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GOL'DBERG ORDER AND GOL'DBERG TYPE OF ENTIRE FUNCTIONS REPRESENTED BY MULTIPLE DIRICHLET SERIES

Md. Feruj Alam

Department of Mathematics, Rajshahi University of Engineering & Technology, Rajshahi

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ABSTRACT

We consider the Hadamard product of the class of entire multiple Dirichlet series in several complex variables having the same sequence of exponents. Our object is to study the nature of Gol'dberg order and Gol'dberg type of these functions.

1. Notations

The n-tuples $(\sigma_1,\cdots,\sigma_n),(m_1,\cdots,m_n),\ (s_1,\cdots,s_n)$ etc. of C^n or R^n will be denoted by their corresponding unsuffixed symbol $\sigma,\ m,\ s$ etc. respectively. By I^n we shall mean the Cartesian product of n copies of I where I is the set of non-negative integers.

For $s,w \in C^n$ and $\alpha \in C$ where,

$s=(s_1,\cdots,s_n)\quad w=(w_1,\cdots,w_n)$

we define

(i) $s+w=(s_1+w_1,\cdots,s_n+w_n)$

(ii) $\alpha s=(\alpha s_1,\cdots,\alpha s_n)$

(iii) $s.w=s_1w_1+\cdots+s_nw_n$

For $a \in R,\quad s \in C^n$

(iv) $s+a=(s_1+a,\cdots,s_n+a)$

The positive hyperoctant R_+^n in R^n will be denoted by

$$R_+^n=\{x : x \in R^n,\quad x_i \geq 0,\ j=1,\cdots,n\}$$

For $t \in R_{+}^{n}$, we set $\|t\| = t_1 + \cdots + t_n$.

For $k \in R$, $\bar{k}$ will denote the real n-tuple $(k, \cdots, k)$. For an entire function *f* with domain C^n, f^k will denote the function $\dfrac{\partial^{\|k\|} f}{\partial_{s_1}^{k_1} \cdots \partial_{s_n}^{k_n}}$, where $k \in I^n$ and $f^{(\bar{0})} = f$.

We denote the n-tuple $(\lambda_{1m_1}, \cdots, \lambda_{nm_n})$ by $\lambda_{n,m}$

Thus, $s.\lambda_{n,m} = s_1\lambda_{1m_1} + \cdots + s_n\lambda_{nm_n}$.

2. Introduction

We consider the multiple Dirichlet series

$$f(s_1, \cdots, s_n) = \sum_{m_1, \cdots, m_n = 1}^{\infty} a_{m_1 \cdots m_n} \exp\{s_1\lambda_{1m_1} + \cdots + s_n\lambda_{nm_n}\}, \quad \text{that is,}$$

$$f(s) = \sum_{m=\bar{1}}^{\infty} a_m \exp\{s.\lambda_{n,m}\}. \tag{2.1}$$

$(s_j = \sigma_j + i\tau_j \in C,\ j = 1, \cdots, n)$, $a_m \in C$, and $\{\lambda_{jm_j}\}_{m_j=1}^{\infty}$, $j = 1, \cdots, n$ are *n* sequences of exponents satisfying the conditions

$$0 < \lambda_{j1} < \lambda_{j2} < \cdots < \lambda_{jk} \to \infty \text{ as } k \to \infty, \quad \text{for } j = 1, \cdots, n. \tag{2.2}$$

Throughout we assume that $\displaystyle\lim_{m_j \to \infty} \frac{\log m_j}{\lambda_j m_j} = 0, \quad j = 1, \cdots, n$ (2.3)

If (2.3) holds then the domain of convergence of the series (2.1) coincides with its domain of absolute convergence [1].

All the multiple Dirichlet series of the form (2.1) having the same sequence of exponents $\{\lambda_{jm_j}\}_{m_j=1}^{\infty}$, $j = 1, \cdots, n.$ satisfying (2.2) are absolutely and uniformly convergent in C^n and hence are entire functions.

For the entire functions f and g, we define Hadamard product [2] $f * g$ by

$$f(s) * g(s) = \sum_{m=\bar{1}}^{\infty} a_m b_m \exp\{s.\lambda_{n,m}\} \tag{2.4}$$

where $f(s)=\sum_{m=\bar{1}}^{\infty} a_m \exp\{s.\lambda_{n,m}\}$ and $g(s)=\sum_{m=\bar{1}}^{\infty} b_m \exp\{s.\lambda_{n.m}\}$

For $k \in I^n$, we define

$$f^k(s)=\sum_{m=\bar{1}}^{\infty} \lambda_{n,m}^{k} a_m \exp\{s.\lambda_{n,m}\} \tag{2.5}$$

$$f^k(s) * g^k(s)=\sum_{m=\bar{1}}^{\infty} \lambda_{n,m}^{2k} a_m b_m \exp\{s.\lambda_{n,m}\} \tag{2.6}$$

Definitions : We define the poly half plane D_l as $D_l=\{S : S \in C^n, \operatorname{Re} s=\sigma << l\}$, where $l \in R^n$. These type of domains are called the fundamental domains. The region $D_l + r$, depending on the parameter $r \in R$, is defined as $D_l + r=\{s+r, s \in D_l\}$. We simply write D instead of D_l .Then for the entire function *f*, given by (2.1) , we define the maximum modulus $M_{f,D}(r)$ with respect to the region D, where $r \in R$ as

$$M_{f,D}(r)=\sup\{\,|\,f(s)\,| : s \in D+r\}.$$

Let f be an entire function and D be a fundamental domain. Also, let S_f be the set of points $\alpha \in R$ such that for every $\alpha \in S_f$, there corresponding an $r_0 \in R$ such that

$$\log M_{f,D}(r) \le e^{r\alpha}, \text{ for } r \ge r_0 .$$

The infimum of the set S_f is called the Gol'dberg order $\rho(D)$ of f with respect to the region D. We say that f is of infinite or finite Gol'dberg order according as S_f is empty or non-empty.

Next, for the Gol'dberg order $\rho(D) > 0$, let $K_f(\rho)$ be the set of all $K \in R$ such that

$$\log M_{f,D}(r) \le K e^{r\alpha}, \text{ for } r \ge r_0 .$$

The infimum of the set $K_f(\rho)$ is called the Gol'dberg type $T(D)$ of f corresponding to $\rho(D)$. As before, we say that f is of infinite or finite Gol'dberg type according as $K_f(\rho)$ is empty or non-empty. We shall call Gol'dberg order and Gol'dberg type simply as G-order and G-type respectively.

From the definition it follows easily that

$$\rho(D) = \lim_{r\to\infty}\sup \frac{\log\log M_{f,D}(r)}{r} \tag{2.7}$$

$$\underset{k}{\rho}(D) = \lim_{r\to\infty}\sup \frac{\log\log M_{f^k,D}(r)}{r} \tag{2.8}$$

$$T(D) = \lim_{r\to\infty}\sup \frac{\log M_{f,D}(r)}{e^{r\rho(D)}}, \quad if\ \rho(D) > 0 \tag{2.9}$$

$$T_k(D) = \lim_{r\to\infty}\sup \frac{\log M_{f^k,D}(r)}{e^{r\rho_k(D)}}, \quad if\ \underset{k}{\rho}(D) > 0 \tag{2.10}$$

We know that the G-order $\rho(D)$ does not dependent on the choice of the domain D while G-type $T(D)$ does [3]. Here we may write ρ instead of $\rho(D)$. It is also know [3] that

$$\rho = \lim_{m\to\overline{\infty}}\sup \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{-\log|a_m|} \tag{2.11}$$

$$\underset{k}{\rho} = \lim_{m\to\overline{\infty}}\sup \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{-\log|a_m\lambda^k_{n,m}|} \tag{2.12}$$

Theorem 1: The function $f^k(s) * g^k(s)$, as defined by (2.6) is an entire function.

Proof : Since $f(s)\ and\ g(s)$ are entire functions so $f^k(s)\ and\ g^k(s)$ are also entire functions. Now $f^k(s) = \sum_{m=\overline{1}}^{\infty} \lambda^k_{n,m} a_m \exp\{s.\lambda_{n,m}\}$ [defined by (2.5)]. But the series $\sum_{m=\overline{1}}^{\infty} |\lambda^k_{n,m} a_m| \exp\{\sigma.\lambda_{n,m}\}$ is convergent for all $\sigma \in R^n$. In particular, it is convergent at $\sigma = \overline{0}$, so that $\sum_{m=\overline{1}}^{\infty} |\lambda^k_{n,m} a_m|$ is convergent. Thus $\lim_{m\to\overline{\infty}} |\lambda^k_{n,m} a_m| = 0$ and hence the n-sequence $\{|\lambda^k_{n,m} a_m|\}$ is bounded. Also, the series

$$\sum_{m=\overline{1}}^{\infty} |\lambda^k_{n,m} b_m| \exp\{\sigma.\lambda_{n,m}\}$$

is convergent for all $\sigma \in R^n$ and consequently,

$$\sum_{m=\overline{1}}^{\infty} \lambda^{2k}_{n,m} a_m b_m \exp\{\sigma.\lambda_{n,m}\}$$

is convergent for all $\sigma \in R^n$. This implies that

$$\sum_{m=1}^{\infty} \lambda_{n,m}^{2k} a_m b_m \exp\{s.\lambda_{n,m}\}$$

is absolutely convergent for all $s \in C^n$

Hence $f^k(s) * g^k(s)$ defined by (2.6) represents an entire function.

Theorem 1.1.2: Let f and g be entire functions where $f^k(s) = \sum_{m=1}^{\infty} \lambda_{n,m}^k a_m \exp\{s.\lambda_{n,m}\}$ and $g^k(s) = \sum_{m=1}^{\infty} \lambda_{n,m}^k b_m \exp\{s.\lambda_{n,m}\}$ having G-order ρ_{k_f} $(0 < \rho_{k_f} < \infty)$ and ρ_{k_g} $(0 < \rho_{k_g} < \infty)$ respectively. Then $f^k(s) * g^k(s)$ is an entire function with G-order ρ_k such that $\rho_k \le (\rho_{k_f} \rho_{k_g})^{1/2}$ provided $\log \frac{1}{|\lambda_{n,m}^{2k} a_m b_m|} \sim \left\{ \log \frac{1}{|\lambda_{n,m}^k a_m|} \log \frac{1}{|\lambda_{n,m}^k b_m|} \right\}^{1/2}$

Proof: $f^k(s) * g^k(s)$ is an entire function by theorem 1. Now by (2.12)

$$\frac{1}{\rho_{k_f}} = \liminf_{m \to \infty} \frac{-\log|\lambda_{n,m}^k a_m|}{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|} \quad \text{and}$$

$$\frac{1}{\rho_{k_g}} = \liminf_{m \to \infty} \frac{-\log|\lambda_{n,m}^k b_m|}{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}$$

For arbitrary $\varepsilon > 0$, it follows that

$$\frac{1}{\rho_{k_f}} - \varepsilon/2 < \frac{\log \frac{1}{|\lambda_{n,m}^k a_m|}}{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|} \quad \text{and}$$

$$\frac{1}{\rho_{k_g}} - \varepsilon/2 < \frac{\log \frac{1}{|\lambda_{n,m}^k b_m|}}{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}$$

Now we can write

$$\frac{\log \frac{1}{|\lambda_{n,m}^k a_m|} \log \frac{1}{|\lambda_{n,m}^k b_m|}}{\left(\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|\right)^2} > \left(\frac{1}{\rho_{k_f}} - \varepsilon/2\right)\left(\frac{1}{\rho_{k_g}} - \varepsilon/2\right)$$

or, $$\frac{\left\{\log\frac{1}{|\lambda_{n,m}^{k}a_m}\log\frac{1}{|\lambda_{n,m}^{k}b_m|}\right\}^{1/2}}{\|\lambda_{n,m}\|\log\|\lambda_{n,m}\|} > \left\{\left(\frac{1}{\rho_{k_f}}-\varepsilon/2\right)\left(\frac{1}{\rho_{k_g}}-\varepsilon/2\right)\right\}^{1/2}$$

for sufficiently large ||m||.

Now if $$\log\frac{1}{|\lambda_{n,m}^{2k}a_m b_m|} \sim \left\{\log\frac{1}{|\lambda_{n,m}^{k}a_m|}\log\frac{1}{|\lambda_{n,m}^{k}b_m|}\right\}^{1/2}$$

then $$\frac{\log\frac{1}{|\lambda_{n,m}^{2k}a_m b_m|}}{\|\lambda_{n,m}\|\log\|\lambda_{n,m}\|} > \left\{\left(\frac{1}{\rho_{k_f}}-\varepsilon/2\right)\left(\frac{1}{\rho_{k_g}}-\varepsilon/2\right)\right\}^{\frac{1}{2}}$$

Therefore $$\limsup_{m\to\infty}\frac{\log\frac{1}{|\lambda_{n,m}^{2k}a_m b_m|}}{\|\lambda_{n,m}\|\log\|\lambda_{n,m}\|} \geq \left(\frac{1}{\rho_{k_f}\rho_{k_g}}\right)^{\frac{1}{2}}$$

Thus $$\frac{1}{\rho_k} \geq \left(\frac{1}{\rho_{k_f}\rho_{k_g}}\right)^{\frac{1}{2}}$$

Hence $\rho_k \leq \left(\rho_{k_f}\rho_{k_g}\right)^{1/2}$.

Theorem 3: Let f^k and g^k be entire function of G-order $\rho_{k_f}\,(0<\rho_{k_f}<\infty)$ and $\rho_{k_g}\,(0<\rho_{k_g}<\infty)$ and finite G-type $T_{k_f}(D)$ and $T_{k_g}(D)$ respectively having the same fundamental domain D. If $f^k * g^k$ is of G-order $\rho_k\left(0<\rho_k<\infty\right)$, where $\log M_{f^k*g^k}(r) \sim \log M_{f^k,D}(r)\log M_{g^k,D}(r)$, then $\rho_k \leq \rho_{k_f}+\rho_{k_g}$.

Also if $T_k(D)$ be the G-type of $f^k * g^k$ with respect to the domain D then $T_k(D) \leq T_{k_f}(D)T_{k_g}(D)$ provided the sign of equality holds in $\rho_k \leq \rho_{k_f}+\rho_{k_g}$.

Proof: From (2.8), we have

$$\rho_{k_f}(D) = \limsup_{r\to\infty}\frac{\log\log M_{f^k,D}(r)}{r} \quad\text{and}\quad \rho_{k_g}(D) = \limsup_{r\to\infty}\frac{\log\log M_{g^k,D}(r)}{r}$$

Hence for an arbitrary $\varepsilon > 0$,

$$\log M_{f^k,D}(r) < \exp\{r(\rho_{k_f} + \varepsilon/2)\}, \quad \text{for } r > r_0 \text{ and}$$

$$\log M_{g^k,D}(r) < \exp\{r(\rho_{k_g} + \varepsilon/2)\}, \quad \text{for } r > r_0 .$$

Hence for $r > r_0$, we have

$$\log M_{f^k,D}(r) \log M_{g^k,D}(r) < exp\{r(\rho_{k_f} + \rho_{k_g} + \varepsilon)\}$$

Thus if $\log M_{f*g,D}(r) \sim \log M_{f^k,D}(r) \log M_{g^k,D}(r)$, then

$$\rho_k(D) = \limsup_{r\to\infty} \frac{\log\log M_{f^k * g^k,D}(r)}{r} \le \rho_{k_f} + \rho_{k_g}$$

That is, $\rho_k \le \rho_{k_f} + \rho_{k_g}$.

Again from (2.10) we have

$$\frac{\log M_{f^k,D}(r)}{e^{r\rho_{k_f}}} < T_{k_f}(D) + \varepsilon, \quad \text{for } r > r' \text{ and}$$

$$\frac{\log M_{g^k,D}(r)}{e^{r\rho_{k_g}}} < T_{k_g}(D) + \varepsilon, \quad \text{for } r > r'$$

Hence,

$$\frac{\log M_{f^k,D}(r) \log M_{g^k,D}(r)}{e^{r\left(\rho_{k_f} + \rho_{k_g}\right)}} < [T_{k_f}(D) + \varepsilon]\,[T_{k_g}(D) + \varepsilon], \textit{for } r > r'$$

Thus if, $\rho_k = \rho_{k_f} + \rho_{k_g}$

$$\limsup_{r\to\infty} \frac{\log M_{f^k * g^k,D}(r)}{e^{r\rho_k}} \le T_{k_f}(D) + T_{k_g}(D)$$

That is $T_k(D) \le T_{k_f}(D)\ T_{k_g}(D)$.

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