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p-Γ-Rings

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ABSTRACT

The purpose of this paper is to introduce *p-Γ*-rings and a few of their most basic properties. Then these have been applied to investigate whether the most important properties like commutativty, being radical class and some other characterizations are preserved under our defined *p-Γ*-rings.

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1. Introduction:

The idea of a *Γ*-ring as the generalization of a ring was introduced by N. Nobusawa [8] and obtained analogues of the Wedderburn Theorem for *Γ*-rings with minimum condition on left ideals. W.E. Barnes [4] improved the idea of N. Nobusawa and gave the definition of *Γ*-rings which are more general than that of N. Nobusawa [8]. The notion of Jacobson radical was introduced by Coppape and Luh [5] and they developed some radical properties.

In this paper, we study various properties of *p-Γ*-rings. In 3, we obtain a basic theorem like: if *I* is an ideal of *M*, then *M/I is a p-Γ*-ring. (Th. 3.2), a *p-Γ*-ring is commutative(Th. 3.8) and if if $\mathfrak{R}$ is a class of all *p-Γ*-ring, then $\mathfrak{R}$ is a radical class (3.12).

In 4, we obtain a couple of necessary and sufficient conditions for *p-Γ*-rings (Th.4.1). By this theorem, we show that every finitely generated ideal is principal and the intersection of any two principal ideals of *R* is principal (4.2). Furthermore, we have seen that (1) The Jacobson radical $\mathfrak{J}(M)$ of *M* is zero, (2) *M* is a semi-simple ring if and only if it is a Noetherian *p-Γ*-ring, (3) The *p-Γ*-ring *M* without zero-divisor is a field, (4) Every ideal of *M* is non-singular, (5) *M* is left and right semi-hereditary (3.3). We have also proved: *M* is closed under homomorphic image; if $\mathfrak{R}$ is a class of all *p-Γ*-ring, then $\mathfrak{R}$ is a radical class.

2. Preliminaries:

Let *M* and *Γ* be additive abelian groups. If there is a mapping $M\times\Gamma\times M\rightarrow M$ satisfying, for all $a, b, c\in M$; $\alpha, \beta, \gamma\in\Gamma$

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$

(ii) $a(\alpha+\beta)b = a\alpha b + a\beta b$

(iii) $a\alpha(b+c) = a\alpha b + a\alpha c$ and

(iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then M is called a Γ-ring. This Γ-ring is due to Barnes [4].

***Ideal of Γ-rings*:** A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M = \{a\alpha b \mid a\in A\ \alpha\in\Gamma,\ b\in M\} \subseteq I(M\Gamma I \subseteq I)$. If I is both a right ideal and a left ideal then we say that I is an ideal, or redundantly, a two-sided ideal of M.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of M is also a left (respectively right or two-sided) ideal of M.

All other definitions and standard results used in this paper are due to Barnes [4].

3. *p-Γ*-Rings:

Definition: A Γ-ring M is said to be a p-Γ-ring if for every $x\in M$, there exists $\gamma\in\Gamma$ such that $(x\gamma)^p x = x$ for some prime $p > 1$ with $px = 0$.

Example. Let $M = (\mathbf{Z}_5, +, .)$ and $\Gamma = (\mathbf{Z}_5, +)$. Then M is a p-Γ-ring.

***Lemma* 3.1.** Let M be a p-Γ-ring. Then every right ideal I of M is a two-sided ideal of M.

Proof. We first observe that M has no nonzero nilpotent elements. For if $x \neq 0$, then $(x\gamma)^p x = x$ implies that $(x\gamma)^p x \neq 0$ for some prime p and some $\gamma\in\Gamma$. Next, let $a\in I$ and suppose $(a\gamma)^p a = a$ for some prime p. Then $\{(a\gamma)^{p-1}a\}\gamma\{(a\gamma)^{p-1}a\} = \{(a\gamma)^{p-1}a\gamma(a\gamma)^{p-1}a = (a\gamma)^p(a\gamma)^{p-1}a = (a\gamma)^p a\gamma(a\gamma)^{p-2}a = a\gamma(a\gamma)^{p-2}a = (a\gamma)^{p-1}a$, so $(a\gamma)^{p-1}a$ is an idempotent element.

Next, we show that an idempotent element commutes with every elements of M. To show this let e be an idempotent element of M. Then for any $x\in M$, $(x\gamma e - e\gamma x\gamma e)\gamma(x\gamma e - e\gamma x\gamma e) = 0 = (e\gamma x - e\gamma x\gamma e)\gamma(e\gamma x - e\gamma x\gamma e)$. Thus, $x\gamma e - e\gamma x\gamma e = e\gamma x - e\gamma x\gamma e = 0$ and so $x\gamma e = e\gamma x$, i.e., e commutes with every elements of M.

Now, for any $r\in M$ and $a\in I$ with $(a\gamma)^p a = a$, $r\gamma a = r\gamma(a\gamma)^p a =$ $r\gamma(a\gamma)^{p-1}(a\gamma)a = (a\gamma)^{p-1}a\gamma r\gamma a = (a\gamma)(a\gamma)^{p-2}a\gamma r\gamma a = a\gamma(a\gamma)^{p-1}r\gamma a = a\gamma r'$, where $r' = (a\gamma)^{p-1}r\gamma a\in M$. Since $a\gamma r'\in I$, so does $r\gamma a$ and so I is a two-sided ideal. ■

Lemma 3.2. Let M be a p-Γ-ring and I an ideal of M. Then M/I is p-Γ-ring.

Proof. Let $x \in M/I$, then $x = m + I$ for all $m \in M$ with $(m\gamma)^p m = m$, $p > 1$ and $\gamma \in \Gamma$. Now, $(x\gamma)^p x = (m + I)\gamma\}^p(m + I) = \{m\gamma + I\}^p(m + I) = \{(m\gamma)^p + I\}(m + I) = (m\gamma)^p m + I = m + I = x$. Thus, M/I is a p-Γ-ring. ■

Lemma 3.3. Let D be a division p-Γ-ring of characteristic $p \neq 0$ and let C be the center of D. Suppose that $a \in D$, $a \notin C$ is such that $(a\gamma)^{p^h} a = a$ for some $h > 0$. Then there exists an element $x \in D$ such that $x\gamma a\gamma x^{-1} \neq a$.

Proof. We define the mapping f: $D \to D$ by $f(x) = x\gamma a - a\gamma x$ for every $x \in D$. Now, $f^2(x) = ff(x) = f(x\gamma a - a\gamma x) = (x\gamma a - a\gamma x)\gamma a - a\gamma(x\gamma a - a\gamma x) = x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x$.

Again, $f^3(x) = f(x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x) = (x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x)\gamma a = (x\gamma a\gamma a\gamma a - 3a\gamma x\gamma a\gamma a - 3a\gamma a\gamma x\gamma a + a\gamma a\gamma a\gamma x)$. Thus, a simple computation yields that

$$f^p(x) = x\gamma(a\gamma)^{p-1}a - (a\gamma)^{p-1}a\gamma x, \text{ where char}D = p, \text{ a prime.}$$

Continuing we obtain that

$$f^{p^h}(x) = x\gamma(a\gamma)^{p^k} a - (a\gamma)^{p^k} a\gamma x,$$

for all $k \geq 0$. Let P denote the prime field of C; since a is algebraic over P, $P(a)$ must be a finite field having p^m elements, say. Hence $(a\gamma)^{p^m} a = a$ and so

$$f^{p^m}(x) = x\gamma(a\gamma)^{p^m} - (a\gamma)^{p^m} a\gamma x = x\gamma a - a\gamma x = f(x).$$

Thus, we see that the function $f^{p^m} = f$.

If $r \in P(a)$, then $f(r\gamma x) = (r\gamma x)\gamma a - a\gamma(r\gamma x) = r\gamma(x\gamma a - a\gamma x) = r\gamma f(x)$, since r commutes with a. If I denotes the identity map on D and rI denotes the map defined by $(rI)(x) = r\gamma x$, we have that $fo(rI) = (rI)of$, for all $r \in P(a)$. Since all elements of $P(a)$ satisfy the polynomial $t^{p^m} - t$, we find that $t^{p^m} - t = \prod_{r \in P(a)}(t - r)$. Since rI commutes with f, we have that $0 = f^{p^m} - f = \prod_{r \in P(a)}(f - r)$, where $(f - rI)(x) = f(x) - r\gamma x$. Now, Let $r_1 = 0$ (one of r's must be zero), and suppose for each $r_i \neq 0$, $(f - r_iI) \neq 0$, all $x \in D$, $x \neq 0$. Then $[(f - r_2I)o(f - r_3I)o\ldots\ldots o(f - r_{p^m}I)](x) \neq 0$, for all $x \in D$, $x \neq 0$. But since

$$0 = f^{p^m} - f = fo(f - r_2I)o(f - r_3I)o\ldots\ldots\ldots o(f - r_{p^m}I),$$

it follows that $f(x) = 0$ for all $x \in D$. Thus, $0 = f(x) = x\gamma a - a\gamma x$, whence $x\gamma a = a\gamma x$ for all $x \in D$. Thus, $a \in C$, contradicting the hypothesis. Thus, there is a $r_i \neq 0$, $r_i \in P(a)$ and $x \neq 0$ in D such that $(f - r_i I)(x) = 0$

i.e. $(f(x) - r_i I)(x) = 0$

i.e. $x\gamma a - a\gamma x - r_i\gamma x = 0$

i.e. $x\gamma a - a\gamma x = r_i\gamma x$

i.e. $x\gamma a\gamma x^{-1} - a\gamma x\gamma x^{-1} = r_i\gamma x\gamma x^{-1}$

i.e. $x\gamma a\gamma x^{-1} = r_i\gamma x\gamma x^{-1} + a\gamma x\gamma x^{-1} \neq a$, since $r_i \neq 0$.

This completes the proof. ■

Lemma 3.4. If D is a division Γ-ring of characteristic $p \neq 0$ and $G \subseteq D$ is a finite multiplicative subgroup of D, then G is commutative.

Proof. Let P be the prime field of D and let $A = \{r_i\gamma g_i / r_i \in D \text{ and } g_i \in G\}$. Clearly A is a finite subgroup of D under addition; moreover, since G is a group under multiplication, A is finite sub-Γ-ring of D. Therefore A is a finite division Γ-ring, hence is commutative. Since $G \subseteq A$, G is also commutative. ■

Lemma 3.5. Let D be a division Γ-ring such that for every $x \in D$ there exists a prime p such that $(x\gamma)^p x = x$. Then D is commutative.

Proof. Suppose a, $b \in D$ are such that $c = a\gamma b - b\gamma a \neq 0$. By hypothesis $(c\gamma)^m c = c$ for some prime $m > 1$. If $r(\neq 0) \in C$, the center of D, then $r\gamma c = r\gamma(a\gamma b - b\gamma a) = (r\gamma a)\gamma b - b\gamma(r\gamma a)$, hence by hypothesis, $\{(r\gamma c)\gamma\}^p (r\gamma c) = r\gamma c$. Let $q = (m-1)(p-1)+1$, $m > 1$, $p > 1$. Then $q > 1$ and q is prime. It follows that $(c\gamma)^q c = c$ and $\{(r\gamma c)\gamma\}^q (r\gamma c) = r\gamma c$, hence

$\{(r\gamma c)\gamma(r\gamma c)\gamma(r\gamma c)\gamma \ldots\ldots \text{up to } q \text{ times}\}(r\gamma c) = r\gamma c$

i.e. $(r\gamma)^q (c\gamma)^q (r\gamma c) = r\gamma c$,

i.e. $(r\gamma)^q (c\gamma)^q (c\gamma r) = r\gamma c$,

i.e. $(r\gamma)^q c\gamma r = r\gamma c$,

i.e. $(r\gamma)^q r\gamma c = r\gamma c$,

i.e. $\{(r\gamma)^q r - r\}\gamma c = 0$.

Since D is a division Γ-ring and $c \neq 0$, so $(r\gamma)^q r = r$ for every $r \in C$, $q > 1$ depending on r and γ. We know that C is of characteristic $p \neq 0$. Let P be the prime field of C. We claim that if D is not commutative, we could have chosen our a, b such that not only is $c = a\gamma b - b\gamma a \neq 0$ but, in fact, c is not even in C. If not, all commutators are in C; hence $c \in C$ and C contains $a\gamma(a\gamma b) - (a\gamma b)\gamma a = a\gamma(a\gamma b) - a\gamma(b\gamma a) = a\gamma(a\gamma b - b\gamma a) = a\gamma c$. This would place $a \in C$ contrary to $c = a\gamma b - b\gamma a \neq 0$. Thus, we assume that $c = a\gamma b - b\gamma a \notin C$. Since $(c\gamma)^m c = c$, c is algebraic over P hence $(c\gamma)^{p^k} c = c$ for some $k > 0$. Thus, all the hypothesis of the *Lemma* 3.3 are satisfied for C. Hence we can find $x \in D$ such that $x\gamma c\gamma x^{-1} = c_1 \neq c$, that is $x\gamma c = c_1\gamma x \neq c\gamma x$. In particular, $d = x\gamma c - c\gamma x \neq 0$; but $d\gamma c = x\gamma c\gamma c - c\gamma x\gamma c = c_1\gamma x\gamma c - c\gamma c_1\gamma x = c_1\gamma x\gamma c - c_1\gamma c\gamma x$ (since $c_1 \in C$) $= c_1\gamma(x\gamma c - c\gamma x) = c_1\gamma d$. As a commutator, $(d\gamma)^t d = d$ for some prime $t > 1$ and $d\gamma c\gamma d^{-1} = c_1$. Thus, the multiplicative subgroup of D generated by c and d is finite. Hence by *Lemma* 3.4, the multiplicative subgroup is abelian. This contradicts $c\gamma d \neq d\gamma c$. and proves the lemma. ■

Lemma 3.6. Let M be a p-Γ-ring with identity 1. Then for x, y∈M, xγy – yγx is in the intersection of the maximal ideals of M.

Proof. We know that every ring has a maximal ideal. Let I be such a maximal ideal. Then the quotient ring M/I has an identity, and since I is a maximal right ideal of M, M/I has no maximal ideals other than 0 and M/I. Thus, M/I is a division ring. Since M is a p-Γ-ring, M/I is a p-Γ-ring (by *Lemma* 3.2). Then by *Lemma* 3.5, M/I is commutative. From this it follows that $x\gamma y - y\gamma x \in I$, for all $x, y \in M$. The conclusion of the lemma is now immediate. ■

Lemma 3.7. Let M be p-Γ-ring with identity 1. Then M is commutative.

Proof. Suppose $x \neq 0$ is in every maximal ideal of M. Then $(x\gamma)^p x = x$, and $(x\gamma)^{p-1}x$ is an idempotent, say $(x\gamma)^{p-1}x = e \neq 0$ for all $p > 1$ and some $\gamma \in \Gamma$ and e must also be in every maximal ideal of M. Now, $1 - e$ can not be in any proper right ideal of M, for if it were, it would be in a maximal ideal K of M. Since $e \in K$, $1 = e + (1 - e)$ would be in K and hence $K = M$, a contradiction. Since $(1 - e)\gamma M \neq 0$ and since $(1 - e)\gamma M$ is a (right) ideal, it follows that $(1 - e)\gamma M = M$, whence $(1 - e)\gamma r = e$ for some $r \in M$. Thus, $0 = e\gamma(1 - e)\gamma r = e$, a contradiction. Thus, x can not be in every maximal ideal in M and the intersection of all the maximal ideals of M is 0. Thus, by *Lemma* 3.6, $x\gamma y - y\gamma x \in 0$, $x, y \in M$, that is, $x\gamma y = y\gamma x$, for all $x, y \in M$. ■

***Remarks*:** Since the intersection of all maximal ideals of a commutative Γ-ring with 1 is the Jacobson radical, so the Jacobson radical of p-Γ-ring with 1 is zero.

Theorem 3.8. If M is a p-Γ-ring, then M is commutative.

Proof. Let e be an idempotent in M. Then, $e\gamma x = x\gamma e$ for all $x\in M$. Thus, $e\gamma M = M\gamma e = T$ is also a p-Γ-ring, but T has an identity, namely e. Hence by *Lemma* 3.7, T is commutative. Now, for all x, $y\in M$, $x\gamma y\gamma e = x\gamma y\gamma e\gamma e = (x\gamma e)\gamma(y\gamma e) = (y\gamma e)\gamma(x\gamma e) = y\gamma x\gamma e$, that is $(x\gamma y - y\gamma x)\gamma e = 0$. Since $\{(x\gamma y - y\gamma x)\gamma\}^p(x\gamma y - y\gamma x) = (x\gamma y - y\gamma x)$ for some prime $p > 1$, so $\{(x\gamma y - y\gamma x)\gamma\}^{p-1}(x\gamma y - y\gamma x)$ is an idempotent, say e_1. Thus,

$$0 = (x\gamma y - y\gamma x)e_1 = \{(x\gamma y - y\gamma x)\gamma\}^p(x\gamma y - y\gamma x) = x\gamma y - y\gamma x,$$

that is, $x\gamma y = y\gamma x$. Hence, M is commutative ■

Lemma 3.9. Let M be a commutative Γ-ring. Let I be an ideal of M such that I a p-Γ-ring. Then $e\gamma\{y - (y\gamma)^p y\} = 0$ for all $y\in M$ and some $\gamma\in\Gamma$ and $e\in I$ an idempotent.

Proof. Let $x\in I$ and $y\in M$. Then $x\gamma y\in I$. Since I is a p-Γ-ring, $(x\gamma)^p x = x$. Also $\{(x\gamma y)\gamma\}^p(x\gamma y) = x\gamma y$ for some prime p and $\gamma\in\Gamma$.

Now, $\{(x\gamma y)\gamma\}^p(x\gamma y) = x\gamma y$,

i.e. $\{(x\gamma y)\gamma(x\gamma y)\gamma\,.\,.\,.\,.\,.\,.\text{up to p times}\}(x\gamma y) = x\gamma y$,

i.e. $(y\gamma)^p(x\gamma)^p(x\gamma y) = x\gamma y$, since M is commutative,

i.e. $(y\gamma)^p x\gamma y = x\gamma y$.

i.e. $\{(y\gamma)^p y - y\}\gamma x = 0$,

so $(x\gamma)^{p-1}x\gamma\{y - (y\gamma)^p y\} = 0$ and hence $e\gamma\{y - (y\gamma)^p y\} = 0$, where $e = (x\gamma)^{p-1}x$ is an idempotent of I. ■

Lemma 3.10. Let M be a Γ-ring and I an ideal of M. Then M is a p-Γ-ring if M/I and I are p-Γ-rings.

Proof. Let M/I and I be p-Γ-rings. Let $x\in M$, then $x + I\in M/I$ and so

$\{(x + I)\gamma\}^p(x + I) = x + I$ for some prime p and $\gamma\in\Gamma$.

i.e. $(x\gamma + I)^p(x + I) = x + I$,

i.e. $\{(x\gamma)^p + I\}(x + I) = x + I$,

i.e. $(x\gamma)^p x + I = x + I$.

Thus, $(x\gamma)^p x - x\in I$. Since I is a p-Γ-ring, $\{((x\gamma)^p x - x)\gamma\}^m\{(x\gamma)^p x - x\} = (x\gamma)^p x - x$ for some prime m. Let $e' = \{((x\gamma)^p x - x)\gamma\}^{m-1}\{(x\gamma)^p x - x\}$. Then e' is an idempotent of I. By *Lemma*

3.9, $e^{/}\gamma\{(x\gamma)^p x - x\} = 0$ for every $x\in M$. Now, $0 = e^{/}\gamma\{(x\gamma)^p x - x\} = \{(x\gamma)^p x - x)\gamma\}^{m-1}\{(x\gamma)^p x - x\}\gamma\{(x\gamma)^p x - x\} = \{(x\gamma)^p x - x)\gamma\}^{m}\{(x\gamma)^p x - x\} = (x\gamma)^p x - x$. Hence $(x\gamma)^p x = x$. Therefore M is a p-Γ-ring. ■

Lemma 3.11. If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ - - - - - - is an ascending chain of ideals which are all p-Γ-rings, then $\cup_\alpha I_\alpha$ is a p-Γ-rings.

Proof. Let $x\in\cup_\alpha I_\alpha$, then $x\in I_\alpha$ for some α. Since I_α is a p-Γ-ring, then $(x\gamma)^p x = x$ for some prime p and $\gamma\in\Gamma$. Hence $\cup_\alpha I_\alpha$ is a p-Γ-ring. ■

Thus, by *Lemma* 3.2, *Lemma* 3.10 and *Lemma* 3.11, we have the following theorem:

Theorem 3.12. The class of all p-Γ-rings is a radical class.

4. Some Characterizations of *p-Γ*-rings

Theorem 4.1. Let M be a ring with 1. Let a, x∈M such that $a = (x\gamma)^{p-2}x$. Then the following statements are equivalent:

M is a p-Γ-ring.

Every principal ideal Mγa is generated by an idempotent.

For every principal ideal Mγa of M, there exists an element b∈M such that M = Mγa ⊕ Rγb.

Every principal ideal Mγa is a direct summand of M.

Proof. (***a***) ⇒ (***b***) Let $x\in M$. Then $(x\gamma)^p x = x$ for some prime p and $\gamma\in\Gamma$. Let $a\in M$ such that $a = (x\gamma)^{p-2}x$. Now, the principal ideal $M\gamma a$ is generated by the element $x\gamma a$ which is idempotent; for $(x\gamma a)\gamma(x\gamma a) = x\gamma\{(x\gamma)^{p-2}x\}\gamma\{x\gamma(x\gamma)^{p-2}x\} = (x\gamma)^p x\gamma(x\gamma)^{p-2}x = x\gamma a$.

(***b***) ⇒ (***c***) Let $M\gamma a = M\gamma e$, where $e\gamma e = e$ and $a = (x\gamma)^{p-2}x$, $x\in M$. Since $1 = e + (1 - e)$, and if there exists $b\in M$ such that $a\gamma e = b\gamma(1 - e)$, then $a\gamma e = a\gamma e\gamma e = b\gamma(1 - e)\gamma e = 0$. So $M = M\gamma e \oplus M\gamma(1 - e)$.

(***c***) ⇒ (***d***) Trivial.

(***d***) $\Rightarrow$ (***a***) Let $a\in M$. Then there exists an ideal I of M such that $M = M\gamma a \oplus I$. Hence $1 = x\gamma a + b$, where $b\in I$, so $x = x\gamma a\gamma x + b\gamma x$. Since $a = (x\gamma)^{p-2}x$, $b\gamma x = x - x\gamma a\gamma x \in M\gamma a \cap I = 0$, and therefore $x = x\gamma\{(x\gamma)^{p-2}x\}\gamma x = (x\gamma)^p x$. Hence M is a p-Γ-ring.

Theorem 4.2. Let M be a p-Γ-ring with 1. Then

Every finitely generated ideal is principal.

The intersection of any two principal ideals of M is principal.

Proof. 1) It is enough to prove that if a, $b\in M$, then $M\gamma a + M\gamma b$ is principal. Since M is a p-Γ-ring, there exists elements x, $y\in M$ with $a = (x\gamma)^{p-2}x$ and $b = (y\gamma)^{p-2}y$ such that the elements $e_1 = x\gamma a$ and $e_2 = y\gamma b$ are the idempotent elements of $M\gamma a$ and $M\gamma b$ respectively and also $M\gamma a = M\gamma e_1$ and $M\gamma b = M\gamma e_2$ by *Theorem* 4.1(*b*). Now, $M\gamma a + M\gamma b = M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma(e_2 - e_2\gamma e_1)$ because $a_1\gamma e_1 + a_2\gamma e_2 = (a_1 + a_2\gamma e_2)\gamma e_1 + a_2\gamma(e_2 - e_2\gamma e_1)$. If $s = \{(e_2 - e_2\gamma e_1)\gamma\}^{p-2}(e_2 - e_2\gamma e_1)\in M$, then

$(e_2 - e_2\gamma e_1)\gamma s\gamma(e_2 - e_2\gamma e_1) = \{(e_2 - e_2\gamma e_1)\gamma\}^p(e_2 - e_2\gamma e_1) = (e_2 - e_2\gamma e_1)$. Then $e'_2 = s\gamma(e_2 - e_2\gamma e_1)$ is an idempotent of $M\gamma b$. Then $M\gamma e_1 + M\gamma e_2 =$

$M\gamma e_1 + M\gamma e'_2$ with $e'_2\gamma e_1 = s\gamma(e_2 - e_2\gamma e_1)\gamma e_1 = 0$.

Finally, we have, $a_1\gamma e_1 + a_2\gamma e'_2 = (a_1\gamma e_1 + a_2\gamma e'_2)\gamma(e_1 + e'_2 - e'_2\gamma e_1)$, a_1, $b_1\in M$. Thus, $M\gamma e_1 + M\gamma e'_2 = M\gamma(e_1 + e'_2 - e'_2\gamma e_1)$. Therefore $M\gamma a + M\gamma b = M\gamma(e_1 + e'_2 - e'_2\gamma e_1)$. Thus, $M\gamma a + M\gamma b$ is a principal ideal.

2) Let $M\gamma a$ and $M\gamma b$ be two principal ideals. Since M is a p-Γ-ring, there exists elements x, $y\in M$ with $a = (x\gamma)^{p-2}x$ and $b = (y\gamma)^{p-2}y$ such that the elements $e_1 = x\gamma a$ and $e_2 = y\gamma b$ are the idempotents of $M\gamma a$ and $M\gamma b$ respectively and also $M\gamma a = M\gamma e_1$ and $M\gamma b = M\gamma e_2$ by *Theorem* 4.1(*b*). Hence $M = M\gamma e_1 \oplus M\gamma(1 - e_1) = M\gamma e_2 \oplus M\gamma(1 - e_2)$, and

$M\gamma e_1 = Ann_M[(1 - e_1)\gamma M] = \{x\in M \mid x\gamma(1 - e_1)\gamma M = 0\}$,

$M\gamma e_2 = Ann_M[(1 - e_2)\gamma M] = \{x\in M \mid x\gamma(1 - e_2)\gamma M = 0\}$.

Indeed obviously $M\gamma e_1 \subseteq Ann_M[(1 - e_1)\gamma M]$.

Conversely, if $x\in M$ and $x\gamma(1 - e_1) = 0$, writing $x = a_1\gamma e_1 + b_1\gamma(1 - e_1)$, a_1, $b_1\in M$, we have

$a_1\gamma e_1\gamma(1 - e_1) + b_1\gamma(1 - e_1)\gamma(1 - e_1) = 0$, and so

$b_1\gamma(1 - e_1) = 0$, hence $x = a_1\gamma e_1\in M\gamma e_1$.

Thus, $M\gamma e_1 \cap M\gamma e_2 = Ann_M[(1 - e_1)\gamma M + (1 - e_2)\gamma M]$. Now, there exists $e_3\in M$ such that $(1 - e_1)\gamma M + (1 - e_2)\gamma M = (1 - e_3)\gamma M$, and from $M\gamma e_3 = Ann_M[(1 - e_3)\gamma M]$ we deduce that $M\gamma e_1 \cap M\gamma e_2 = M\gamma e_3$. Thus, $M\gamma e_1 \cap M\gamma e_2 = M\gamma a \cap M\gamma b$ is a principal ideal. ■

***Semihereditary*:** A Γ-ring M is said to be **semihereditary** if every finitely generated right ideal of M is projective M-module.

Nonsingular Γ-ring: An ideal I of a Γ-ring M is called essential if for every nonzero ideal A in M, $I \cap A \neq 0$. Let $\varphi(M)$ be the class of all essential ideals in M and $Z_r(M) = \{x \in M \mid x\Gamma I = 0 \text{ for some } I \in \varphi(M)\}$. M is called a **nonsingular** Γ-ring if $Z_r(M) = 0$. For the case of a classical ring R, we define $Z_r(R) = \{x \in R \mid xI = 0 \text{ for some } I \in \varphi(R)$. Then R is called a non-singular if $Z_r(R) = 0$.

Theorem 4.3. Let M be a p-Γ-ring with unity 1. Then

a) The Jacobson radical $\mathfrak{I}(M)$ of M is zero.

b) M is a semisimple ring if and only if it is a Noetherian p-Γ-ring.

c) The centre of M is also a p-Γ-ring.

d) The p-Γ-ring M without zero divisor is a field.

e) Every ideal of M is nonsingular.

f) For any idempotent element e of M, $(1 - e)\gamma M\gamma e = 0$.

g) If $(M_i)_{i \in I}$ is a family of p-Γ-rings then $\prod M_i$ is a p-Γ-ring.

h) M is semihereditary.

Proof. a) Let $a \in \mathfrak{I}(M)$. Then $M\gamma a \subseteq \mathfrak{I}(M)$. Since $M\gamma a = M\gamma e$, where $e = x\gamma a$ is an idempotent with $a = (x\gamma)^{p-2}x$, so $e \in \mathfrak{I}(M)$. It follows that $(1 - e)$ is inevitable. So there exists $y \in M$ such that $1 = y\gamma(1 - e) = y - y\gamma e$. Hence $e = y\gamma e - y\gamma e\gamma e = y\gamma e - y\gamma e = 0$ and therefore $a = 0$. Thus, $\mathfrak{I}(M) = 0$.

b) First suppose that M is finitely generated. Then every ideal of M is finitely generated and hence a direct summand. So M is a semi-simple.

Conversely, let M be a semisimple ring. Then every principal ideal of M is a direct summand of M and hence M is a p-Γ-ring by *Theorem* 4.1(*d*). Since Jacobson radical $\mathfrak{I}(M)$ is the largest ideal of M and since in a p-Γ-ring, $\mathfrak{I}(M) = 0$, so any ascending chain of ideals of M must be finite. Hence M is Noetherian.

c) Since p-Γ-ring is abelian, so centre of M is M itself, i.e. $C(M) = M$.

d) Let $a \in M$ with $a \neq 0$. Then $(a\gamma)^p a = a$ for some prime p. Then $(a\gamma)^p a - a = 0 \Rightarrow a\gamma\{(a\gamma)^{p-1}a - 1\} = 0$. Since $a \neq 0$, so $(a\gamma)^{p-1}a - 1 = 0$ and so $(a\gamma)^{p-2}a$ is the inverse of a. Since p-Γ-ring M is abelian, so M is a field.

e) Suppose that $x\gamma I = 0$ for some $x \in M$ and $I \subseteq M$ is an ideal of M. Let

$M\gamma x$ be a principal ideal of M. Then there is an idempotent $e \in M$ such that $M\gamma x = M\gamma e$. Now, since $M\gamma e\gamma I = M\gamma x\gamma I = 0$, we see that $I \subseteq M\gamma(1 - e)$. Then $I \cap e\gamma M = 0$, whence $M\gamma e = 0$ and consequently $x = 0$. Thus, M is nonsingular.

f) Since $M\gamma e$ is a two-sided ideal, so $(1 - e)\gamma M\gamma e = M\gamma e - M\gamma e\gamma e = M\gamma e - M\gamma e = 0$.

g) Proof is obvious

h) Since a finitely generated ideal of M is a direct summand of M and

so is projective. Hence M is semihereditary. ■

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