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A STANDARD MODULO PRIESTLEY TOPOLOGICAL QUASI-VARIETY

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ABSTRACT

We construct a topological quasi-variety which is not standard but standard modulo Priestley.

Key words: Standard topological quasi-variety, Boolean model, standard modulo Priestley.

1. Introduction

A structure $\widetilde{M} =< M; G, H, R, T >$ is said to be a **topological structure** if $< M; T >$ is a topological space, G is a set of finitary total operations, H is a set of finitary partial operations and R is a set of finitary relations on M. If the topological space $< M; T >$ is compact, then we say that $\widetilde{M}$ is **compact topological structure**. A **topological quasi-variety** generated by a finite topological structure $\widetilde{M} =< M; G, H, R, T >$ with discrete topology T is the class $IS_c P^+(\widetilde{M})$ of isomorphic copies of closed substructure of non-empty direct power of $\widetilde{M}$. For convenience of notation we will write $Q_T(\widetilde{M})$ for $IS_c P^+(\widetilde{M})$.

1.1 Definition of standardness

Let $\widetilde{M}$ be a topological structure. We say that a topological structure $\widetilde{X} =< X; G^X, H^X, R^X, T^X >$ is the **same type** as $\widetilde{M}$ if

(i) $< X; T^X >$ is a topological space,

(ii) for each $g \in G \cup H \cup R$, there is a corresponding $g^X \in G^X \cup H^X \cup R^X$ of the same arity as that of g.

A topological structure $\widetilde{X} =< X; G^X, H^X, R^X, T^X >$ is said to be a **Boolean structure** (or **Boolean model**) of type $< G, H, R >$ if

(i) $< X; T^X >$ is a Boolean space (that is, compact totally disconnected space),

(ii) for each n-ary $h \in G \cup H$, the set $dom(h^X)$ is a closed subset of X^n and h^X: $dom(h^X) \to X$ is continuous, and

(iii) if $r \in R$ is n-ary, then r^X is a closed subset of X^n.

We will omit the superscripts on G^X, H^X, R^X, T^X where there is no danger of ambiguity.

An **atomic formula** of type $< G, H, R >$ is an expression of either of the forms

$$t_1 \approx t_2 \text{ or } r(t_1, t_2, \cdots, t_n)$$

where $t_1, t_2, \cdots, t_n$ are terms built from the function symbols in $G \cup H$ and $r \in R$ is an n-ary relation symbol. A **quasi-atomic formula** of type $< G, H, R >$ is an expression of one of the forms

$$\alpha \;\; or \;\; \underset{i \in I}{\vee} \neg \beta_i \;\; or \;\; \underset{i \in I}{\wedge} \beta_i \Rightarrow \alpha$$

where α and each β_i are atomic formulæ and I is a finite set.

Let Σ be a set of quasi-atomic formulæ. We denote by $Mod_T(\Sigma)$ the class of all Boolean structures which satisfy each quasi-atomic formula in Σ. The collection of all quasi-atomic formulæ that hold in $\widetilde{M}$ forms the **quasi-atomic theory** of $\widetilde{M}$ which is denoted by $Th_{qa}(\widetilde{M})$.

We say that $Q_T(\widetilde{M})$ is a **standard topological quasi-variety**, or that $\widetilde{M}$ is standard, if $Q_T(\widetilde{M})$ is exactly the class of all Boolean models of the quasi-atomic theory of $\widetilde{M}$, in symbols,

$$Q_T(\widetilde{M}) = Mod_T(Th_{qa}(\widetilde{M}))$$

We say that a subset $\Sigma \subseteq Th_{qa}(\widetilde{M})$ axiomatizes $Q_T(\widetilde{M})$ provided that $Q_T(\widetilde{M}) = Mod_T(\Sigma)$.

If $Q_T(\widetilde{M})$ is axiomatizable, then it is certainly standard, and the axioms provide a description of its members.

Most of the cases $Q_T(\widetilde{M})$ are not standard, but we can often describe them by assuming that the underlying ordered set is a Priestley space (that is, compact totally order-disconnected space). This is the reason that we are interested in studying standardness modulo Priestley.

1.2 Definition of Standard modulo Priestley

Let $\widetilde{M} =< M; G, H, \leq, T >$ be a finite topological ordered partial algebra. A topological structure $\widetilde{X} =< X; G^X, H^X, \leq^X, T^X >$ of the same type $< G, H, \leq >$ as $\widetilde{M}$ is said to be a **Priestley structure** (or **Priestley model**) of type $< G, H >$ if

(i) $\widetilde{X} =< X; \leq^X, T^X >$ is a Priestley space,

(ii) If $g \in G \cup H$ is an n-ary operation, then $dom(g^X)$ is a closed subset of X^n and g^X: $dom(g^X) \to X$ is continuous.

Let Σ be a set of quasi-atomic formulæ satisfied by $\underset{\sim}{M}$. We denote the class of all Priestley models which satisfy each quasi-atomic formula in Σ by $Mod_P(\Sigma)$. The class of all Priestley models of the quasi-atomic theory of $\underset{\sim}{M}$ is denoted by $Mod_P(Th_{qa}(\underset{\sim}{M}))$. We say that $Q_T(\underset{\sim}{M})$ is **standard modulo Priestley**, or that $\underset{\sim}{M}$ is **standard modulo Priestley**, if $Q_T(\underset{\sim}{M})$ is exactly the class of all Priestley models of the quasi-atomic theory of $\underset{\sim}{M}$. In symbols,

$$Q_T(\underset{\sim}{M}) = Mod_P(Th_{qa}(\underset{\sim}{M}))$$

Throughout this paper, if $< X;T >$ is a finite topological space, then it will be assumed that T is the discrete topology.

A natural question is: *Which finite topological ordered partial algebras $\underset{\sim}{M}$ generate a topological quasi-variety that is standard modulo Priestley?* By the definition, the category of the Priestley space is standard modulo Priestley. By [6, Theorem 4.2], every finite Boolean unar is standard. Every finite anti-chain is term equivalent to a finite Boolean unar and hence every finite anti-chain is standard. Begum [3] proved that all two element topological ordered unars are standard modulo Priestley. She also proved that all three and four-element topological chain with an order-preserving operation is standard modulo Priestley.

In Section 2, we give a detail background which we need in this paper. In Section 3, we consider a quasi-variety generated by a four element Boolean ordered unar which is neither a chain nor an anti-chain. We show that the quasi-variety is not standard but standard modulo Priestley.

2. Preliminaries

The notion of standardness was first introduced in [6]. The standardness problem arises from the question: Which structures are in and what do they look like? The Preservation Theorem gives us a set of axioms which satisfies each member of $IS_cP^+(\underset{\sim}{M})$.

Theorem 2.1 (The Preservation Theorem 1.4.3 [5]) *Let $\underset{\sim}{M}$ be a finite topological structure and let $X \in IS_cP^+(\underset{\sim}{M})$. Then* **X** *is a Boolean structure which satisfies every quasi-atomic formula that is satisfied by $\underset{\sim}{M}$.*

The following Lemma is compiled from [7, Lemma 11.2 and Exercise 11.14].

Lemma 2.2 *Let $\underset{\sim}{X} =< X;\leq,T >$ be a Priestley space.*

(i) let Y be a close downset in X and let $x \notin Y$. Then there exists a clopen downset U such that $Y \subseteq U$ and $x \notin U$.

(ii) ${}^{\downarrow}y$ and ${}^{\uparrow}y$ are closed for each $y \in X$.

(iii) If $Y \subseteq X$ is closed in X, then ${}^{\downarrow}Y$ and ${}^{\uparrow}Y$ are *closed in X.*

To prove a topological structure $\widetilde{M} = < M; g, \leq, T >$ is non-standard, we will often use the following lemma which is due to [3].

Lemma 2.3 *Let* $\widetilde{M} = < M; G, \leq, T >$ *be a finite topological ordered unary algebra and let* $\widetilde{X} = < X; G, \leq, T >$. *Then the following are equivalent:*

(i) $\widetilde{X} \in Q_T(\widetilde{M})$,

(ii) $\widetilde{X}$ *is a Boolean ordered unary algebra* (*that is,* $< X; T >$ *is a Boolean space and for all* $g \in G$ *the map* $g : X \to X$ *is a continuous and* $\leq$ *is a closed order relation*) *such that*

(Sep) *for all* $x, y \in X$ *with* $x \not\leq y$, *there exist a continuous g-preserving and* $\leq$-*preserving map* $\alpha : X \to M$ *with* $\alpha(x) \not\leq \alpha(y)$.

(iii) each $g \in G$ *is a unary map on X, the binary relation* $\leq$ *on X is an anti-symmetric and* (Sep) *holds.*

Note that this lemma shows that provided we can establish (Sep) then we do not need to prove that the maps $g : X \to X$ are continuous! We may now use these tools to show that specific Boolean ordered unars are non-standard.

Let $\widetilde{M} = < M; g, \leq, T >$ be a finite topological ordered unar. Then we say that $\widetilde{M}$ is **non-standard via Priestley** if there exists a Boolean structure $\widetilde{X} = < X; g, \leq, T >$ such that

(a) $\widetilde{X}$ is locally finite,

(b) every finite substructure of $\widetilde{X}$ is in $Q_T(\widetilde{M})$, and

(c) $< X; \leq, T >$ is not a Priestley space.

Since every underlying ordered space of every member of $Q_T(\widetilde{M})$ is a Priestley space, it follows that non-standard via Priestley implies non-standard. Indeed, one of the standard ways to prove that a finite topological ordered unar is non-standard is to prove that it is non-standard via Priestley.

Proving that $\widetilde{M}$ is non-standard via Priestley has the added advantage that it is an *inherent* property, that is, it goes up to larger quasi-varieties.

Lemma 2.4 ([1]) *Let* $\widetilde{M} = < M; g, \leq, T >$ *and* $\widetilde{N} = < N; g, \leq, T >$ *be finite topological ordered unars. If* $\widetilde{M}$ *is non-standard via Priestley and* $\widetilde{M} \in Q_T(\widetilde{N})$, *then* $\widetilde{N}$ *is also non-standard via Priestley.*

Proof. Let $\widetilde{X}$ be an example that shows that $\widetilde{M}$ is non-standard via Priestley. Since $\widetilde{M} \in Q_T(\widetilde{N})$ implies that $Q_T(\widetilde{M}) \subseteq Q_T(\widetilde{N})$ it follows immediately that $\widetilde{X}$ also shows that $\widetilde{N}$ is non-standard via Priestley. □

Corollary 2.5 ([1]). *Let $\widetilde{M}$ and $\widetilde{N}$ be finite topological ordered unars. If $\widetilde{M}$ is non-standard via Priestley and $\widetilde{M}$ is a substructure of $\widetilde{N}$, then $\widetilde{N}$ is also non-standard via Priestley.*

Lemma 2.6 *Let $\widetilde{M} =< M; g, \leq, T >$ be a finite topological ordered unary algebra. Let Σ be a set of axioms satisfied by $\widetilde{M}$. If $\widetilde{X} \in Q_T(\widetilde{M})$, then $\widetilde{X} \in Mod_T(\Sigma)$ and $< X; \leq, T >$ is a Priestley space.*

Proof. Let $\widetilde{X} \in Q_T(\widetilde{M})$. Then by the Preservation Theorem 2.1, $\widetilde{X} \in Mod_T(\Sigma)$. To prove $< X; \leq, T >$ is a Priestley space, let $x, y \in X$ with $x \not\leq y$. Since $\widetilde{X} \in Q_T(\widetilde{M})$, by the Lemma 2.3, there exists a morphism $\alpha : \widetilde{X} \to \widetilde{M}$ such that $\alpha(x) \not\leq \alpha(y)$. The clopen downset $\alpha^{-1}({\downarrow}\alpha(y))$ in X contains y but not x. Hence $< X; \leq, T >$ is a Priestley space. □

Because of the above Lemma we are guaranteed, for topological unary algebras, that if $\widetilde{X} \in Q_T(\widetilde{M})$, then $\widetilde{X} \in Mod_P(Th_{qa}(\widetilde{M}))$ so that to show $Q_T(\widetilde{M}) = Mod_P(Th_{qa}(\widetilde{M}))$ we need only

- write down a set Σ of axioms that is satisfied by $\widetilde{M}$, and
- show that if $\widetilde{X} =< X; G, \leq, T >$ is a Boolean model of Σ such that $< X; \leq, T >$ is Priestley space, then condition (Sep) of Lemma 2.3 holds.

This is the "standard method" to show that a finite topological ordered unary algebra $\widetilde{M} =< M; G, \leq, T >$ is standard modulo Priestley.

3. A non-standard structure which are Standard modulo priestley

In this section we consider the topological quasi-varieties generated by a four-element Boolean ordered unar M_4 (see Figure 2) which is neither a chain nor an anti-chain. We show that it is non-standard but standard modulo Priestley.

If we consider the structure M_2 given in Figure 1 where line indicate the relation and arrow line indicate the unary operation, then by [1, 2], we have M_2 is non-standard.

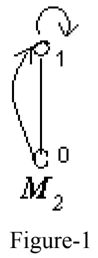

Figure-1

Now we construct our main topological quasi-variety. Consider the topological structure $\widetilde{M}_4 =< \{0, a, b, 1\}; g, \leq, T >$ given in the following Figure 2.

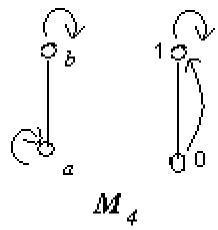

Figure 2: Four-element topological ordered unars

Observe that each $\widetilde{X} \in Q_T(\widetilde{M}_4)$ satisfies the following set Σ_{M_4} of axioms:

(i) $\leq$ is an order relation (i.e., reflexive, symmetry and transitive),

(ii) $x \leq y \Rightarrow g(x) \leq g(y)$

(iii) $g(x) = g^2(x)$

(iv) $g(x) \leq g(y)$ and $g(y) = y \Rightarrow x \leq y$

(v) $g(x) = x$ and $x \leq y \Rightarrow g(y) = y$

Theorem 3.1 M_4 *is non-standard.*

Proof. Observe that M_2 is a substructure of M_4. By [1] M_2 is non-standard via Priestley. Then by Corollary 2.5, M_4 is non-standard via Priestley and hence non-standard. □

Theorem 3.2 $Q_T(\widetilde{M}_4) = Mod_P(\Sigma_{M_4})$ *and hence* $Q_T(\widetilde{M}_4)$ *is standard modulo Priestley.*

Proof. Let $\widetilde{X} \in Mod_T(\Sigma_{M_4})$ and $< X;\leq,T >$ be a Priestley space. In order to apply Lemma 2.3, let $x, y \in X$ with $x \not\leq y$.

Case-1 $g(x) \not\leq g(y)$. Since $< X;\leq,T >$ is a Priestley space, there exists a clopen upset U containing $g(x)$ but not $g(y)$. Thus $g^{-1}(U)$ contains x but not y. Define a map $\alpha : X \to M$ by

$$\alpha(z) = \begin{cases} b \text{ if } z \in g^{-1}(U) \\ a \text{ if } z \in X \setminus g^{-1}(U). \end{cases}$$

Since U is an upset, by (i), we have $g^{-1}(U)$ is an upset and hence $X \setminus g^{-1}(U)$ is a downset. Therefore, α preserves $\leq$. By (ii), we have $g(g^{-1}(U)) \subseteq g^{-1}(U)$ and $g(X \setminus g^{-1}(U)) \subseteq X \setminus g^{-1}(U)$. Hence α preserves g. Moreover, $\alpha(x) = b \not\leq a = \alpha(y)$ and therefore α is the required separating morphism.

Case-2 $g(x) \leq g(y)$. By (iv), we have $g(y) \neq y$. Assume $g(x) = x$. Let $U := fix(g)$ be the set of fixed points of X. Since g is continuous, U is closed. Using (v) we can prove that U

is an upset. Hence, by Lemma 2.2, there is a clopen upset *V* containing *U* but not *y*. Define a map $\alpha : X \to M$ by

$$\alpha(z) = \begin{cases} 1 \ \text{if } z \in V \\ 0 \ \text{if } z \in X \setminus V. \end{cases}$$

Here *V* is an upset and $X \setminus V$ is a down set whence *g* preserves $\leq$. Now for all $u \in X$ we have, by (ii), $g(u) \in U$ and hence $g(u) \in V$. Therefore, $g(V) \subseteq V$ and $g(X \setminus V) \subseteq V$. Thus, α preserves *g*. Moreover $\alpha(x) = 1 \not\leq \alpha(y) = 0$ and so α is the required separating morphism.

Again assume $g(x) \neq x$. and again let $U := fix(g)$. As above, *U* is a closed upset. Define $V = U \cup {\uparrow}x$ By Lemma 2.2, we have ${\uparrow}x$ is closed. Hence *V* is a closed upset containing *x* but not *y*. Hence, by Lemma 2.2, there exists a clopen upset *W* containing *V* but not *y*.

Define a map $\alpha : X \to M$ by

$$\alpha(z) = \begin{cases} 1 \ \text{if } z \in W \\ 0 \ \text{if } z \in X \setminus W. \end{cases}$$

Since for all $u \in X$, we have $g(u) \in U \subseteq V \subseteq W$. Therefore, α is the required separating morphism of *x* and *y* with $\alpha(x) = 1 \not\leq \alpha(y) = 0$. Hence M_4 is standard modulo Priestley.□

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