

$\left\|\int_0^t \lambda(\mathrm{R_n}-\mathrm{R_m})d\varepsilon\right\| \le \boldsymbol{\delta_1}.\frac{t^\theta}{\Gamma(\theta+1)}\|\boldsymbol{\Psi_n}-\Psi_m\|$; $\|\boldsymbol{\Psi_n}-\Psi_m\|$ is constant with respect to ε

From initial condition (i) we have,

$$\|\Psi_n(\xi,0)-\Psi_m(\xi,0)\| \le \delta_0\|\boldsymbol{\Psi_n}-\Psi_m\|$$

So combining all these we have, from (3.8)

$\|\boldsymbol{\Psi_{n+1}}-\Psi_{m+1}\| \le \delta_0\|\boldsymbol{\Psi_n}-\Psi_m\| + \boldsymbol{\delta_1}.\frac{t^\theta}{\Gamma(\theta+1)}\|\boldsymbol{\Psi_n}-\Psi_m\| = \gamma\|\boldsymbol{\Psi_n}-\Psi_m\|$; here, $\gamma = \delta_0 + \boldsymbol{\delta_1}.\frac{t^\theta}{\Gamma(\theta+1)} < 1$

therefore, we have, $\|\mathbf{T}\boldsymbol{\Psi_n}-\mathrm{T}\Psi_m\| = \|\boldsymbol{\Psi_{n+1}}-\Psi_{m+1}\| \le \gamma\|\boldsymbol{\Psi_n}-\Psi_m\|$ (3.9)

$\|\mathbf{T}\boldsymbol{\Psi_n}-\mathrm{T}\Psi_m\| < \beta\|\boldsymbol{\Psi_n}-\Psi_m\| + \gamma\|\boldsymbol{\Psi_n}-\Psi_m\|;\ \beta \ge 0$

$\beta\|\boldsymbol{\Psi_n}-\Psi_m\|$ reflects how much this residual influences stability.

Hence, by the theorem (2.1) LVIM is Picard T-stable if $\gamma < 1$.

In this section, we establish the convergence of the Laplace Variational Iteration Method using the Banach fixed point theorem.

Theorem 3.2 *Let* (*X*, ·) *be a Banach space and* $T: X \to X$, *a mapping associated with LTVIM*, *be defined by* $\psi_{n+1}(\xi,t) = T\psi_n(\xi,t)$ *Then T has a unique fixed point and the sequence* $\{\Psi_n\}_{n=1}^{\infty}$ *generated by LTVIM with an initial value* $\Psi_0 \in X$ *converges to this fixed point* [26]

Proof: Combination of theorem (2.1), (2.2), and equation (3.9) implies that *T* has a unique fixed point. Let $\Psi_0 \in X$ and consider the sequence $\{\Psi_n\}_{n=1}^{\infty}$ generated by LVIM with $\Psi_{n+1} = \Psi_n$, observe that

$$\xi_1 = T\xi_0,\quad \xi_2 = T\xi_1 = T^2\xi_0,\ \cdots,\quad \xi_n = T^n\xi_0, \cdots \quad (3.10)$$

Equation (3.10) illustrates a sequence of iterates obtained by repeatedly applying the operator $\boldsymbol{T}$ to the initial point ξ_0. The convergence of the LTVIM can be ensured by proving that this sequence is Cauchy sequence.

By combining (3.9) and (3.10) we obtain

$$\begin{aligned}\|\Psi_{n+1}-\Psi_n\| &= \|T\Psi_n - \mathrm{T}\Psi_{n-1}\| \\ &\le \gamma\|\Psi_n-\Psi_{n-1}\| \\ &= \gamma\,\|\mathrm{T}\Psi_{n-1}-\mathrm{T}\Psi_{n-2}\| \\ &\le \gamma^2\,\|\Psi_{n-1}-\Psi_{n-2}\| \\ &\cdots\quad \cdots\quad \cdots \\ &\le \gamma^n\,\|\Psi_1-\Psi_0\| \end{aligned} \quad (3.11)$$

By the triangle inequality and equation (3.10), for any $m,n \in N$ such that$n > m$, we have

$$\begin{aligned}\|\Psi_m-\Psi_n\| &= \|\Psi_m-\Psi_{m+1}+\Psi_{m+1}-\Psi_{m+2}+\Psi_{m+2}+\cdots+\Psi_{n-1}-\Psi_n\| \\ &\le \|\Psi_m-\Psi_{m+1}\| + \|\Psi_{m+1}-\Psi_{m+2}\| + \cdots \|\Psi_{n-1}-\Psi_n\| \\ &= \|\mathrm{T}\Psi_{m-1}-\mathrm{T}\Psi_m\| + \|\mathrm{T}\Psi_m-\mathrm{T}\Psi_{m+1}\| + \cdots + \|\mathrm{T}\Psi_{n-2}-\mathrm{T}\Psi_{n-1}\| \\ &\le \gamma^m\,\|\Psi_1-\Psi_0\| + \gamma^{m+1}\,\|\Psi_1-\Psi_0\| + \cdots\gamma^{n-1}\,\|\Psi_1-\Psi_0\| \\ &= (\gamma^m+\gamma^{m+1}+\gamma^{m+2}+\cdots+\gamma^{n-1})\|\Psi_1-\Psi_0\| \\ &= \gamma^m(1+\gamma+\gamma^2+\cdots+\gamma^{n-m-1})\|\Psi_1-\Psi_0\|\end{aligned}$$

Since $0 \le \gamma < 1$, the sum $1+\gamma+\gamma^2+\cdots+\gamma^{n-m-1}$ represents a finite geometric progression whose total sum is $\frac{1-\gamma^{n-m}}{1-\gamma}$. Therefore,

$$\|\Psi_\mathrm{m}-\Psi_\mathrm{n}\| \le \gamma^m\left(\frac{1-\gamma^{n-m}}{1-\gamma}\right)\|\Psi_1-\Psi_0\| \le \frac{\gamma^m}{1-\gamma}\|\Psi_1-\Psi_0\|, as\ 1-\gamma^{n-m} \le 1 \quad (3.12)$$

Since $0 \le \gamma < 1$ and $\|\Psi_1-\Psi_0\|$ is fixed, by choosing m sufficiently large, the right-hand side of equation (3.12) can be made arbitrarily small. Consequently, the sequence $\{\Psi n\}$ is Cauchy and thus convergent.

Let, $\{\Psi_n\}$ converge to $\Psi \in X$. To complete the proof, it is necessary to demonstrate that the limit Ψ satisfies $T\Psi = \Psi$ thereby establishing that it is the unique fixed point of the operator T. From the triangle inequality and equation (3.10), we have,

$$\|\Psi - T\Psi\| \leq \|\Psi - \Psi_n\| + \|\Psi_n - T\Psi\|$$
$$= \|\Psi - \Psi_n\| + \|T\Psi_{n-1} - T\Psi\|$$
$$\leq \|\Psi - \Psi_n\| + \gamma\|\Psi_{n-1} - \Psi\|$$
$$= 0 \; as \; n \to \infty$$

This implies that $\|\Psi - T\Psi\| = 0$. Since $\|.\|$ is a metric, we have $T\Psi = \Psi$, i.e., Ψ is the fixed point of T which is unique.

Illustrative Examples:

In this section five illustrative examples are solved by LVIM to demonstrate the efficiency of this method. The results are presented in a couple of tables (Table 1-Table 10) in order to understanding the behavior of the result and finally shown by graphically and compared them with existing method also that is Fractional Power Series Method (FPSM) or Fractional Variational Homotopy Perturbation Iteration Method (FVHPIM).

Example 1: The Time Fokker-Planck Equation

$$\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = \frac{\partial \Psi(x,t)}{\partial x} + \frac{\partial^2 \Psi(x,t)}{\partial x^2} \; ; \; 0 < \alpha \leq 1 \tag{4.1}$$

with initial condition $\Psi(x, 0) = x$ [27]

Solution: Using the initial condition, our primary iteration is $\Psi_0(x,t) = \Psi(x,0) + tu_t(x,0) = x$;

Using Laplace transform in (4.1) $L[\Psi(x,t)] = \frac{1}{S^\alpha}.S^{\alpha-1}.\Psi^0(x,0) + \frac{1}{S^\alpha} L\left[\frac{\partial^2\Psi(x,t)}{\partial x^2} + \frac{\partial\Psi(x,t))}{\partial x}\right]$

$$= \frac{x}{S} + \frac{1}{S^\alpha} L[\frac{\partial^2\Psi(x,t)}{\partial x^2} + \frac{\partial\Psi(x,t))}{\partial x}]$$

Taking inverse Laplace transform on both sides, we get, $\Psi(x,t) = x + L^{-1}\left\{\frac{1}{S^\alpha} L[\frac{\partial^2\Psi(x,t)}{\partial x^2} + \frac{\partial\Psi(x,t)}{\partial x}]\right\}$ (4.2)

Differentiating both sides of (4.2) with respect to t, $\frac{\partial\Psi(x,t)}{\partial t} = \frac{\partial}{\partial t} L^{-1}\left\{\frac{1}{S^\alpha} L[\frac{\partial^2\Psi(x,t)}{\partial x^2} + \frac{\partial\Psi(x,t)}{\partial x}]\right\}$

Now the correction functional for this problem is

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \lambda \int_0^t \left[\frac{\partial\Psi_n(x,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L\left[\frac{\partial^2\Psi_n(x,t)}{\partial x^2} + \frac{\partial\Psi_n(x,t)}{\partial x}\right]\right\}\right] d\varepsilon \tag{4.3}$$

From variation theory, λ can be found using by $1 + \lambda|_{\varepsilon=t} = 0$ so, $\lambda = -1$

So our 1st iteration will be taking $n = 0$ in (4.3)

$$\Psi_1(x,t) = \Psi_0(x,t) - \int_0^t \left[\frac{\partial\Psi_0(x,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L\left[\frac{\partial^2\Psi_0(x,t)}{\partial x^2} + \frac{\partial\Psi_0(x,t)}{\partial x}\right]\right\}\right] d\varepsilon$$
$$= x + \frac{t^\alpha}{\Gamma(\alpha+1)}$$

Similarly, $\Psi_2(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)}$; $\Psi_3(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)}$

The nth term will be then $\Psi_n(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)}$

The general solution of equation (4.1) can be found taking by $n \to \infty$

$$\psi(x,t) = \sum_{n=0}^{\infty} \Psi_n(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{4.4}$$

Table 1: Numerical values of $\psi(x,t)$ at $x = 0.2$

	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.0$
$\alpha = 0.2$	1.025402	1.148138	1.228229	1.289124
$\alpha = 0.4$	0.847326	1.054152	1.204550	1.327060
$\alpha = 0.6$	0.687149	0.938380	1.141748	1.319175
$\alpha = 0.8$	0.554179	0.816662	1.052944	1.273671
$\alpha = 1.0$	0.450000	0.700000	0.950000	1.200000

Table 2: Numerical values of $\psi(x,t)$ at $x = 0.5$

	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.00$
$\alpha = 0.2$	1.325402	1.448138	1.528229	1.589124
$\alpha = 0.4$	1.147326	1.354152	1.504550	1.627060
$\alpha = 0.6$	0.987149	1.238380	1.441748	1.619175
$\alpha = 0.8$	0.854179	1.116662	1.352944	1.573671
$\alpha = 1.0$	0.750000	1.000000	1.250000	1.500000

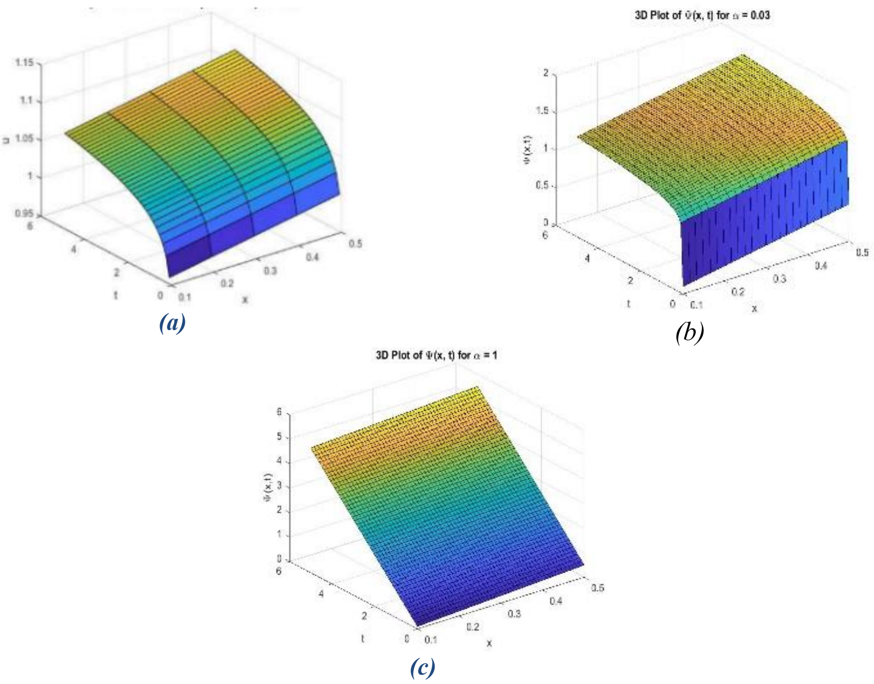


Figure 2: Three-dimensional graph of equation (4.4) by (a) FPSM at $\alpha = 0.03$ (b) LVIM at $\alpha = 0.03$ (c) Exact result at $\alpha = 1.0$ by LVIM

Example 2: The time fractional Fokker Planck equation

$$\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = (1+x)\frac{\partial \Psi(x,t)}{\partial x} + e^t x^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2} \;;\; 0 < \alpha \leq 1 \tag{4.5}$$

with initial condition $\Psi(x,0) = 1 + x$ [27]

Solution: Taking the Laplace transform for fractional differential equation (4.5)

$$L[\Psi(x,t)] = \frac{1}{S}\Psi(x,0) + \frac{1}{S^\alpha} L\left[(1+x)\frac{\partial \Psi(x,t)}{\partial x} + e^t x^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2}\right]$$
$$= \frac{1+x}{S} + \frac{1}{S^\alpha} L\left[(1+x)\frac{\partial \Psi(x,t)}{\partial x} + e^t x^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2}\right]$$

Taking inverse Laplace transform in the both sides

$$\Psi(x,t) = 1 + x + L^{-1}\left\{\frac{1}{S^\alpha} L\left[(1+x)\frac{\partial \Psi(x,t)}{\partial x} + e^t x^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2}\right]\right\} \tag{4.6}$$

Differentiating both sides of (4.6) with respect to t

$$\frac{\partial \Psi(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[L^{-1}\left\{\frac{1}{S^\alpha} L\left[(1+x)\frac{\partial \Psi(x,t)}{\partial x} + e^t x^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2}\right]\right\}\right] \tag{4.7}$$

The Correction functional of (4.7) for LVIM is given by

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \lambda \int_0^t \left[\frac{\partial \Psi_n(x,\varepsilon)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L\left[e^t x^2 \frac{\partial^2 \Psi_n(x,t)}{\partial x^2} + (1+x)\frac{\partial \Psi_n(x,t)}{\partial x}\right]\right\}\right] d\varepsilon \tag{4.8}$$

Now taking $n = 0$ and $\lambda = -1$, we get, the 1st iteration is

$$\Psi_1(x,t) = \Psi_0(x,t) - \int_0^t \left[\frac{\partial \Psi_0(x,\varepsilon)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L\left[e^t x^2 \frac{\partial^2 \Psi_0(x,\varepsilon)}{\partial x^2} + (1+x)\frac{\partial \Psi_0(x,\varepsilon)}{\partial x}\right]\right\}\right] d\varepsilon$$
$$= (1+x) - \int_0^t \left[\frac{\partial (1+x)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L\left[e^t x^2 \frac{\partial^2 (1+x)}{\partial x^2} + (1+x)\frac{\partial (1+x)}{\partial x}\right]\right\}\right] d\varepsilon$$
$$= (1+x) + \int_0^t \left[\frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L[e^t x^2 . 0 + (1+x). 1]\right\}\right] d\varepsilon$$
$$= (1+x) + \int_0^t \left[\frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{S^\alpha} L[(1+x)]\right\}\right] d\varepsilon$$
$$= 1 + x + \frac{(1+x)t^\alpha}{\Gamma(\alpha+1)}$$

Similarly, $\Psi_2(x,t) = 1 + x + \frac{(1+x)t^\alpha}{\Gamma(\alpha+1)} + \frac{(1+x)t^{2\alpha}}{\Gamma(2\alpha+1)}$;

$$\Psi_3(x,t) = 1 + x + \frac{(1+x)t^\alpha}{\Gamma(\alpha+1)} + \frac{(1+x)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(1+x)t^{3\alpha}}{\Gamma(3\alpha+1)}$$

The nth term is

$$\Psi_n(x,t) = 1 + x + \frac{(1+x)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(1+x)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(1+x)t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{(1+x)t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= (1+x)(1 + \frac{t^{\alpha}}{\alpha!} + \frac{(t^{\alpha})^2}{2\alpha!} + \frac{(t^{\alpha})^3}{3\alpha!} + \cdots)$$

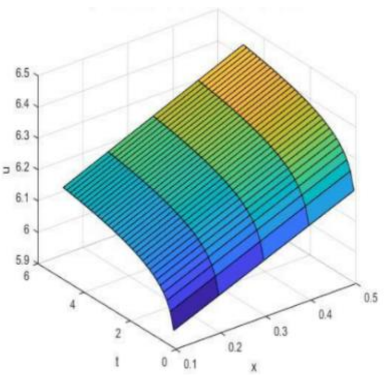
The general solution of (4.5) is $\Psi(x,t) = \sum_{n=0}^{\infty} \frac{(1+x)t^{n\alpha}}{\Gamma(n\alpha+1)} = (1+x)\sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = (1+x)e^{\alpha t}$ (4.9)

Table 3: value of $\psi(x,t)$ for $x = 0.5$

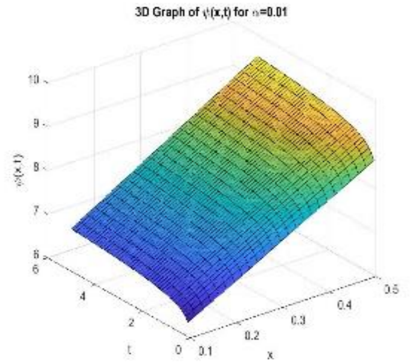
	$t=2$	$t=4$	$t=6$	$t=8$
$\alpha = 0.7$	12.5202	42.3102	99.7506	192.4925
$\alpha = 0.8$	11.9175	46.3715	123.5618	263.2825
$\alpha = 0.9$	11.2223	49.4483	148.1629	347.9500
$\alpha = 1.0$	10.5000	51.5000	172.5000	445.5000

Table 4: value of $\psi(x,t)$ for $x = 1.0$

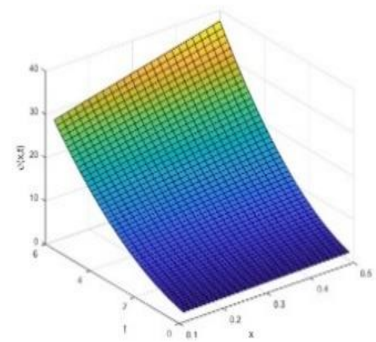
	$t=2$	$t=4$	$t=6$	$t=8$
$\alpha = 0.7$	16.6936	56.4136	133.0008	256.6567
$\alpha = 0.8$	15.8900	61.8287	164.7490	351.0433
$\alpha = 0.9$	14.9631	65.9311	197.5506	463.9333
$\alpha = 1.0$	14.0000	68.6667	230.0000	594.0000



(a)



(b)



(c)

Figure 3: Third order approximate solution of equation (4.9) (a) by FPSM at $\alpha = 0.01$ (b) by LVIM at $\alpha = 0.01$ (c) Exact solution of equation (4.9) by LVIM at $\alpha = 0.01$

Example 3: One-dimensional time-fractional diffusion equation

$$\frac{\partial^{\alpha}\Psi(x,t)}{\partial t^{\alpha}} = \frac{\partial\Psi(x,t)}{\partial x} + \frac{\partial^2\Psi(x,t)}{\partial x^2}; \quad 0 < \alpha \leq 1 \tag{4.10}$$

with initial condition $\Psi(x,0) = e^x$; $x\epsilon[0,1]$; $u(0,t) = E_{\alpha}(2t^{\alpha})$ [28]

Solution: Taking Laplace transform in (4.10)

$$L[\Psi(x,t)] = \frac{e^x}{s} + \frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial x^2}\Psi(x,t) + \frac{\partial}{\partial x}\Psi(x,t)\right]$$

$$\therefore \frac{\partial}{\partial t}[\Psi(x,t)] = \frac{\partial}{\partial t}\left[L^{-1}\left\{\frac{1}{s^{\alpha}}\left[L\left\{\frac{\partial\Psi(x,t)}{\partial x} + \frac{\partial^2\Psi(x,t)}{\partial x^2}\right\}\right]\right\}\right]$$

The correction functional of LVIM for this problem can be constructed as follows,

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \int_0^t \lambda \left[\frac{\partial}{\partial \varepsilon}\Psi_n(x,\varepsilon) - \frac{\partial}{\partial \varepsilon}\left[L^{-1}\left\{\frac{1}{s^\alpha}\left[L\left\{\frac{\partial \Psi_n(x,\varepsilon)}{\partial x} + \frac{\partial^2 \Psi_n(x,\varepsilon)}{\partial x^2}\right\}\right]\right\}\right]\right] d\varepsilon \quad (4.11)$$

Now, putting $n = 0, \lambda = -1$ in (4.11) for getting 1st approximation

$$\Psi_1(x,t) = \Psi_0(x,t) - \int_0^t \left[\frac{\partial}{\partial \varepsilon}\Psi_0(x,\varepsilon) - \frac{\partial}{\partial \varepsilon}\left[L^{-1}\left\{\frac{1}{s^\alpha}\left[L\left\{\frac{\partial \Psi_0(x,\varepsilon)}{\partial x} + \frac{\partial^2 \Psi_0(x,\varepsilon)}{\partial x^2}\right\}\right]\right\}\right]\right] d\varepsilon$$

$$= e^x + \frac{2e^x t^\alpha}{\Gamma(\alpha+1)}$$

Similarly, $\Psi_2(x,t) = e^x + \frac{2e^x t^\alpha}{\Gamma(\alpha+1)} + \frac{4e^x t^{2\alpha}}{\Gamma(2\alpha+1)}$; $\Psi_3(x,t) = e^x + \frac{2e^x t^\alpha}{\Gamma(\alpha+1)} + \frac{4e^x t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{6e^x t^{3\alpha}}{\Gamma(3\alpha+1)}$

Then, the nth term will be, $\Psi_n(x,t) = \frac{e^x 2^n t^{n\alpha}}{\Gamma(n\alpha+1)}$ (4.12)

The complete and general solution to the problem (4.10) can be developed by considering, $n \to \infty$

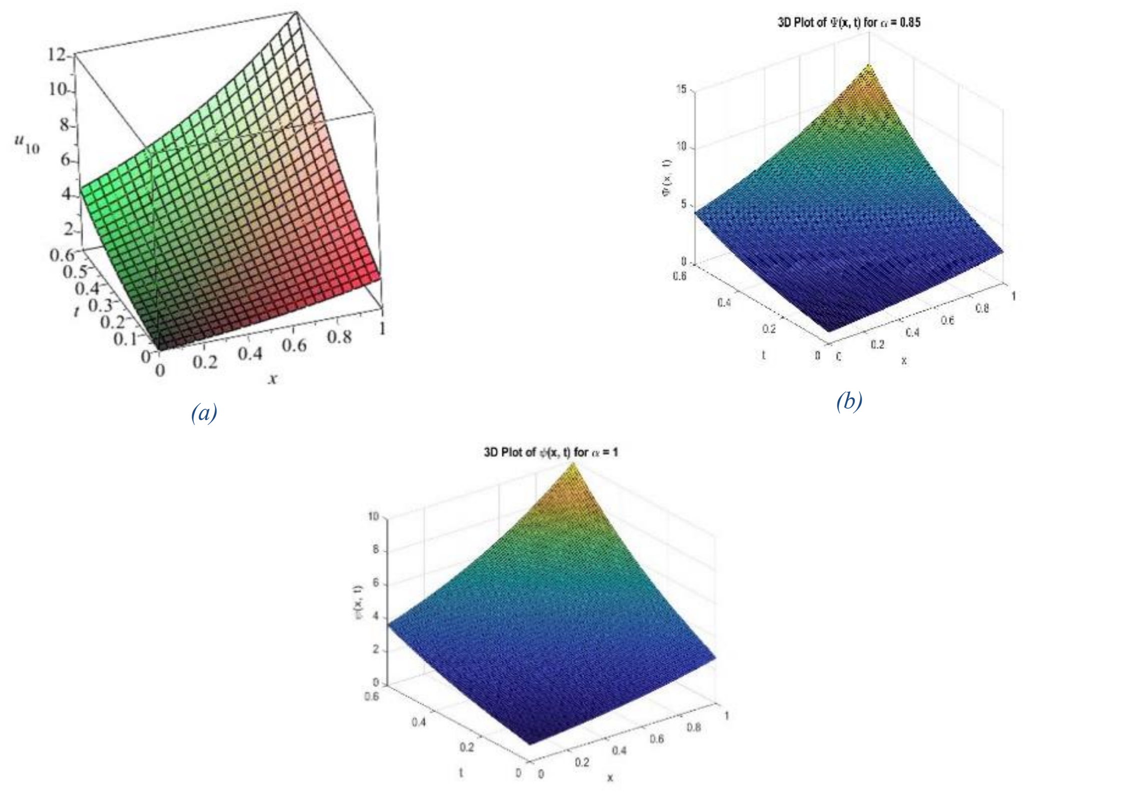
$\Psi(x,t) = \sum_{n=0}^{\infty} \frac{e^x 2^n t^{n\alpha}}{\Gamma n\alpha+1} = e^x E_\alpha(2t^\alpha)$, Where, $E_\alpha(2t^\alpha)$ is Mittag - Leffler function (4.13)

Table 5: value of $\psi(x,t)$ for $x = 0.5$

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$\alpha = 0.7$	2.6424	3.6701	4.9623	6.6258
$\alpha = 0.8$	2.3457	3.0938	4.0127	5.1624
$\alpha = 0.9$	2.1495	2.7191	3.4129	4.2670
$\alpha = 1.0$	2.0138	2.4596	3.0042	3.6693

Table 6: value of $\psi(x,t)$ for $x = 1.0$

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$\alpha = 0.7$	2.6424	3.6701	4.9623	6.6258
$\alpha = 0.8$	2.3457	3.0938	4.0127	5.1624
$\alpha = 0.9$	2.1495	2.7191	3.4129	4.2670
$\alpha = 1.0$	2.0138	2.4596	3.0042	3.6693



(a) (b) (c)

Figure 4: The tenth-order approximate solution of Eq. (4.13) by (a) FVHPIM at $\alpha = 0.85$ (b) LVIM at $\alpha = 0.85$ (c) Exact solution of Eq. (4.13) by LVIM at $\alpha = 1.0$

Example 4: **Two-dimensional time-fractional diffusion equation [28]**

$$\frac{\partial^\alpha \Psi(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 \Psi(x,y,t)}{\partial x^2} + \frac{\partial^2 \Psi(x,y,t)}{\partial y^2} + x\frac{\partial \Psi(x,y,t)}{\partial x} + y\frac{\partial \Psi(x,y,t)}{\partial y} + 2\Psi(x,y,t)\ ;0 < \alpha \le 1 \quad (4.14)$$

Subject to the initial condition, $\Psi(x,y,0) = x + y$

and the boundary conditions $\Psi(0,y,t) = yE_\alpha(3t^\alpha), \Psi(1,y,t) = (1+y)E_\alpha(3t^\alpha); \Psi(x,0,t) = xE_\alpha(3t^\alpha);$ $\Psi(x,1,t) = (x+1)E_\alpha(3t^\alpha); t \ge 0;$

Solution: First of all, we consider the initial guess by using the following formula

$\Psi_0(x,y,t) = \Psi(x,y,0) + t\Psi_t(x,y,0)$; Therefore, $\Psi_0(x,y,t) = x + y$

Now, taking the Laplace transform in both sides of the equation (4.14), we get

$$L[\Psi(x,y,t)] = \frac{x+y}{s} + \frac{1}{s^\alpha}L\left[\frac{\partial^2 \Psi(x,y,t)}{\partial x^2} + \frac{\partial^2 \Psi(x,y,t)}{\partial y^2} + x\frac{\partial \Psi(x,y,t)}{\partial x} + y\frac{\partial \Psi(x,y,t)}{\partial y} + 2\Psi(x,y,t)\right]$$

Taking inverse Laplace transform,

$$\frac{\partial \Psi(x,y,t)}{\partial t} = \frac{\partial}{\partial t}\left[L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{\partial^2 \Psi(x,y,t)}{\partial x^2} + \frac{\partial^2 \Psi(x,y,t)}{\partial y^2} + x\frac{\partial \Psi(x,y,t)}{\partial x} + y\frac{\partial \Psi(x,y,t)}{\partial y} + 2\Psi(x,y,t)\right]\right\}\right] \quad (4.15)$$

Now, constructing the correction functional for solving the problem (4.15) taking by $n = 0, \lambda = -1$ is given by

$$\Psi_1(x,y,t) = \Psi_0(x,y,t) - \int_0^t \left[\frac{\partial \Psi_0(x,y,\epsilon)}{\partial \epsilon} - \frac{\partial}{\partial \epsilon}L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{\partial^2 \Psi_0(x,y,\epsilon)}{\partial x^2} + \frac{\partial^2 \Psi_0(x,y,\epsilon)}{\partial y^2} + x\frac{\partial \Psi_0(x,y,\epsilon)}{\partial x} + y\frac{\partial \Psi_0(x,y,\epsilon)}{\partial y} + 2\Psi_0(x,y,\epsilon)\right]\right\}\right]d\epsilon$$

$$= x + y - \int_0^t \left[0 - \frac{\partial}{\partial \epsilon}\left\{L^{-1}\left\{\frac{1}{s^\alpha}L(0 + 0 + x + y + 2x + 2y)\right\}\right\}\right]d\epsilon$$

$$= x + y + \int_0^t \left[\frac{\partial}{\partial \epsilon}\left\{L^{-1}\left(\frac{3x+3y}{s^{\alpha+1}}\right)\right\}\right]d\epsilon$$

$$= x + y + \int_0^t \left[\frac{\partial}{\partial \epsilon}\left\{\frac{\epsilon^\alpha(3x+3y)}{s^{\alpha+1}}\right\}\right]d\epsilon$$

$$= x + y + \frac{3(x+y)t^\alpha}{\Gamma(\alpha+1)}$$

Proceeding in this way,

$$\Psi_2(x,y,t) = x + y + \frac{3(x+y)t^\alpha}{\Gamma(\alpha+1)} + \frac{9(x+y)t^{2\alpha}}{\Gamma(2\alpha+1)}\ ;$$

$$\Psi_3(x,y,t) = x + y + \frac{3(x+y)t^\alpha}{\Gamma(\alpha+1)} + \frac{9(x+y)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{27(x+y)t^{3\alpha}}{\Gamma(3\alpha+1)}$$

We can find the k-the term, $\Psi_k(x,y,t) = \frac{(x+y)3^k t^{k\alpha}}{\Gamma(k\alpha+1)}$ (4.16)

So, the required solution to the problem is that means the general solution of equation (4.14) can be found by taking, $k \to \infty$

$$\Psi(x,y,t) = \sum_{k=1}^{\infty}\frac{(x+y)3^k t^{k\alpha}}{\Gamma(k\alpha+1)} = (x+y)E_\alpha(3t^\alpha) \quad (4.17)$$

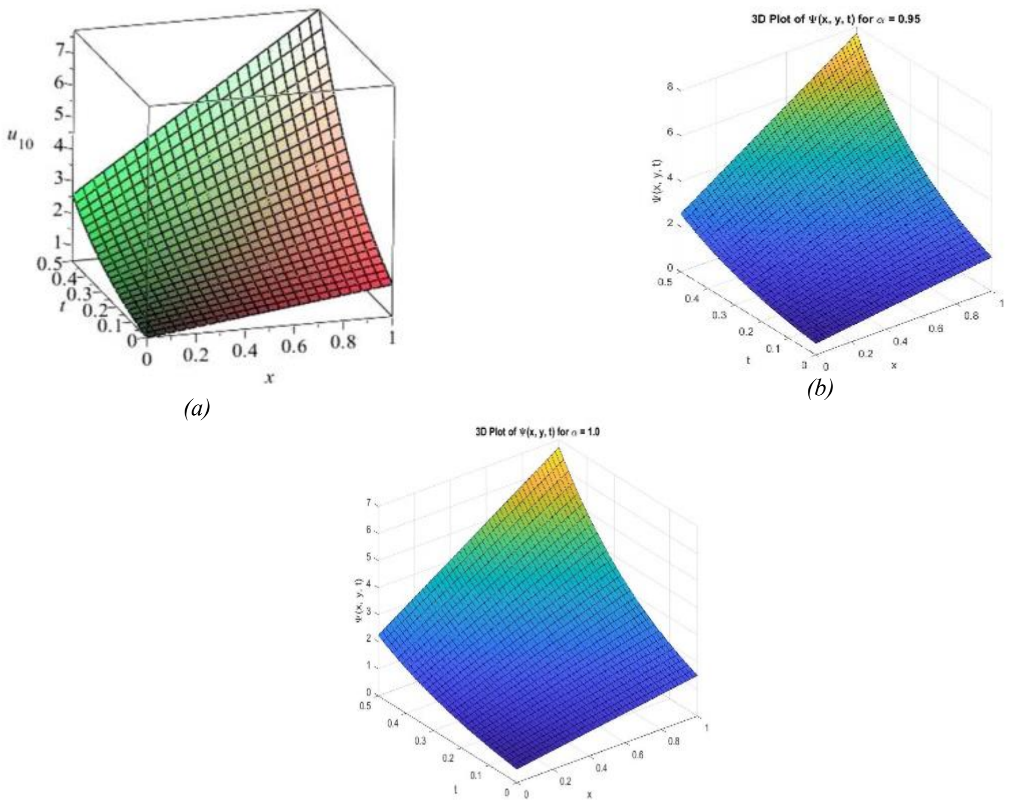
$E_\alpha(3t^\alpha)$ is Mittag-Leffler function; which is the exact solution [28] of the problem (4.14) at $\alpha = 1.0$

Table 7: value of $\Psi(x,y,t)$ for different values of $x = 0.3$ & $y = 0.6$

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$\alpha = 0.75$	1.6837	2.7192	4.2824	6.6748
$\alpha = 0.8$	1.5412	2.376	3.591	5.3819
$\alpha = 0.95$	1.2726	1.7657	2.4387	3.3611
$\alpha = 1.0$	1.2149	1.6399	2.2136	2.9881

Table 8: value of $\Psi(x,y,t)$ for different values of $x = 0.45$ & $y = 0.75$

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$\alpha = 0.75$	2.2449	3.6256	5.7099	8.8998
$\alpha = 0.8$	2.0549	3.1679	4.788	7.1758
$\alpha = 0.95$	1.6967	2.3543	3.2516	4.4815
$\alpha = 1.0$	1.6198	2.1865	2.9515	3.9841



(c)

Figure 5: The tenth-order approximate solution of Eq. (4.17) by (a) FVHPIM at $\alpha = 0.95$ and $y = 0.5$ (b) LVIM at $\alpha = 0.95$ and $y = 0.5$ (c) Exact solution of eq. (4.17) by LVIM at $\alpha = 0.95$ and $y = 0.5$

Example 5: Three-dimensional time-fractional diffusion equation [28]

$$\frac{\partial^\alpha \Psi(x,y,z,t)}{\partial t^\alpha} = \frac{\partial^2 \Psi(x,y,z,t)}{\partial x^2} + \frac{\partial^2 \Psi(x,y,z,t)}{\partial y^2} + \frac{\partial^2 \Psi(x,y,z,t)}{\partial z^2} + x\frac{\partial \Psi(x,y,z,t)}{\partial x} + y\frac{\partial \Psi(x,y,z,t)}{\partial y} + z\frac{\partial \Psi(x,y,z,t)}{\partial z} + 3\Psi(x,y,z,t) \quad (4.18)$$

The initial condition is given by $\Psi(x,y,z,t) = (x+y+z)^2$ and the boundary conditions $\Psi(0,y,z,t) = (3+(y+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha)$; $\Psi(1,y,z,t) = (3+(1+y+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha)$; $\Psi(x,0,z,t) = (3+(x+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha)$; $\Psi(x,1,z,t) = (3+(x+1+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha)$; $\Psi(x,y,0,t) = (3+(x+y)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha)$; $\Psi(x,y,1,t) = (3+(x+y+1)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha)$; $t \geq 0$

Solution: The initial guess is then $\Psi_0(x,y,z,t) = (x+y+z)^2$

Let, $\Psi = \Psi(x,y,z,t)$

To solve the fractional differential equation, firstly, we take Laplace transformation on both sides of the equation (4.18)

$$L[\Psi(x,y,z,t)] = \frac{1}{s^\alpha}\sum_{n=\alpha}^{m-1} s^{\alpha-1-n}\,\Psi^n(x,y,z,0) + \frac{1}{s^\alpha}\left\{L\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y} + z\frac{\partial\Psi}{\partial z} + 3\Psi\right]\right\} \quad (4.19)$$

Taking inverse Laplace on both sides, and differentiating, we get,

$$\frac{\partial}{\partial t}\Psi(x,y,z,t) = \frac{\partial}{\partial t}L^{-1}\left\{\frac{1}{s^\alpha}\,L\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y} + z\frac{\partial\Psi}{\partial z} + 3\Psi\right]\right\} \quad (4.20)$$

Now, we construct correction functional of LVIM for (4.20)

$$\Psi_{n+1}(x,y,z,t) = \Psi_n(x,y,z,t) + \int_0^t \lambda\left[\frac{\partial\Psi_n(x,y,z,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left\{\frac{1}{s^\alpha}\left\{L\left(\frac{\partial^2\Psi_n(x,y,z,\varepsilon)}{\partial x^2} + \frac{\partial^2\Psi_n(x,y,z,\varepsilon)}{\partial y^2} + \frac{\partial^2\Psi_n(x,y,z,\varepsilon)}{\partial z^2} + x\frac{\partial\Psi_n(x,y,z,\varepsilon)}{\partial x} + y\frac{\partial\Psi_n(x,y,z,\varepsilon)}{\partial y} + z\frac{\partial\Psi_n(x,y,z,\varepsilon)}{\partial z} + 3\Psi_n(x,y,z,\varepsilon)\right)\right\}\right\}\right\}\right]d\varepsilon \quad (4.21)$$

So, 1st iteration taking by $\lambda = -1$ & $n = 0$ in (4.21)

$$\Psi_1(x,y,z,t) = (x+y+z)^2 + \int_0^t \left[\frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left\{\frac{1}{s^\alpha}\{L\{6 + 2x(x+y+z) + 2y(x+y+z) + 2z(x+y+z)\}\}\right\}\right\}\right]d\varepsilon$$

$$= (x+y+z)^2 + \left\{\frac{6}{\Gamma(\alpha+1)} + \frac{5(x+y+z)^2}{\Gamma(\alpha+1)}\right\} t^{\alpha}$$

In this way, 2nd and 3rd iteration

$$\Psi_2(x,y,z,t) = (x+y+z)^2 + \left\{\frac{6}{\Gamma(\alpha+1)} + \frac{5(x+y+z)^2}{\Gamma(\alpha+1)}\right\} t^{\alpha} + \left\{\frac{48}{\Gamma(2\alpha+1)} + \frac{25(x+y+z)^2}{\Gamma(2\alpha+1)}\right\} t^{2\alpha}$$

$$\Psi_3(x,y,z,t) = (x+y+z)^2 + \left\{\frac{6}{\Gamma(\alpha+1)} + \frac{5(x+y+z)^2}{\Gamma(\alpha+1)}\right\} t^{\alpha} + \left\{\frac{48}{\Gamma(2\alpha+1)} + \frac{25(x+y+z)^2}{\Gamma(2\alpha+1)}\right\} t^{2\alpha} + \frac{\{(294+125(x+y+z)^2\}t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Proceeding in this way, the nth term will be

$$\Psi_n(x,y,z,t) = \frac{(3(5^n-3^n)+5^n(x+y+z)^2 t^{n\alpha})}{\Gamma(n\alpha+1)}$$

The general solution to the problem can be found in taking, $n \to \infty$

$$\Psi(x,y,z,t) = \lim_{k\to\infty} \sum_{k=0}^{n} \Psi_n(x,y,z,t) = \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{(3(5^n-3^n)+5^n(x+y+z)^2 t^{n\alpha})}{\Gamma(n\alpha+1)}$$

This result is also written in the form of the Mittag-Leffler function as follows,

$$\Psi(x,y,z,t) = (3+(x+y+z)^2)E_{\alpha}(5t^{\alpha}) - 3E_{\alpha}(3t^{\alpha}) \qquad (4.22)$$

Which provides the exact solution [28] to the problem at $\alpha = 1.0$

Table 9: value of $\Psi(x,y,t)$ for different values of $x = 0.2$, $y = 0.4$, $z = 0.6$

	$t = 0.1$	$t = 0.2$	$t = 0.3$
$\alpha = 0.7$	10.145690	35.298330	107.975532
$\alpha = 0.8$	6.071240	16.453191	40.005178
$\alpha = 0.9$	4.245446	9.719085	20.426793
$\alpha = 1.0$	3.270746	6.602815	12.519890

Table 10: *value of* $\Psi(x,y,t)$ *for different values of* $x = 0.3$, $y = 0.5$, $z = 0.7$

	$t = 0.1$	$t = 0.2$	$t = 0.3$
$\alpha = 0.7$	13.135242	43.681975	130.892917
$\alpha = 0.8$	8.116044	20.899623	49.487124
$\alpha = 0.9$	5.836724	12.660829	25.809748
$\alpha = 1.0$	4.606210	8.804623	16.150058

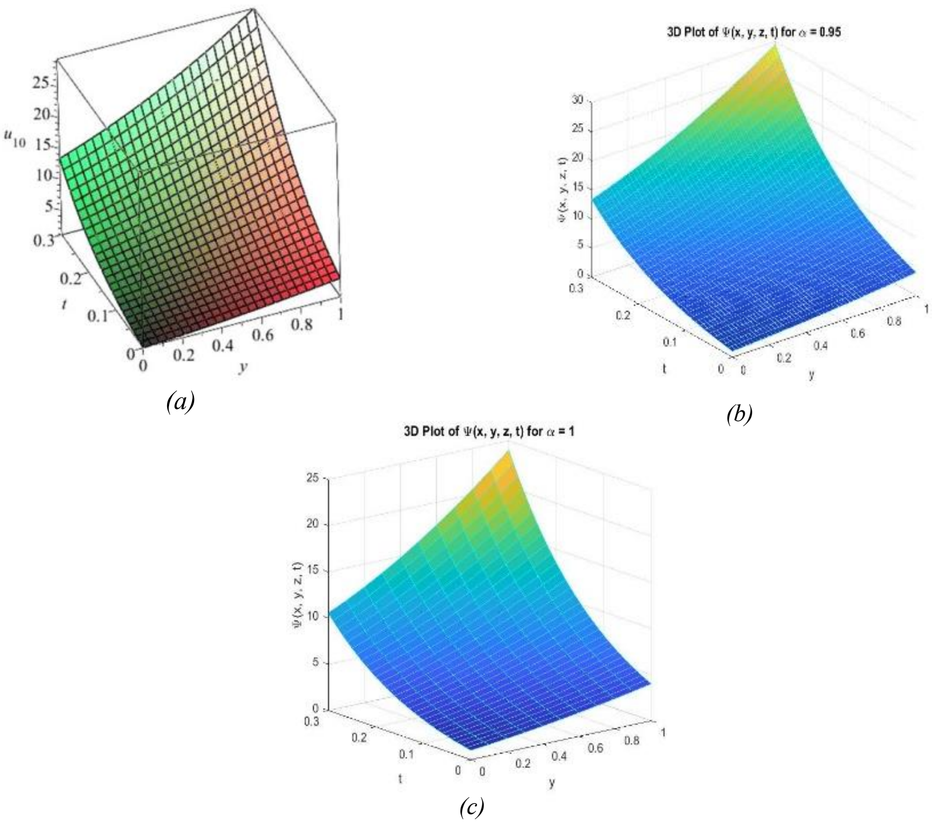


Figure 6: The tenth-order approximate solution of Eq. (4.22) by (a) FVHPIM at α = 0.95 and x = z = 0.5 (b) LVIM at α = 0.95 and x = z = 0.5 (c) Exact solution of eq. (4.22) by LVIM at α = 1.0 and x = z = 0.5

Results and Discussion:

If we consider the five mathematical examples (combining first, second, and third-order cases), our numerical solutions are found to be very close to the exact solutions, shown in Figure 2 - Figure 6. Once the numerical solutions were obtained, we compiled the results into a couple of tables for each problem, displaying the outcomes for various values of the fractional order and the independent variable. This approach enabled us to observe how changes in the fractional order affected the behavior of the solutions. Following the tabular presentation, we visually represented the results through 3-D plots (see, Figure2-Figure6). These graphical representations not only highlighted the numerical solutions but also facilitated direct comparisons with existing methods. The combination of numerical, tabular, and graphical analyses provided a comprehensive understanding of the behavior of these equations under various conditions.

The results show that the method is both accurate and stable, and it yields the exact solution when the fractional order $\alpha = 1$. Moreover, the method is easy to implement and computationally efficient, often producing the exact solution within just 2–3 iterations. In each case, we compared our solutions with those obtained by the Fractional Power Series Method (FPSM) or the Fractional Variational Homotopy Perturbation Iteration Method (FVHPIM). In all examples, our numerical solutions using LVIM were found to be very close to those produced by FPSM or FVHPIM.

Conclusion:

This paper introduces the Laplace Variational Iteration Method as an accessible approach for solving fractional Fokker-Planck and fractional diffusion equations. Through tables and graphical representations, it provides accurate values for both fractional and non-fractional cases, illustrating how the function behaves with different values of the independent variables. The results show a strong agreement with those obtained by existing methods, indicating that the proposed method is highly effective in significantly reducing the computational cost of solving such problems. The method does not need any linearization or perturbation, which helps it produce more reliable series solutions that usually converge fast in real physical problems.

Future research may focus on applying LVIM to real-world problems across various scientific and engineering fields. Its ability to handle complex nonlinear systems without linearization makes it a valuable tool for solving a wide range of fractional models.

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