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JORDAN *k*-HOMOMORPHISMS ONTO COMPLETELY PRIME Γ_N-RINGS

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ABSTRACT

By introducing the notions of *k*-homomorphism, anti-*k*-homomorphism and Jordan *k*-homomorphism of Nobusawa Γ-rings, we establish some significant results related to these concepts. If M_1 is a Nobusawa Γ_1-ring and M_2 is a 2-torsion free completely prime Nobusawa Γ_2-ring, then we prove that every Jordan *k*-homomorphism θ of M_1 onto M_2 such that $k(\Gamma_1) = \Gamma_2$ is either a *k*-homomorphism or an anti-*k*-homomorphism.

1. Introduction

The notion of a Γ-ring was first introduced by N. Nobusawa [12] and then it was generalized by W. E. Barnes [1]. A number of significant properties of Γ-rings were obtained by them as well as by S. Kyuno [8, 9, 10], J. Luh [11], G. L. Booth [3] and others. We begin with the following definition.

Let M and Γ be additive abelian groups. If there exists a mapping $(a,\alpha,b) \to a\alpha b$ of $M \times \Gamma \times M \to M$ which satisfies the following conditions for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$:

(a) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ and

(b) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then M is called a Γ-ring in the sense of Barnes [1], or simply, a Γ-ring.

It is obvious that *every ring is a* Γ-*ring*, but the converse is not true in general.

Example 1.1 Let R be a ring with identity 1 and $M_{m,n}(R)$ the set of all $m \times n$ matrices

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with entries in R. If we set $M = M_{m,n}(R)$ and $\Gamma = M_{n,m}(R)$, then M is clearly a Γ-ring with respect to the matrix addition and multiplication.

In addition to the definition of a Γ-ring given above, if there exists another mapping $(\alpha, a, \beta) \to \alpha a \beta$ of $\Gamma \times M \times \Gamma \to \Gamma$ satisfying the following conditions

(a*) $(\alpha + \beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a+b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta+\gamma) = \alpha a\beta + \alpha a\gamma$,

(b*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and

(c*) $a\alpha b = 0$ implies $\alpha = 0$

for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$, then M is called a Γ-ring in the sense of Nobusawa [12], or simply, a Nobusawa Γ-ring and then we say that M is a Γ_N-ring. This notation was first used by G. L. Booth in [3].

Example 1.2 Let $D_{m,n}$ be the set of all rectangular $m \times n$ matrices over some division ring D. If we take $M = D_{m,n}$ and $\Gamma = D_{n,m}$, then it is obvious that M is a Γ_N-ring under the usual addition and multiplication of matrices.

Clearly, M is a Γ-ring does not imply that Γ is an M-ring in general, but M is a Γ_N-ring always implies that Γ is an M-ring.

Note that the notions of a prime Γ-ring and a completely prime Γ-ring were introduced by J. Luh [11] and some analogous results corresponding to prime Γ-rings were obtained by him and S. Kyuno [10].

Let M be a Γ-ring. Then M is called a *prime* Γ-ring if $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in M$. And, M is called a *completely prime* Γ*-ring* if $a\Gamma b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in M$. Obviously, the primeness and completely primeness are equivalent in case of Γ_N-rings.

Example 1.3 Let R be an integral domain with identity 1. Take $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \text{ is an integer} \right\}$. Then M is a Γ-ring. If we assume $N = \{(a, a) : a \in R\} \subset M$, then it is easy to verify that N is a Γ-ring (in fact, N is then a Γ-subring of M) and also that N is a completely prime Γ-ring.

In a Γ-ring M, an additive subgroup U of M is called a *left* (or, *right*) *ideal* of M if $M\Gamma U \subset U$ (or, $U\Gamma M \subset U$). Here, U is said to be a (*two-sided*) *ideal* of M if U is both a left and a right ideal of M. If $a, b \in M$ and $\alpha \in \Gamma$, then we *write* $(a \circ b)_\alpha = a\alpha b + b\alpha a$ and it is known as the *Jordan product* of a and b with respect to α. Besides, M is said to be a *2-torsion free* Γ-ring if $2a = 0$ implies $a = 0$ for all $a \in M$.

The notion of Jordan homomorphism of rings was introduced by N. Jacobson and C. E. Rickart in [7], where they have proved that *every Jordan homomorphism of a ring into an integral domain is either a homomorphism or an anti-homomorphism*. Afterwards, I. N.

Herstein [5] has modified this result stating that *every Jordan homomorphism of a ring onto a prime ring of characteristic different from two and three is either a homomorphism or an anti-homomorphism*. But, M. F. Smiley [13] has given a brief proof of this result and at the same time has removed the requirement that the characteristic should be different from three. Thus, the said Herstein's result has reduced by him to the form: *every Jordan homomorphism of a ring onto a 2-torsion free prime ring is either a homomorphism or an anti-homomorphism*.

Note that the notion of homomorphism for Γ-rings was first introduced by W. E. Barnes [1] and then generalized by W. E. Coppage and J. Luh in [4]. Later, S. Kyuno [9] has specified this concept for Γ_N-rings; but, he has given a more general definition of a homomorphism for Γ_N-rings in [8] in the following way:

Let (Γ_1, M_1) and (Γ_2, M_2) be Γ_N-rings. Then an ordered pair (θ, φ) of mappings is called a homomorphism from (Γ_1, M_1) to (Γ_2, M_2) if it satisfies the following properties:

(1) θ is a group homomorphism from M_1 to M_2,

(2) φ is a group homomorphism from Γ_1 to Γ_2, and

(3) for every $x, y \in M_1$ and $\alpha \in \Gamma_1$, $\theta(x\alpha y) = (\theta x)(\varphi\alpha)(\theta y)$.

Here, we *redefine* this concept in a new shape keeping its originality unaltered and then we *introduce* a new name of this as a *k-homomorphism* in the following way.

Definition 1.1 Let M_1 be a Nobusawa Γ_1-ring and M_2 a Nobusawa Γ_2-ring. If $\varphi : M_1 \to M_2$ and $k : \Gamma_1 \to \Gamma_2$ are additive mappings such that $\varphi(a\alpha b) = \varphi(a)k(\alpha)\varphi(b)$ for all $a, b \in M_1$ and $\alpha \in \Gamma_1$, then φ is called a *k-homomorphism*.

Remark 1.1 If $\varphi : M_1 \to M_2$ is a *k-homomorphism* of a Nobusawa Γ_1-ring M_1 into another Nobusawa Γ_2-ring M_2, then we have $\varphi(a\alpha b) = \varphi(a)k(\alpha)\varphi(b)$ for all $a, b \in M_1$ and $\alpha \in \Gamma_1$. Let $c \in M_1$ and $\beta, \gamma \in \Gamma_1$. Put $\beta c\gamma$ for α to get $\varphi(a\beta c\gamma b) = \varphi(a)k(\beta c\gamma)\varphi(b)$. Then simplifying it by definition, we obtain $\varphi(a)[k(\beta)\varphi(c)k(\gamma) - k(\beta c\gamma)]\varphi(b) = 0$. Since M_2 is a Nobusawa Γ_2-ring and $\varphi(a), \varphi(b) \in M_2$ for all $a, b \in M_1$, by applying the Nobusawa condition (c*) in the last equality, we get $k(\beta c\gamma) = k(\beta)\varphi(c)k(\gamma)$ for all $c \in M_1$ and $\beta, \gamma \in \Gamma_1$.

Some authors have used this property as the second defining condition in addition with the first one that we have mentioned in Definition 1.1. Since the first condition implies the second one by Remark 1.1, the second condition may be removed from the definition, and thus, the Definition 1.1 is justified.

Example 1.4 Let M be a Nobusawa Γ-ring. Consider $M' = \{m' : m \in M\}$ and $\Gamma' = \{\gamma' : \gamma \in \Gamma\}$ such that the operations of addition and multiplication on M' are defined by $m' + n' = (m+n)'$ and $m'\gamma'n' = (m\gamma n)'$ along with those on Γ' by $\gamma' + \delta' = (\gamma + \delta)'$ and $\gamma' m' \delta' = (\gamma m \delta)'$, respectively, for all $m', n' \in M'$ such that $m, n \in M$ and for all $\gamma', \delta' \in \Gamma'$ such that $\gamma, \delta \in \Gamma$. Then M' is a Nobusawa Γ'-ring under these operations. Now, if the map $\varphi : M \to M'$ is defined by $\varphi(x) = x'$ for all $x \in M$ and the map $k : \Gamma \to \Gamma'$ is defined by $k(\alpha) = \alpha'$ for all $\alpha \in \Gamma$, then it can be shown that each of them is an additive map, and consequently, φ is a k-homomorphism.

Following the definition of k-homomorphism for Γ_N-rings, we then *introduce* the concept of *anti-k-homomorphism* for Γ_N-rings in the following way.

Definition 1.2 Let M_1 be a Nobusawa Γ_1-ring and M_2 a Nobusawa Γ_2-ring. Suppose $\psi : M_1 \to M_2$ and $k : \Gamma_1 \to \Gamma_2$ are additive mappings such that $\psi(a\alpha b) = \psi(b)k(\alpha)\psi(a)$ for all $a, b \in M_1$ and $\alpha \in \Gamma_1$. Then ψ is called an *anti-k-homomorphism*.

Remark 1.2 Suppose $\psi : M_1 \to M_2$ is an anti-k-homomorphism of a Nobusawa Γ_1-ring M_1 into another Nobusawa Γ_2-ring M_2. Then we get $\psi(a\alpha b) = \psi(b)k(\alpha)\psi(a)$ for all $a, b \in M_1$ and $\alpha \in \Gamma_1$. Placing $\beta c \gamma$ for α (where $c \in M_1$ and $\beta, \gamma \in \Gamma_1$), we have $\psi(a\beta c\gamma b) = \psi(b)k(\beta c\gamma)\psi(a)$. It gives, $\psi(b)[k(\gamma)\psi(c)k(\beta) - k(\beta c \gamma)]\psi(a) = 0$. But, since M_2 is a Nobusawa Γ_2-ring and $\psi(a), \psi(b) \in M_2$ for all $a, b \in M_1$, using the Nobusawa condition (c*), we obtain $k(\beta c\gamma) = k(\gamma)\psi(c)k(\beta)$ for all $c \in M_1$ and $\beta, \gamma \in \Gamma_1$.

One may use this condition as the second defining property along with the first one we have given in Definition 1.2. But, since the second one follows from the first property by Remark 1.2, the second property must not be added as a defining condition, and therefore, it justifies the Definition 1.2.

Example 1.5 Let D be a division ring and $M_{n,m}(D)$ the set of all $n \times m$ matrices with entries in D. Choose $M = M_{n,m}(D)$ and $\Gamma = M_{m,n}(D)$. Then M is a Nobusawa Γ-ring with respect to addition and multiplication of matrices. Define $N = \{A^T : A \in M_{n,m}(D)\}$ and $\Gamma' = \{\alpha^T : \alpha \in M_{m,n}(D)\}$, where X^T denotes the transpose of the matrix X. Then N is a Nobusawa Γ'-ring with respect to addition and multiplication of matrices. Now, let the map $\psi : M \to N$ be defined by $\psi(A) = A^T$ for all $A \in M$, and let the map $k : \Gamma \to \Gamma'$ be defined by $k(\alpha) = \alpha^T$ for all $\alpha \in \Gamma$. Then it can be shown that both the maps are additive, and accordingly, ψ is an anti-k-homomorphism.

Now, we introduce the notion of *Jordan k-homomorphism* for Γ_N-rings as below:

Definition 1.3 Let M_1 be a Nobusawa Γ_1-ring and M_2 a Nobusawa Γ_2-ring, and let $k : \Gamma_1 \to \Gamma_2$ be an additive map. Then an additive map $\theta : M_1 \to M_2$ is called a *Jordan k-homomorphism* if $\theta((a \circ b)_\alpha) = (\theta(a) \circ \theta(b))_{k(\alpha)}$, that is, if $\theta(a\alpha b + b\alpha a)$

$=\theta(a)k(\alpha)\theta(b)+\theta(b)k(\alpha)\theta(a)$ for all $a, b\in M_1$ and $\alpha\in\Gamma_1$.

In particular, if M_2 is 2-torsion free and $a=b$, then the above defining condition reduces to $\theta(a\alpha a)=\theta(a)k(\alpha)\theta(a)$ for all $a\in M_1$ and $\alpha\in\Gamma_1$. Thus, *alternatively*, we can say that an additive map $\theta: M_1\to M_2$ is a Jordan *k*-homomorphism of a Nobusawa Γ_1-ring M_1 into a 2-torsion free Nobusawa Γ_2-ring M_2 if $\theta(a\alpha a)=\theta(a)k(\alpha)\theta(a)$ for all $a\in M_1$ and $\alpha\in\Gamma_1$.

Example 1.6 Let M_i be a Nobusawa Γ_i-ring for $i=1, 2, 3, 4$. Consider $M=M_1\times M_2$ and $\Gamma=\Gamma_1\times\Gamma_2$. Let the operations of addition and multiplication on *M* be defined by

$$(x_1, x_2)+(y_1, y_2)=(x_1+y_1, x_2+y_2) \text{ and}$$

$$(x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)=(x_1\alpha_1 y_1, x_2\alpha_2 y_2)$$

along with those on Γ by

$$(\alpha_1, \alpha_2)+(\beta_1, \beta_2)=(\alpha_1+\beta_1, \alpha_2+\beta_2) \text{ and}$$

$$(\alpha_1, \alpha_2)(x_1, x_2)(\beta_1, \beta_2)=(\alpha_1 x_1\beta_1, \alpha_2 x_2\beta_2)$$

for all $x_1, y_1\in M_1$, $x_2, y_2\in M_2$, $\alpha_1, \beta_1\in\Gamma_1$ and $\alpha_2, \beta_2\in\Gamma_2$, respectively. Then it can be verified that *M* is a Nobusawa Γ-ring under these operations.

Similarly, if we consider $M'=M_3\times M_4$ and $\Gamma'=\Gamma_3\times\Gamma_4$, and define the operations of addition and multiplication on M' by

$$(x_3, x_4)+(y_3, y_4)=(x_3+y_3, x_4+y_4) \text{ and}$$

$$(x_3, x_4)(\alpha_3, \alpha_4)(y_3, y_4)=(x_3\alpha_3 y_3, x_4\alpha_4 y_4)$$

including those on Γ' by

$$(\alpha_3, \alpha_4)+(\beta_3, \beta_4)=(\alpha_3+\beta_3, \alpha_4+\beta_4) \text{ and}$$

$$(\alpha_3, \alpha_4)(x_3, x_4)(\beta_3, \beta_4)=(\alpha_3 x_3\beta_3, \alpha_4 x_4\beta_4)$$

for all $x_3, y_3\in M_3$, $x_4, y_4\in M_4$, $\alpha_3, \beta_3\in\Gamma_3$ and $\alpha_4, \beta_4\in\Gamma_4$, respectively, then M' is a Nobusawa Γ'-ring under these operations.

Let $k_1:\Gamma_1\to\Gamma_3$ be an additive map for which $\varphi: M_1\to M_3$ is a k_1-homomorphism, and let $k_2:\Gamma_2\to\Gamma_4$ be another additive map for which $\psi: M_2\to M_4$ is an anti-k_2-homomorphism. Define an additive map $\theta: M\to M'$ by $\theta((x, y))=(\varphi(x), \psi(y))$ for all $x\in M_1$ and $y\in M_2$, and another additive map $k:\Gamma\to\Gamma'$ by $k((\alpha, \beta))=(k_1(\alpha), k_2(\beta))$ for all $\alpha\in\Gamma_1$ and $\beta\in\Gamma_2$. Then for all $x, z\in M_1$, $y, w\in M_2$, $\alpha\in\Gamma_1$ and $\beta\in\Gamma_2$, we have

$$\theta((x,y)(\alpha,\beta)(z,w)+(z,w)(\alpha,\beta)(x,y))$$
$$=\theta(x\alpha z+z\alpha x, y\beta w+w\beta y)$$
$$=(\varphi(x\alpha z+z\alpha x),\psi(y\beta w+w\beta y))$$
$$=(\varphi(x)k_1(\alpha)\varphi(z),\psi(y)k_2(\beta)\psi(w))+(\varphi(z)k_1(\alpha)\varphi(x),\psi(w)k_2(\beta)\psi(y))$$
$$=(\varphi(x),\psi(y))(k_1(\alpha),k_2(\beta))(\varphi(z),\psi(w))+(\varphi(z),\psi(w))(k_1(\alpha),k_2(\beta))(\varphi(x),\psi(y))$$
$$=\theta((x,y))k((\alpha,\beta))\theta((z,w))+\theta((z,w))k((\alpha,\beta))\theta((x,y)),$$

which indicates that θ is a Jordan *k*-homomorphism.

Our objective is to show that every Jordan k-homomorphism $\theta: M_1 \to M_2$ such that $k(\Gamma_1)=\Gamma_2$ is either a k-homomorphism or an anti-k-homomorphism if we choose M_1 as a Nobusawa Γ_1-ring and M_2 as a 2-torsion free completely prime Nobusawa Γ_2-ring. To establish this result we build up the following important lemmas.

2. Main Results

Lemma 2.1 Let *M* be a Γ_1-ring and *N* a 2-torsion free Γ_2-ring. If $\theta: M \to N$ is a Jordan *k*-homomorphism, then for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma_1$, the following statements hold:

(i) $\theta(a\alpha b\beta a + a\beta b\alpha a)=\theta(a)k(\alpha)\theta(b)k(\beta)\theta(a)+\theta(a)k(\beta)\theta(b)k(\alpha)\theta(a)$;

(ii) $\theta(a\alpha b\alpha a)=\theta(a)k(\alpha)\theta(b)k(\alpha)\theta(a)$;

(iii) $\theta(a\alpha b\alpha c + c\alpha b\alpha a)=\theta(a)k(\alpha)\theta(b)k(\alpha)\theta(c)+\theta(c)k(\alpha)\theta(b)k(\alpha)\theta(a)$.

Proof. Since θ is a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*, by definition, we have $\theta(a\alpha b + b\alpha a)=\theta(a)k(\alpha)\theta(b)+\theta(b)k(\alpha)\theta(a)$ for all $a,b \in M$ and $\alpha \in \Gamma_1$. Replacing *b* by $a\beta b + b\beta a$ in this equality, we get (i). Then (ii) is easily obtained by replacing α for β in (i), and (iii) is obtained by replacing $a+c$ for *a* in (ii).

Lemma 2.2 If θ is a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*, then $k(\beta b\beta)=k(\beta)\theta(b)k(\beta)$ for all $b \in M$ and $\beta \in \Gamma_1$.

Proof. For all $a \in M$ and $\alpha \in \Gamma_1$, we have $\theta(a\alpha a)=\theta(a)k(\alpha)\theta(a)$. Let $b \in M$ and $\beta \in \Gamma_1$. Putting $\beta b\beta$ for α, we get $\theta(a\beta b\beta a)=\theta(a)k(\beta b\beta)\theta(a)$. Expanding the LHS by Lemma 2.1(ii), we obtain $\theta(a)(k(\beta)\theta(b)k(\beta)-k(\beta b\beta))\theta(a)=0$. Since *N* is a Γ_2-ring and $\theta(a)\in N$ for all $a \in M$, applying the Nobusawa condition (c*) in the last equality, we get the proof.

Lemma 2.3 If θ is both a Jordan k_1- and a Jordan k_2-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*, then $k_1 = k_2$.

Proof. Obvious.

Hence, it follows that if θ is a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*, then *k* is uniquely determined.

Definition 2.1 Let θ be a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*. Then for $a,b \in M$ and $\alpha \in \Gamma_1$, we define $F_\alpha(a,b) = \theta(a\alpha b) - \theta(a)k(\alpha)\theta(b)$.

Lemma 2.4 If θ is a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*, then for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma_1$:

(i) $F_\alpha(a,b) + F_\alpha(b,a) = 0$; (ii) $F_\alpha(a+b,c) = F_\alpha(a,c) + F_\alpha(b,c)$;

(iii) $F_\alpha(a,b+c) = F_\alpha(a,b) + F_\alpha(a,c)$; (iv) $F_{\alpha+\beta}(a,b) = F_\alpha(a,b) + F_\beta(a,b)$.

Proof. Clear.

Note that θ is a *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N* if and only if $F_\alpha(a,b) = 0$ for all $a,b \in M$ and $\alpha \in \Gamma_1$.

Definition 2.2 Let θ be a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*. Then for $a,b \in M$ and $\alpha \in \Gamma_1$, we define $G_\alpha(a,b) = \theta(a\alpha b) - \theta(b)k(\alpha)\theta(a)$.

Lemma 2.5 If θ is a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N*, then for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma_1$:

(i) $G_\alpha(a,b) + G_\alpha(b,a) = 0$; (ii) $G_\alpha(a+b,c) = G_\alpha(a,c) + G_\alpha(b,c)$;

(iii) $G_\alpha(a,b+c) = G_\alpha(a,b) + G_\alpha(a,c)$; (iv) $G_{\alpha+\beta}(a,b) = G_\alpha(a,b) + G_\beta(a,b)$.

Proof. Obvious.

Note that θ is an anti-*k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N* if and only if $G_\alpha(a,b) = 0$ for all $a,b \in M$ and $\alpha \in \Gamma_1$.

Lemma 2.6 Let θ be a Jordan *k*-homomorphism of a Γ_1-ring *M* into a 2-torsion free Γ_2-ring *N* and $a,b \in M$, $\alpha,\gamma \in \Gamma_1$. Then $F_\alpha(a,b)k(\gamma)G_\alpha(a,b) + G_\alpha(a,b)k(\gamma)F_\alpha(a,b) = 0$.

Proof. We put $X = a\alpha b\gamma b\alpha a + b\alpha a\gamma a\alpha b$ and look what happens here. We determine $\theta(X) = \theta(a\alpha(b\gamma b)\alpha a) + \theta(b\alpha(a\gamma a)\alpha b)$ and $\theta(X) = \theta((a\alpha b)\gamma(b\alpha a) + (b\alpha a)\gamma(a\alpha b))$ by using Lemma 2.1, separately. Upon equating them, cancel the like terms from both sides of the equality and then rearrange the remaining terms using Lemma 2.4(i) and Lemma 2.5(i) to obtain the proof.

Lemma 2.7 Let *N* be a 2-torsion free completely prime Γ_2-ring and $x,y \in N$ such that $x\delta y + y\delta x = 0$ for all $\delta \in \Gamma_2$. Then $x\delta y = y\delta x = 0$.

Proof. By repeated application of the hypothesis, we have

$$(x\delta y)\delta(x\delta y) = -(y\delta x)\delta(x\delta y) = -(y(\delta x\delta)x)\delta y) = (x(\delta x\delta)y)\delta y)$$
$$= x\delta(x\delta y)\delta y = -x\delta(y\delta x)\delta y = -(x\delta y)\delta(x\delta y).$$

This implies, $2(x\delta y)\delta(x\delta y) = 0$. Since N is 2-torsion free, we get $(x\delta y)\delta(x\delta y) = 0$. By the completely primeness of N, we obtain $x\delta y = 0$. Hence we get, $x\delta y = y\delta x = 0$ for all $x, y \in N$ and $\delta \in \Gamma_2$.

Corollary 2.1 Let θ be a Jordan k-homomorphism of a Γ_1-ring M into a 2-torsion free completely prime Γ_2-ring N, and let $a, b \in M$ and $\alpha, \gamma \in \Gamma_1$. Then

$$F_\alpha(a,b)k(\gamma)G_\alpha(a,b) = 0 = G_\alpha(a,b)k(\gamma)F_\alpha(a,b).$$

Proof. Using Lemma 2.7 in the result of Lemma 2.6, we obtain the proof.

Lemma 2.8 Let $A_1, A_2, ..., A_n$ be additive groups and let N be a completely prime Γ_2-ring. Suppose the maps $f : A_1 \times A_2 \times ... \times A_n \to N$ and $g : A_1 \times A_2 \times ... \times A_n \to N$ are additive in each argument. If $f(a_1, a_2, ..., a_n)\beta g(a_1, a_2, ..., a_n) = 0$ for all $a_i \in A_i$ with $i = 1,2,...,n$ and $\beta \in \Gamma_2$, then $f(a_1, a_2, ..., a_n)\beta g(b_1, b_2, ..., b_n) = 0$ for all $a_i, b_i \in A_i$ with $i = 1,2,...,n$ and $\beta \in \Gamma_2$.

Proof. It suffices to prove the case for $n = 1$. In this case, the mappings are $f : A_1 \to N$ and $g : A_1 \to N$ such that $f(a_1)\beta g(a_1) = 0$ for all $a_1 \in A_1$ and $\beta \in \Gamma_2$. Then $f(a_1 + b_1)\beta g(a_1 + b_1) = 0$ for all $a_1, b_1 \in A_1$ and $\beta \in \Gamma_2$. This implies,

$$f(a_1)\beta g(a_1) + f(a_1)\beta g(b_1) + f(b_1)\beta g(a_1) + f(b_1)\beta g(b_1) = 0,$$

and therefore, $f(a_1)\beta g(b_1) + f(b_1)\beta g(a_1) = 0$. Then we have

$$(f(a_1)\beta g(b_1))\beta(f(a_1)\beta g(b_1)) = -f(a_1)\beta g(b_1)\beta f(b_1)\beta g(a_1) = 0.$$

Hence, $f(a_1)\beta g(b_1) = 0$ (by the completely primeness of N), and it finishes the proof.

Corollary 2.2 If θ is a Jordan k-homomorphism of a Γ_1-ring M into a 2-torsion free completely prime Γ_2-ring N, then $F_\alpha(a,b)k(\gamma)G_\alpha(c,d) = 0$ for all $a, b, c, d \in M$ and $\alpha, \gamma \in \Gamma_1$.

Proof. Following Definition 2.1, Lemma 2.4, Definition 2.2 and Lemma 2.5, we see that the mappings $(a,b) \to F_\alpha(a,b)$ and $(a,b) \to G_\alpha(a,b)$ are additive in each argument. Hence, from Corollary 2.1, Lemma 2.8 gives this result.

We are now *ready* to prove our main claim as follows:

Theorem 2.1 Every Jordan k-homomorphism θ of a Γ_1-ring M onto a 2-torsion free completely prime Γ_2-ring N such that $k(\Gamma_1) = \Gamma_2$ is either a k-homomorphism or an anti-k-homomorphism.

Proof. Let θ be a Jordan *k*-homomorphism of a Γ_1-ring M onto a 2-torsion free completely prime Γ_2-ring *N* such that $k(\Gamma_1) = \Gamma_2$. Then, from Corollary 2.2, we have

$$F_\alpha(a,b)k(\gamma)G_\alpha(c,d) = 0 \text{ for all } a,b,c,d \in M \text{ and } \alpha,\gamma \in \Gamma_1.$$

Here, (i) $F_\alpha(a,b) \in N$ for all $a,b \in M$ and $\alpha,\gamma \in \Gamma_1$, (ii) $k(\gamma) \in \Gamma_2$ for all $\gamma \in \Gamma_1$, and (iii) $G_\alpha(c,d) \in N$ for all $c,d \in M$ and $\alpha \in \Gamma_1$. Since *N* is a completely prime Γ_2-ring and $k(\Gamma_1) = \Gamma_2$, therefore, we conclude that either $F_\alpha(a,b) = 0$ for all $a,b \in M$ and $\alpha,\gamma \in \Gamma_1$ or $G_\alpha(c,d) = 0$ for all $c,d \in M$ and $\alpha \in \Gamma_1$, which implies (respectively) that either θ is a *k*-homomorphism or θ is an anti-*k*-homomorphism (by definition). This completes the proof of the theorem.

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