GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) 29 (2009) 139-146

THE HOMOLOGY AND COHOMOLOGY GROUPS OF H_3

Subrata Majumdar[1] and Quazi Selina Sultana[2]

[12]*Department of Mathematics, Rajshahi University, Rajshahi*
E-mail:-majumdar_subrata@hotmail.com

Received 07.06.08 Accepted 27.06.09

ABSTRACT

A free resolution of **Z** for the integral group ring of the three-dimensional Heisenberg group H_3 has been constructed by extending Lyndon's partial resolution. The integral homology and cohomology have been calculated from there.

AMS Classification: 18G, 20J.

Key words: Fox derivatives, free resolution.

1. Introduction

Here we shall construct a full free resolution of **Z** for the integral group ring of the three dimensional Heisenberg group, using Majumdar-Akhter [10] technique of extending Lyndon's partial free resolution [8] to a full free resolution. We compute the integral homology and cohomology from the resolution obtained.

H_3, the three dimensional Heisenberg group has a presentation

$$H_3 = \langle x, y, z : [x,z] = [y,z] = 1, [x,y] = z \rangle, \text{ (Burillo[1], p.2).}$$

It is a member of widely studied important class of Lie Groups called the Heisenberg groups. It is nilpotent of class 2. Huebschmann [6] used his sophisticated perturbation theory technique to determine the cohomology of the generalized Heisenberg group given by

$$G = \left\langle x, y, z : [x,z] = [y,z] = 1, [x,y] = z^k \right\rangle.$$

$$G = H_3, \text{ if } k = 1.$$

The $(2n+1)$ Heisenberg group H_{2n+1} is the group of upper triangular $(n+2)\times(n+2)$ matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y^T \\ 0 & 0 & 1 \end{pmatrix},$$

where $x = (x_1, x_2 \cdots, x_n)$, $y = (y_1, y_2 \cdots, y_n)$, I is the $n \times n$ unite matrix. Thus H_3 is the group of all upper triangular matrices:

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}, \ (x,y,z \in R).$$

The Heisenberg group H_3 has a cubic Dehn function ([1], p.1) the latter being a best possible choice for isoperimetric function. Isoperimetric inequalities have been used fruitfully in the study of hyperbolic groups and automatic groups (Gromov [4], Epstien [2]). The cubic nature of Dehn function for H_3 shows that it is neither hyperbolic nor automatic, since these have respectively a linear Dehn function and a quadratic Dehn function. Thurston proved this fact by combinatorial methods. He also shows that H_3 is not combable

2. Before constructing our free resolution for H_3, we give a few definitions, state a few known results and prove a number of results that will be needed for construction and its proof.

Lemma 2.1

H_3 is torsion free.

Proof

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix} : \ (x,y,z \in R) \right\}.$$

Now
$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nx & nz + nxy \\ 0 & I & ny \\ 0 & 0 & 1 \end{pmatrix}.$$

So
$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

if and only if $n = 0$.

Hence H_3 is torsion free.

The following definitions are due to Higman [5].

Definition 2.2

A group G is said to be *indexed* if it can be mapped homomorphically onto a non-zero subgroup of **Z.**

Definition 2.3

A group G is said to be i*ndicable throughout* if every subgroup $H\ (\neq 1)$ of G can be indexed.

We state two theorems due to Higmann:

Theorem 2.4 ([5], p.242)

If G is indicable throughout and R is a ring with , and has no zero divisors, then RG has no zero divisors.

Theorem 2.5 ([5], p. 243)

If G is indicable throughout and R is a ring with I and has no zero divisors, then the units of RG are trivial.

We shall use these results to prove:

Theorem 2.6

H_3 is indicable throughout.

Proof

We write G for H_3 and let $H(\neq 1)$ be a subgroup of G.

Case I

First suppose that $H \subseteq G'$. By the definition of G, $G'' \subseteq Z(G)$. This implies that G' is abelian. G' is the normal subgroup generated by the commutator $[h_1, h_2]$, h_1, h_2 are the images of x_1, x_2 in G. So a typical element g' of G' is $\prod\left(g_i^{-1}\left([h_1, h_2]^{e_i}\right)\right)$, $e_i = 1$.

$$\therefore g' = \prod_{i=1}^{n} [h_1, h_2]^{e_i}, \text{ since } [h_1, h_2] \text{ is a commutator of } h_1 \text{ and } h_2.$$

Since G' is infinite cyclic, and since $H \neq 1$, H too infinite cyclic. So H can be indexed.

Case II

Suppose $H \subseteq G'$. Then $HG' \neq G'$, and so, $\frac{HG'}{G'}$ is a subgroup of $\frac{G'}{G}$, and is not the identity subgroup. Hence $\frac{HG'}{G'}$ is free abelian. Then there is a homomorphism $f : \frac{HG'}{G'} \to Z$ such that $Im f \neq \{0\}$. If φ is the canonical homomorphism $\varphi : H \to \frac{HG'}{G'}$, then φ is onto and $Im \overline{f} \neq \{0\}$, where $\overline{f} : H \to Z$ is the composite $\overline{f} = f\varphi$. Thus H can be indexed.

Hence G is indicable throughout.

As a consequence of Theorem 2.4, Theorem 2.5 and Theorem 2.6 implies the following:

Corollary 2.7

***ZG** has no zero divisors.*

Corollary 2.8

The units of ***ZG*** *are trivial.*

3. Free resolution of Z

Let $G = H_3 = \frac{F}{R}$, where F is the free group generated by x_1, x_2 and R is the normal closure of r_1, r_2, where $r_1 = [\, x_1, [x_1, x_2\,]\,]$ and $r_2 = [\, x_2, [\, x_1, x_2\,]\,]$.

i.e.,

$$r_1 = x_1^{-1}x_2^{-1}x_1^{-1}x_2x_1x_1x_1^{-1}x_2^{-1}x_1x_2;$$
$$r_2 = x_2^{-1}x_2^{-1}x_1^{-1}x_2x_1x_2x_1^{-1}x_2^{-1}x_1x_2.$$

Then the Fox derivatives of r_1, r_2 are:

$$\frac{\partial r_1}{\partial x_1} = \quad -r_1 - x_2x_1r_1 + x_2^{-1}x_1x_2 + x_2;$$
$$\frac{\partial r_1}{\partial x_2} = \quad -x_1r_1 + x_1x_2^{-1}x_1x_2 - x_2^{-1}x_1x_2 + 1;$$
$$\frac{\partial r_2}{\partial x_1} = \quad -x_2^2r_2 + x_2x_1^{-1}x_2^{-1}x_1x_2 - x_1^{-1}x_2^{-1}x_1x_2 + x_2;$$
$$\frac{\partial r_2}{\partial x_2} = \quad -r_2 - x_2r_2 + x_2^{-1}x_1x_2^2r_2 + x_1^{-1}x_2^{-1}x_1x_2 - x_2^{-1}x_1x_2 + 1.$$

Writing $\pi(x_i) = h_i, i = 1,2,$ we have

$$\frac{\partial r_1}{\partial x_1} = \quad -1 - h_2h_1 + h_2^{-1}h_1h_2 + h_2;$$
$$\frac{\partial r_1}{\partial x_2} = \quad -h_1 + h_1h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 + 1;$$
$$\frac{\partial r_2}{\partial x_1} = \quad -h_2^2 + h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2;$$
$$\frac{\partial r_2}{\partial x_2} = \quad -1 - h_2 + h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 + 1.$$

To construct the free resolution of **Z** we proceed as follows. In Lyndon's partial free resolution of **Z**, let $\beta_1\gamma_1 + \beta_2\gamma_2 \in Ker d_1$, where $\gamma_1, \gamma_2 \in ZG$.

Then $\quad d_1(\beta_1\gamma_1 + \beta_2\gamma_2) = 0.$

$$\therefore \left[\alpha_1\left(h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2\right) + \alpha_2\left(1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2\right)\right]\gamma_1$$
$$+\left[\alpha_1\left(-h_2^2 - h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2\right) + \alpha_2\left(h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2\right)\right]\gamma_2$$
$$= 0$$

or, $$\alpha_1\left[\left(h_2 - 1 - h_1h_2 - h_2^{-1}h_1h_2\right)\gamma_1 + \left(-h_2^2 - h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2\right)\gamma_2\right] \\ + \alpha_2\left[\left(1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2\right)\gamma_1 + \left(h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2\right)\gamma_2\right] = 0$$

Since Y_0 is free on α_1, α_2, we have

$$\left.\begin{array}{ll} \left(h_2 - 1 - h_1h_2 - h_2^{-1}h_1h_2\right)\gamma_1 & \\ + \left(-h_2^2 - h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2\right)\gamma_2 = 0 & (\mathrm{i}) \\ \left(1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2\right)\gamma_1 & \\ + \left(h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2\right)\gamma_2 = 0 & (\mathrm{ii}) \end{array}\right\}. \qquad (3.1)$$

We write Equation (3.1) as

$$\left.\begin{array}{ll} a\gamma_1 + b\gamma_2 = 0 & (i) \\ c\gamma_1 + d\gamma_2 = 0 & (ii) \end{array}\right\}. \qquad (\mathrm{i})$$

Solving the equations (I) in **Q**, we have $\gamma_1 = \gamma_1'$, where γ_1' is an arbitrary element of **ZG**,

and $\gamma_2 = -b^{-1}a\gamma_1' = -d^{-1}c\gamma_1'$, so that $-b^{-1}a_1 = -d^{-1}c$. γ_1' is an arbitrary element of ZG, $-b^{-1}a_1 = -d^{-1}c \in ZG$. Since $b^{-1}a$ has a inverse $a^{-1}b$, $b^{-1}a = g_0$, for some $g_0 \in G$, by Corollary 2.8.

Define Y_2 as the right **ZG-**module freely generated by δ and define

$$d_2 : Y_2 \to Y_1 \quad \text{by}$$

$$d_2(\delta) = \beta_1 - \beta_2 g_0.$$

Then $$d_2(\delta\gamma_1) = (\beta_1 - \beta_2 g_0)\gamma_1 = \beta_1\gamma_1 - \beta_2 g_0\gamma_1 = \beta_1\gamma_1 - \beta_2\gamma_2.$$

Hence $$\beta_1\gamma_1 + \beta_2\gamma_2 \in Kerd_2.$$

Also $$\begin{aligned}(d_1d_2)(\delta) &= d_1(\beta_1 - \beta_2 g_0) \\ &= (\alpha_1 a\gamma_1 + \alpha_2 a\gamma_2) - (\alpha_1 bg_0\gamma_1 + \alpha_2 dg_0\gamma_2) \\ &= \alpha_1(a - a)\gamma_1 + \alpha_2(c - c)\gamma_2 \\ &= 0\end{aligned}$$

$$\therefore Kerd_2 \supseteq \operatorname{Im} d_1.$$

Now let $\delta \in Kerd_2$, then

$$\alpha_1\gamma - \alpha_2 g_0\gamma = 0$$

$$\Rightarrow \gamma = 0.$$

Thus we have

Theorem 3.1

The following is a free **Z***G-resolution of* **Z***:*

$$0 \to Y_2 \xrightarrow{d_2} Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} ZG \xrightarrow{\varepsilon} Z \to 0,$$

where Y_0, Y_1, Y_2 *are right* ***ZG****-modules freely generated by* $\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}, \delta$ *and* $\varepsilon, d_0, d_1, d_2$ *are defined by*

$$\varepsilon(g) = 1, \forall g \in G;$$

$$d_0(\alpha_1) = h_1 - 1;$$

$$d_0(\alpha_2) = h_2 - 1;$$

$$d_1(\beta_1) = \alpha_1\left(h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2\right) + \alpha_2\left(1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2\right)$$

$$d_1(\beta_2) = \alpha_1\left(-h_2^2 - h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2\right)$$
$$+ \alpha_2\left(h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2\right).$$

$$d_2(\delta) = \beta_1 - \beta_2 g_0.$$

4. Homology

For a left **ZG**-module A, the homology groups $H_n(G, A)$ are given by the homology of the complex of abelian groups

$$0 \to A \xrightarrow{\overline{d_2}} A^2 \xrightarrow{\overline{d_1}} A^2 \xrightarrow{\overline{d_0}} A \to 0,$$

where $\overline{d_0}, \overline{d_1}, \overline{d_2}$ are given by

$$\overline{d_0}(a_2, a_2) = (h_1 - 1)a_1 + (h_2 - 1)a_2;$$

$$\overline{d_1}(a_1, a_2) = \Big(\left[h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2\right]a_1,$$
$$\left[-h_2^2 - h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2\right]a_2\Big)$$

$$\overline{d_2}(a) = (a, g_0a).$$

If A is trivial, then

$$\overline{d_0}(a_2, a_2) = 0;$$

$$\overline{d_1}(a_1, a_2) = (0, 0);$$

$$\overline{d_2}(a) = (a, a).$$

If $A = Z$, then

$$H_0(G,Z) \cong Z,$$
$$H_1(G,Z) \cong Z \oplus Z,$$
$$H_2(G,Z) \cong Z,$$
$$H_3(G,Z) \cong 0.$$

5. Cohomology

For a right **ZG**-module *A*, the cohomology groups $H^n(G,A)$ are given by the homology of the complex

$$0 \leftarrow A \xleftarrow{d_2^*} A^2 \xleftarrow{d_1^*} A^2 \xleftarrow{d_0^*} A \leftarrow 0,$$

where d_0^*, d_1^*, d_2^* are given by

$$d_0^*(a) = (a(h_1 - 1), a(h_2 - 1));$$
$$d_1^*(a_1, a_2) = \left(a_1\left[h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2\right], a_2\left[-h_2^2 - h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2\right]\right),$$
$$d_2^*(a) = a_1 - g_0a_1.$$

If *A* is trivial, then

$$d_0^*(a) = (0,0);$$
$$d_1^*(a_1, a_2) = (0,0)$$
$$d_2^*(a) = 0.$$

If $A = Z$, then

$$H^0(G,Z) \cong Z,$$
$$H^1(G,Z) \cong Z \oplus Z,$$
$$H^2(G,Z) \cong Z \oplus Z,$$
$$H^3(G,Z) \cong Z.$$

REFERENCES

1. Burillo J., Lower bound isoperimetric function for nilpotent groups, *Dimacs series in discrete Mathematics and theoretical computer science* **25**, 1-8, (1996).

2. Epstien A. B. D., J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Peterson, W. P. Thurston, Word processing in groups, *Jones and Bartlett, Boston -London. (*1992).

3. Fox R. H., Free Differential Calculus I, *Ann. Math.* **57**, 3,547-560, (1953).

4. Groves J. R. J., Rewriting systems and Homology of groups, *Proceedings of the third International conference on the theory of Groups, Canberra 1456 LNM, Springer-Verlag,* (1989).

5. Higman G., The zero divisors and units in a group ring, *Proc. Lond. Math. Soc.* **46(2)**, 231-248, (1940).

6. Huebschmann J., Perturbation theory and free resolutions for nilpotent groups of class 2, *J. Algebra*, **126** 2, 348-399, (1984).

7. Lyndon R. C. and P. Schupp, Combinatorial group theory, *Springer-Verlag, Berlin*, (1997).

8. Lyndon R. C., Cohomology theory of groups with a single defining relation, *Ann. Math.* **52**, 3, 656-665, (1953).

9. Majumdar S. and N. Akhter, On a simple method of determining the homology and the cohomology for finitely presented group. *Bull. Cal. Math. Soc.* **100,** (2008).

1. Majumdar S., A free resolution for a class of groups, *J. Lond.Math. Soc.* **2**, 615-619, (1970).