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ON JACOBSON RADICAL FOR Γ-RINGS

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ABSTRACT

Jacobson radical of gamma rings is one of the most significant concept in the ring theory. In this paper we consider the Jacobson radical for gamma rings due to A.C. Paul and T.M. Abul Kalam Azad [5]. Some new characterizations are developed in this radical. The Jacobson Density Theorem and its converse Theorem are also proved here.

1. Introduction

S.Kyuno [3] introduced the Jacobson radical J(M) using the right quasi-regularity. He also proved that the right Jacobson radical is equal to that of the left one.

A.C. Paul and T. M. Abul Kalam Azad [5] gave the notion of Jacobson radical for Γ-rings by means of annihilators of the ΓM-modules. They have developed some characterizations of this radical.

T.S. Ravisankar and U.S. Shukla [6] studied Jacobson radicals in the setting of modules. They obtained a number of remarkable properties of these radicals.

Jacobson radicals for Γ-rings are also studied may other authors such as Kyuno, Coppage and Luh etc.

In this paper, we have depeloped some properties of Jacobson radicals which are different and significant in the Mathematical interest. We have also proved Jacboson Density Theorem and its converse Theorem.

2. Preliminaries

2.1. Definitions.

Gamma Ring: Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that

i) $(x + y)\alpha z = x\alpha z + y\alpha z$
$x(\alpha + \beta)z = x\alpha z + x\beta z$
$x\alpha(y + z) = x\alpha y + x\alpha z$ 0

ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ-ring.

Ideal of Γ-rings: A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c \in M, \alpha \in \Gamma, a \in A\}$ $(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.

Matrix Gamma Ring: Let M be a Γ-ring and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the sets of $m \times n$ matrices with entries from M and of $n \times m$ matrices with entries from Γ, then $M_{m,n}$ is a $\Gamma_{n,m}$ -ring and multiplication defined by

$(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}$. If $m = n$, then M_n is a Γ_n-ring.

Division gamma ring: Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non zero ideal is itself.

Nilpotent element: Let M be a Γ-ring. An element x of M is called nilpotent if for any $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = (x\gamma x\gamma ... \gamma x\gamma)x = 0$.

Nil ideal: An ideal A of a Γ-ring M is a nil ideal if every element of A is nilpotent i.e. for all $x \in A$ and $\gamma \in \Gamma$, $(x\gamma)^n x = (x\gamma x\gamma \ldots \gamma x\gamma)x = 0$, where n depends on the particular element x of A .

Nilpotent ideal: An ideal A of a Γ-ring M is called nilpotent if $(A\Gamma)^n A = (A\Gamma A\Gamma \ldots . \Gamma A\Gamma)A = 0$, where n is the least positive integer.

Maximal ideal: An ideal R in a Γ-ring M is called a maximal

ideal in M if (i) $R \subset M$ and (ii) whenever L is an ideal in M such that $R \subseteq L \subseteq M$, then either $L = R$ or $L = M$.

Annihilator of a subset of a Γ-ring: Let M be a Γ-ring. Let S be a sub set of M. Then the left annihilator $l(S)$ of S is defined by $l(S) = \{m \in M \mid m\Gamma S = 0\}$, where as the right annihilator $r(S)$ is defined by $r(S) = \{m \in M \mid S\Gamma m = 0\}$.

Idempotent element: Let M be a Γ-ring. An element e of M is called idempotent if $e\gamma e = e \neq 0$ for some $\gamma \in \Gamma$.

Primitive idempotent: Let M be Γ-ring. An idempotent e of M is called primitive if it is impossible to express as the sum of two orthogonal idempotent elements.

Internal direct sum: Let M be a Γ-ring and $N_1$and N_2 be two left ideals of M such that

(i) $M = N_1 + N_2 = \{n_1 + n_2 \mid n_1 \in N_1, n_2 \in N_2\}$

(ii) $N_1 \cap N_2 = \{0\}$

Then we say M is the internal direct sum or simply direct sum of N_1 and N_2 and we write $M = N_1 \oplus N_2$.

ΓM-module. Let M be a Γ-ring and let $(P, +)$ be an abelian group. Then P is called a left ΓM-module if there exists a Γ-mapping (Γ-composition) from $M \times \Gamma \times P$ to P sending (m, α, p) to $m\alpha p$ such that

i) $(m_1 + m_2)\alpha p = m_1\alpha p + m_2\alpha p$

ii) $m\alpha(p_1 + p_2) = m\alpha p_1 + m\alpha p_2$

iii) $(m_1\alpha m_2)\beta p = m_1\alpha(m_2\beta p)$, for all p, p_1, $p_2 \in P$, m, m_1, $m_2 \in M$, α, $\beta \in \Gamma$.

If in addition, M has an identity 1 and $1\gamma p = p$ for all $p \in P$ and some $\gamma \in \Gamma$, then P is called a unital ΓM-module.

Sub ΓM-module: Let M be a Γ-ring. Let P be a left ΓM-module. Let $(Q, +)$ be a subgroup of $(P, +)$. We call Q, a sub left ΓM-module of P if $m\gamma q \in Q$ for all $m \in M$, $q \in Q$ and $\gamma \in \Gamma$.

Irreducible Γ*M*-module: Let M be a Γ-ring and P be left ΓM-module. We say that P is an irreducible left ΓM-module if

(i) $P \neq 0$ and $P \supseteq Q \supseteq 0$, Q is a sub ΓM- module of P, implies $Q = P$ or $Q = 0$, and

(ii) $m\gamma p \neq 0$ for some $m \in M$, $p \in P$ and $\gamma \in \Gamma$.

ΓM-homomorphism: Let M be a Γ-ring. Let P and Q be two left ΓM-modules. Let φ be a map of P into Q. Then φ is called a ΓM-homomorphism if and only if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(m\gamma x) = m\gamma\varphi(x)$ for all $x, y \in P$, $m \in M$ and $\gamma \in \Gamma$. If φ is one-one and onto, then φ is a ΓM-isomorphism and is denoted by $P \cong Q$. If φ is a ΓM-homomorphism of P into Q, then kernel of φ, i.e., $\ker\varphi = \{x \in P \mid \varphi(x) = 0\}$, which is a left sub ΓM-module of P and image of φ i.e., $\mathrm{Im}\varphi = \{y \in Q \mid y = \varphi(x) \text{ for some } x \in P\}$ is a left sub ΓM-module of Q. We use the notation $\mathrm{Hom}_{\Gamma M}(P, Q)$ to denote the set all ΓM-homomorphisms of P into Q. If $Q = P$, then φ is called a ΓM-endomorphism. Clearly $\mathrm{Hom}_{\Gamma M}(P, P)$ forms a Γ-ring. We call $\mathrm{Hom}_{\Gamma M}(P, P)$ the Γ-ring of ΓM -endomorphism of P.

Let M be a Γ-ring and A is an ideal of M. Since every ideal A is a ΓM-module, then the homomorphism between two ideals are the same as that of given above.

Γ-vector space: Let $(V, +)$ be an abelian group. Let Δ be a division Γ-ring with identity 1 and let $\varphi: \Delta\times\Gamma\times V \to V$, where we denote $\varphi(m,\gamma,v)$ by $m\gamma v$. Then V is called a left Γ-vector space over Δ, if for all $\delta_1, \delta_2 \in \Delta$, $v_1, v_2 \in V$ and $\gamma \in \Gamma$, the following hold:.

i) $\delta_1\gamma(v_1 + v_2) = \delta_1\gamma v_1 + \delta_2\gamma v_2$

ii) $(\delta_1 + \delta_2)\gamma v_1 = \delta_1\gamma v_1 + \delta_2\gamma v_1$

iii) $(\delta_1\beta\delta_2)\gamma v_1 = \delta_1\beta(\delta_2\gamma v_1)$

iv) $1\gamma v_1 = v_1$ for some $\gamma \in \Gamma$.

We call the elements v of V vectors and the elements δ of Δ scalars. We also call $\delta\gamma v$ the scalar multiple of v by δ. Similarly, we can also define right Γ-vector space over **Δ.**

Sub Γ-Space: Let V be a left Γ-vector space over Δ. A non empty sub set U of V is called a sub Γ-Space of V if

(i) $(U, +)$ is a sub group of $(V, +)$

(ii) $\delta\gamma u \in U$ for all $\delta \in \Delta$, $\gamma \in \Gamma$, $u \in U$.

It is clear that U is a sub Γ-space of V provided that U is closed with respect to the operations of addition in V and scalar multiplication of vectors by scalars.

Linear Γ-combination: Let V be a left Γ-vector space over a division Γ-ring Δ. Let v_1, v_2, . . ., $v_n \in V$ and let $\gamma \in \Gamma$, then the vector $v = \delta_1\gamma v_1 + \delta_1\gamma v_2 + \ldots + \delta_n\gamma v_n$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$ is called a linear γ-combination of the v_i's over Δ. If v is a linear γ-combination for some $\gamma \in \Gamma$, then v is called a linear Γ-combination of the v_i's over Δ.

Linearly Γ-independent and linearly Γ-dependent: Let V be a left Γ-vector space over a division Γ-ring Δ. Let $\gamma \in \Gamma$, then the set of vectors $\{v_i \mid i \in \Lambda\}$ is called linearly γ-independent over Δ (or simply γ-independent) if for each finite sub set of vectors $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ of $\{v_i \mid i \in \Lambda\}$, $\delta_1\gamma v_{i_1} + \delta_2\gamma v_{i_2} + \ldots + \delta_n\gamma v_{i_n} = 0$ implies $\delta_1 = \delta_2 = \ldots = \delta_n = 0$. Otherwise, the set $\{v_i \mid i \in \Lambda\}$ is called linearly γ-dependent (or simply γ-dependent). If $\{v_i \mid i \in \Lambda\}$ is γ-independent for some $\gamma \in \Gamma$, then $\{v_i \mid i \in \Lambda\}$ is called linearly Γ-independent. Otherwise the set $\{v_i \mid i \in \Lambda\}$ is called linearly Γ-dependent.

Generators of a Γ-vector space: Let V be a left Γ-vector space over a division Γ-ring Δ. Let G be a sub set of V. Let G = $\{v_i\}$. Then G is said to be a set of generators for V or G **spans** V, if any $v \in V$ is a linear Γ-combination of vectors in G.

Basis of a Γ-vector space: Let V be a left Γ-vector space over a division Γ-ring Δ. *A* basis B for V is a subset of V such that

(i) B spans V and

(ii) B is Γ-independent

Dimension of a Γ-vector space: Let V be a left Γ-vector space over a division Γ-ring Δ. If V has a basis with n elements, then we say that V is finite dimensional of dimension n over Δ and we denote this by $[V : \Delta] = n$. If V does not have a finite basis, then we say that V is infinite dimensional and write $[V : \Delta] = \infty$. We note that if $V = \{0\}$, then $[V : \Delta] = 0$, since empty set is a basis for {0}.

Linear Γ-transformation: Let V and U be a left Γ-vector spaces over a division Γ-ring Δ. Let $T: V \to U$ satisfy

(i) $T(v_1+v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$

(ii) $T(\delta\gamma v) = \delta\gamma T(v)$ for all $\delta \in \Delta$, $\gamma \in \Gamma$, $v \in V$.

We call T a linear Γ-transformation from V to U and we denote the set of all linear Γ-transformations from V to U by $\mathrm{Hom}_\Delta(V, U)$. $\mathrm{Hom}_\Delta(V, U)$ is an additive group.

For all $T, S \in \mathrm{Hom}_\Delta(V,U)$, $T + S$ and $T\gamma S$ are respectively defined by

$(T + S)(x) = T(x) + S(x)$ and

$(T\gamma S)(x) = T(\gamma S(x))$ for all $x \in V$ and $\gamma \in \Gamma$.

2.2 Theorem. Every Unital irreduicible left ΓM-module is cycle.

2.3 Theorem (Ist Isomorphism Theorem of ΓM-homomorphism). Let M be a Γ-ring and P and Q be the left ΓM-modules. Let $\varphi: P \rightarrow Q$ be a ΓM-homomorphism. Then $M/\ker\phi \cong \mathrm{Im}\ \varphi$

3. Definition of the Jacobson Radical

3.1 Definition

The **Jacobson radical** of M is written as $J(M)$ and is defined as

$J(M) = \{m \in M \mid m\gamma P = 0 \text{ for all } \gamma \in \Gamma\}$, where P is an irreducible left ΓM-module. If M has no irreducible left ΓM-module, then $J(M) =$ M and M is called **radical** Γ-ring.

3.2 Lemma. Let P be an irreducible left ΓM-module. Then $P \cong M/R$, where R is a maximal left ideal of M. Furthermore, there is an $a \in M$ such that $M\gamma(1 - a) = \{x - x\gamma a | x \in M\} \subset R$ for some $\gamma \in \Gamma$.

Proof. By Theorem **2.2,** P is cyclic, say $P = M\gamma p$ for some $p \in P$ and $\gamma \in \Gamma$.

We define $\varphi : M \rightarrow P$ by $\varphi(m) = m\gamma p$ for all $m \in M$. Let $m_1, m_2 \in M$, then $\varphi(m_1 + m_2) = (m_1 + m_2)\gamma p = m_1\gamma p + m_2\gamma p = \varphi(m_1) + \varphi(m_2)$ Let $m \in M$, then $\varphi(m\alpha m_1) = (m\alpha m_1)\gamma p = m\alpha(m_1\gamma p) = m\alpha\varphi(m_1)$ for all $\alpha \in \Gamma$. Hence φ is a ΓM-homomorphism. Also we have seen that φ is one-one and φ is onto. So by Theorem **2.3,** $M/R \cong P$, where R is the kernel of φ.

Now let R' be a left ideal of M such that $R \subseteq R' \subset M$. Then R'/R is isomorphic to a non zero sub ΓM-module of P. But P is an irreducible left ΓM-module, then non zero sub ΓM-module of P is itself.

Hence $R'/R \cong P$. Therefor $R' = M$. Hence R is a maximal ideal of M.

Since $M\gamma p = P$, then there is an $a \in M$ such that $a\gamma p = p$. Then for each $x \in M$, $(x - x\gamma a)\gamma p = x\gamma p - x\gamma a\gamma p = x\gamma p - x\gamma p = 0$, so $x - x\gamma a \in R$. Thus $M\gamma\ (1 - a) \subset R$. Hence the lemma is proved.

3.3 Definition. A left ideal R of M is called a **regular ideal** if there is an $a \in M$ such that $M\gamma(1 - a) \subset R$ for some $\gamma \in \Gamma$.

3.4 Lemma. Every regular proper left ideal of M is contained in a maximal regular left ideal of M.

Proof. Let R be a regular proper left ideal of M and let $a \in M$ such that $M\gamma(1 - a) \subset R$. Suppose $a \in R$ and let $x \in M$. Then $x - x\gamma a \in R$ and $x\gamma a \in R$. Hence $x = (x - x\gamma a) + x\gamma a \in R$. Then $M \subset R$, a contradiction. Thus $a \notin R$.

By Zorn's Lemma, there is an ideal R' maximal with respect to the properties:

i) $R \subset R'$

ii) $a \notin R'$.

We claim that R' is actually a maximal left ideal of M. For suppose

$R' \subseteq R'' \subset M$. Then $a \in R'$ and futher more $x - x\gamma a \in R''$, $x\gamma a \in R''$. Hence $x = (x - x\gamma a) + x\gamma a \in R''$. Therefore $M \subseteq R''$. Hence $R'' = M$.

Finally, since $M\gamma(1 - a) \subset R \subset R'$, then R' is regular. Hence the lemma is proved.

3.5 Definition. If R is a left ideal of M, then $(R{:}M) = \{m \in M \mid m\Gamma M \subset R\}$.

It is easy to verify that **(R:M)** is also a left ideal of M.

3.6 Theorem. $J(M) = \cap R$, where the intersection is over all maximal regular left ideal of the Γ-ring M.

Proof. Let $x \in J(M)$ and let R be a maximal regular left ideal of M. Then $M\gamma(1 - a) \subset R$ for some $a \in M$ and $\gamma \in \Gamma$. Thus $x - x\gamma a \in R$. Since $x \in J(M)$, we have $x \in (M{:}R)$, that is, $x\gamma M \subset R$. Hence $x\gamma a \in R$ and so $x \in R$. Thus $J(M) \subset \cap R$.

We now suppose that $x \in \cap R$. $M\gamma(1+x)$ is a regular left ideal of M. If it is proper, it is contained in a maximal regular left ideal R'. But $x \in R'$ and thus for all $y \in M$, $y\gamma x \in R'$, and $y + y\gamma x - y\gamma x = y \in R'$. Hence $R' = M$, a contradiction. Therefore $M\gamma(1+x) = M$. Hence $-x \in M\gamma(1+x)$, that is, there is a $y \in M$ such that $x + y + y\gamma x = 0$.

We let P be an irreducible left ΓM-module and suppose $\cap R \not\subset l(P)$. Then $(\cap R)\gamma P \neq 0$ and $(\cap R)\gamma p \neq 0$ for some $p \in p$. But then $(\cap R)\gamma p = P$. Hence $r\gamma p = -p$ for some $r \in \cap R$. We now let $s \in M$ be such that $r + s + s\gamma r = 0$. Then

$$0 = (r + s + s\gamma r)\gamma p = r\gamma p + s\gamma p + s\gamma r\gamma p = -p + s\gamma p + s\gamma(-p) = -p + s\gamma p - s\gamma p = -p.$$

Thus $\mathbf{p} = 0$. So that $(\cap \mathbf{R})\gamma \mathbf{p} = 0$, a contradiction. Thus $\cap \mathbf{R} \subset l(\mathbf{P})$ and hence $\cap \mathbf{R} \subset \mathbf{J}(\mathbf{M})$. Therefore $\mathbf{J}(\mathbf{M}) = \cap \mathbf{R}$. Hence the theorem is proved.

3.7 Definition. An element $a \in M$ is **left quasi-regular** if there is an element $a' \in M$ such that

$a + a' + a'\gamma a = 0$ for all $\gamma \in \Gamma$. Then is a' called a **left quasi-inverse** of a. A left ideal R of M is left quasi-regular if each of its elements is left quasi-regular.

We can define right quasi-regularity similarly; an element a is quasi-regular if there exists $a' \in M$ such that $a + a' + a\gamma a' = a + a' + a'\gamma a = 0$ for all $\gamma \in \Gamma$. We note that if R is a left quasi-regular ideal of a Γ-ring M and a' is a left quasi-inverse of $a \in R$, then $a + a' + a'\gamma a = 0$ so that $a' = -a - a'\gamma a \in R$.

3.8 Theorem. $J(M)$ is a left quasi-regular ideal of M and contains every left quasi-regular left ideal of M.

Proof. Let R be a left quasi-regular left ideal of M. Let P be an irreducible left ΓM-module. Suppose $R\gamma P \neq 0$ for some $\gamma \in \Gamma$. Then there exists a $p \in P$ such that $R\gamma p = P$.

Hence there exists an $a\in P$ such that $a\gamma p = -p$. Let $a'\in R$ be a left quasi inverse of a. Then

$0 = (a + a' + a'\gamma a)\gamma p = a\gamma p + a'\gamma p + a'\gamma a\gamma p = a\gamma p + a'\gamma p + a'\gamma(-p) = - p + a'\gamma p - a'\gamma p = -p$.

Therefore $p = 0$. Thus $R\gamma p=0$, a contradiction. Hence $R\gamma P = 0$, that is, $R\subset l(P)$. Therefore $R\subset J(M)$. By Theorem **3.6**, $J(M)$ is the intersection over all maximal regular left ideal of the Γ-ring M. Hence all elements of $J(M)$ is left quasi-regular. Theorefare $J(M)$ is a left quasi-regular left ideal. Hence the theorem is proved.

3.9 Lemma. If an element a of *a* Γ-ring M has a left quasi-inverse c and a right quasi-inverse b, then $b = c$.

Proof. Since c and b are respectively left and right quasi inverse of a, then we have

$$a + c + c\gamma a = 0 \text{ and } a + b + a\gamma b = 0 \text{ for all } \gamma\in\Gamma.$$

Now $(a + c + c\gamma a)\gamma b = 0\gamma b$

$$\Rightarrow a\gamma b + c\gamma b + c\gamma a\gamma b = 0.$$

Also, $c\gamma(a + b + a\gamma b) = c\gamma 0$. So $c\gamma a + c\gamma b + c\gamma a\gamma b = 0$.

Thus $c\gamma a + c\gamma b + c\gamma a\gamma b = a\gamma b + c\gamma b + c\gamma a\gamma b$.

$\Rightarrow a\gamma b = c\gamma a$. Therefore $c\gamma a - a\gamma b = 0$.

Now $c - b = (c - b) + (a - a) + (c\gamma a - a\gamma b) = (a + c + c\gamma a) - (a + b + a\gamma b) = 0$.

Therefore $c = b$. Hence the lemma is proved.

3.10 Lemma. Every element of $J(M)$ is right quasi-regular.

Proof. Let $a\in J(M)$. Then there is an $a'\in J(M)$ such that $a +a'+ a'\gamma a =0$ for all $\gamma\in\Gamma$. Then $a'\in J(M)$, so a'' is a left quasi-inverse of a'. But a is a right quasi-inverse of a' and so by Lemma **3.9**, $a = a''$. Thus $a +a'+ a\gamma a' =0$, that is, a' is a right quasi-inverse of a. Hence every element of $J(M)$ is right quasi-regular. Thus the theorem is proved.

The lemmas give us immediately:

3.11 Theorem. $J(M)$ is a right quasi-regular ideal and thus a quasi-regular ideal of M.

Proof. Let $J'(M)$ is a left quasi-regular ideal of M. Then $J'(M)$ contains every right quasi-regular right ideal, that is, $J(M)\subset J'(M)$. We have $J(M)$ is a right quasi-regular ideal of M, then $J(M)$ contains every left quasi-regular ideal, that is $J'(M) \subset J(M)$. Hence $J'(M) = J(M)$. Thus $J(M)$ is a quasi-regular ideal of M. Hence the theorem is proved.

3.12 Theorem. $J(M) = \{z\in M \mid b\gamma z\gamma a \text{ is quasi-regular for all } a, b\in M \text{ and some } \gamma\in\Gamma\}$.

Proof. Since $J(M)$ is an ideal of M and if $z\in J(M)$, then $b\gamma z\gamma a\in J(M)$ for all a, $b\in M$ and some $\gamma\in\Gamma$. Since $J(M)$ is a quasi-regular ideal, then $b\gamma z\gamma a$ is quasi-regular.

Conversely, let z be an element of M such that $b\gamma z\gamma a$ is quasi-regular for all a, $b\in M$. Let P be an irreducible left ΓM-module. Then as in the proof of Theorem **3.8**, $z\gamma a\in l(P)$ for all

$a \in M$. If $0 \neq u \in M$, then $P = M\gamma u$ and $z\gamma P = z\gamma M\gamma u = 0$, so that $z \in l(P)$. Hence $z \in J(M)$. Thus the theorem is proved.

3.13 Theorem. $\bar{J}\left(M/J(M)\right) = \bar{0}$.

Proof. Let $\bar{R}$ be a left quasi-regular left ideal in $M/J(M)$. Let R be its inverse image in M. Let $a \in R$ and $\bar{a} = a + J(M)$. Let $b \in R$ be such that $a + b + b\gamma a = 0$ and $\bar{a} + \bar{b} + \bar{b}\gamma\, \bar{a} = \bar{0}$ for all $\gamma \in \Gamma$. Then $a + b + b\gamma a \in J(M)$ and so is left quasi-regular. Let c be such that.

$$a + b + b\gamma a + c + c\gamma(a + b + b\gamma a) = 0$$

Then $a + b + b\gamma a + c + c\gamma a + c\gamma b + c\gamma b\gamma a = 0$

$\Rightarrow \; a + b + c + c\gamma b + b\gamma a + c\gamma a + c\gamma b\gamma a = 0$

$\Rightarrow \; a + (b + c + c\gamma b) + (b + c + c\gamma b)\gamma a = 0$

and thus a is left quasi-regular. Hence R is a left quasi-regular left ideal of M, and so $R \subset J(M)$, that is, $\bar{R} = \bar{0}$. But $\bar{J}\left(M/J(M)\right)$ is a left quasi-reqular left ideal of $M/J(M)$. Hence $\bar{J}\left(M/J(M)\right) = \bar{0}$. Thus the theorem is proved.

3.14 Definition. A Γ-ring M is called **(Jacobson) semi-simple** if $J(M) = 0$.

We need the following two theorems due to Paul and Kalam [5]

3.15 Theorem. Every nil left ideal (and hence every nilpotent ideal) of a Γ-ring M is contained in $J(M)$.

3.16 Theorem. If M is a left Artinian Γ-ring, then $J(M)$ is a nilpotent ideal of M.

3.17 Definition. An element a of a Γ-ring M is **called regular** (in the sense of von Neumann) if there exists an element u in M such that $a\gamma u\gamma a = a$ for some $\gamma \in \Gamma$; u is called a relative inverse for a. If every element of a Γ-ring M is regular, then M is called a **regular Γ-ring.**

3.18 Theorem. Any regular Γ-ring is semi-simple.

Proof. Let M be a regular Γ-ring. Suppose $a \in J(M)$. Let u is a relative inverse of a. Then $-u\gamma a$ has a quasi-inverse v such that $-u\gamma a + v + (-u\gamma a)\gamma v = 0$ for some $\gamma \in \Gamma$. Now

$a\gamma\, 0 = a\gamma(-u\gamma a + v + (-u\gamma a))\gamma v$

$\Rightarrow 0 = -a\gamma u\gamma a + a\gamma v - a\gamma u\gamma a\gamma v$

$\Rightarrow 0 = -a + a\gamma v - a\gamma v = -a$. Hence $a = 0$. Therefore $J(M) = 0$. Hence M is semi-simple.

We recall that a classically Γ-ring must have an identity element. We have a related result for Jacobson semi-simplicity. First we note that if M is an arbitrary Γ-ring, it can be embedded in a Γ-ring M* with identity 1 such that $M^* = M \oplus \langle 1 \rangle$, where $\langle 1 \rangle$, the Γ-ring generated by the identity is isomorphic to Z.

3.19 Theorem. Let M be an arbitrary Γ-ring and $M^* = M + \langle 1\rangle$, 1 an identity for M^*. Then $J(M) = J(M^*)\cap M$. If in addition $M\cap\langle 1\rangle = \{0\}$ and $\langle 1\rangle \cong Z$, then $J(M) = J(M^*)$.

Proof. Let $z\in J(M^*)\cap M$. Then z has a quasi-inverse $z'\in M^*$. So $z + z' + z'\gamma z = 0$ for some $\gamma\in\Gamma$. Hence $z' = -z - z'\gamma z$. Since $z' = -z - z'\gamma z$, then z' is in M and z is quasi-regular in M. Hence $J(M^*)\cap M\subset J(M)$. Since $M^* = M + \langle 1\rangle$, then any left ideal of M is a left ideal of M^*. Hence $J(M)\subset J(M^*)\cap M$. Therefore $J(M) = J(M^*)\cap M$.

Suppose now that $M\cap\langle 1\rangle = \{0\}$ and $\langle 1\rangle \cong Z$. Then if $z^*\in J(M^*)$, then the coset z^* of z^* in M^*/M is in the radical of the quotient Γ-ring. But $J(Z) = 0$, being the intersection of the maximal ideals of Z. So $\overline{z^*} = \overline{0}$ and $z^*\in M$. Therefore $J(M^*)\subset M$. Hence $J(M) = J(M^*)$.

3.20 Theorem. If $J(M)$ is the radical of a Γ-ring M, then the radical of M_n is $J(M)_n$, where M_n is the Γ_n-ring, whose entries come from M and $J(M)_n$ is matrix Γ_n-ring, whose entries come from $J(M)$.

Proof. Consider a matrix of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 \ldots 0 \\ a_{21} & 0 & 0 \ldots 0 \\ a_{31} & 0 & 0 \ldots 0 \\ \text{------} & & \\ a_{n1} & 0 & 0 \ldots 0 \end{pmatrix}$$

where a_{11} is left quasi-regular. Then there exists a'_{11} such that $a_{11} + a'_{11} + a'_{11}\gamma_{11}a_{11} = 0$ for $\gamma_{11}\in\Gamma$. Moreover $M\gamma_{11}(1 - a_{11}) = M$ so there exist a'_{i1}, i = 2, 3, . . .n, such that $a'_{i1} - a'_{i1}\gamma_{11}a_{11} = -a_{i1}$. Then if

$$A' = \begin{pmatrix} a'_{11} & 0 & 0 \ldots 0 \\ a'_{21} & 0 & 0 \ldots 0 \\ a'_{31} & 0 & 0 \ldots 0 \\ \text{------} & & \\ a'_{n1} & 0 & 0 \ldots 0 \end{pmatrix},$$

$A + A' + A'\Gamma_n A = 0$; thus A is left quasi-regular.

Now let J_j be the set of elements $A\in M_n$ with entries except possibly the jth column zero and jth column consisting of elements of $J(M)$. Each J_j is a left ideal and by arguments analogous to the one above for $j = 1$, they are left quasi-regular. Hence $J_j\subset J(M_n)$, for $j = 1, 2, \ldots, n$. Thus $J(M)_n = J_1 + J_2 + \ldots, J_n \subset J(M_n)$.

On the other hand, let $C = (c_{ij})$ belong to $J(M_n)$. If $a\in M$, let A_{pq} be the matrix with a in the (p,q) position and zeros elsewhere. Let a, b be arbitrary elements of M. Form

$$\Sigma A_{kp}\,\Gamma_p C\Gamma_q\,B_{qk} = \begin{matrix} a\gamma_{pp}c_{pq}\,\gamma_{qq}\;b & 0 & 0\ldots 0 \\ 0 & 0 & 0\ldots 0 \\ 0 & 0 & 0\ldots a\gamma_{pq}c\gamma_{pq}\,b \end{matrix}$$

But $C\in J(M_n)$ and hence $\Sigma A_{kp}\;\Gamma_p C\Gamma_q\;B_{qk}\in J(M_n)$. Let (C'_{ij}) be the quasi-inverse of ΣA_{kp} $\Gamma_p C\Gamma_q\,B_{qk}$, so that

$$a\gamma_{pp}c_{pq}\,\gamma_{qq}\,b + c'_{11}+ c'_{11}\gamma_{11}a\gamma_{1p}\,c\,\gamma_{qq}\,b = 0 = a\gamma_{pp}c_{pq}\,\gamma_{qq}\,b + c'_{11}+ a\gamma_{pp}\;c_{pq}\gamma_{qq}b\gamma_{q1}\,c'_{11}$$

Thus $a\gamma_{pp}c_{pq}\,\gamma_{qq}\,b$ is a quasi-regular for all a, $b\in M$. But then by Theorem **3.12,** $c_{pq}\in J(M)$. Thus $J(M_n)\subset J(M)_n$. Hence $J(M_n) = J(M)_n$. Thus the theorem is proved.

Let M be a Γ-ring and let be e an idempotent in M. Since any element $m\in M$ can be written as $m = e\gamma m + (m - e\gamma m)$ for some $\gamma\in\Gamma$,

we have $M = e\gamma M + (1- e)\gamma M$, where $(1 - e)\gamma M = \{m - e\gamma m \mid e\in M\}$ as before. But $e\gamma b = b$ for all $b\in e\gamma M$ and $e\gamma b = 0$ for all $b\in(1- e)\gamma M$, so that $e\gamma M\cap(1- e)\gamma M = 0$

and thus $M = e\gamma M \oplus(1- e)\gamma M = M\gamma e \oplus M\gamma(1-e)$.

We can also write $M = e\gamma M\gamma e \oplus e\gamma M\gamma(1- e) \oplus(1-e)\gamma M\gamma e \oplus(1-e)\gamma\, M\gamma(1- e)$.

These representations are called the right, left and two sided Peirce decomposition of M relative to e, repectively. We note that the terms of the first two are right and left ideals, repectively, while those of the third are sub Γ-rings of M. Moreover,

$$e\gamma M\gamma e = e\gamma M\cap M\gamma e,$$
$$e\gamma M\gamma(1 - e) = e\gamma M\cap M\gamma(1 - e),$$
$$(1 - e)\gamma M\gamma e = (1 - e)\gamma M\cap M\gamma e,$$
$$(1 - e)\gamma M\gamma(1 - e) = (1 - e)\gamma M\cap M\gamma(1 - e).$$

3.21 Theorem. Let M be an arbitrary Γ-ring and $J(M)$ its Jacobson radical. Then $e\gamma J(M)\gamma e = e\gamma M\gamma e\cap J(M)$ is the radical of $e\gamma M\gamma e$ and $(1 - e)\gamma\; J(M)\gamma(1 - e) = (1 - e)\gamma M\gamma(1 - e) \cap J(M)$ is the radical of $(1 - e)\gamma M\gamma(1 - e)$ for some $\gamma\in\Gamma$.

Proof. It is clear that $e\gamma M\gamma e\cap J(M) = e\gamma J(M)\gamma e$ and that this is a quasi-regular ideal in $e\gamma M\gamma e$. Hence $e\gamma J(M)\gamma e\subset J(e\gamma M\gamma e)$. Suppose $z\in J(e\gamma M\gamma e)$. Using the two sided Peirce decomposition of M, we write $m\in M$ as $m = m_{11} + m_{10} + m_{01} + m_{00}$, where $m_{11}\in e\gamma M\gamma e$, $m_{10}\in e\gamma M\gamma(1 - e)$, $m_{01}\in(1 - e)\gamma M\gamma e$ and $m_{00}\in(1 - e)\gamma M\gamma\,(1 - e)$. Then

$$z\gamma m = z\gamma(m_{11} + m_{10} + m_{01} + m_{00}) = z\gamma m_{11} + z\gamma m_{10} + z\gamma m_{01} + z\gamma m_{00} = z\gamma m_{11} + z\gamma m_{10},$$

since $z\gamma m_{01} = z\gamma e\gamma m_{01} = z\gamma m_{00} = z\gamma e\gamma m_{00} = 0$.

Now let z' is a quasi-inverse of $z\gamma m_{11}$ in $e\gamma M\gamma e$. Since $z\gamma m_{10}\gamma z' = 0$ then we have

$z\gamma m + z' + z\gamma m\gamma z' = z\gamma m_{10}$.

Moreover, $(z\gamma m_{10})\gamma(z\gamma m_{10}) = 0$ and hence by Theorem **3.15**, $z\gamma m_{10}$ is quasi-regular. Therefore $z\gamma m$ is quasi-regular for every $m\in M$. Thus $z\gamma M\subset J(M)$. Hence $b\gamma z\gamma a$ is quasi-regular for every a, $b \in M$. But then $z\in J(M)$ and $z\in e\gamma M\gamma e\cap J(M) = e\gamma J(M)\gamma e$. Thus $J(e\gamma M\gamma e)= e\gamma J(M)\gamma e$. The proof for $(1 - e)\gamma M\gamma (1 - e)$ is analogous.

4. Primitive Γ-rings

4.1 Definition. A left ΓM-module P is **faithful** if $a\gamma P = 0$ implies that $a = 0$.

4.2 Defintion. A Γ-ring M is **primitive** if it has a faithful, irreducible left ΓM-module. An ideal R of M is primitve if the Γ-ring M/R is primitive.

We need the following theorem which is in [5].

4.3 Theorem. $J(M) = \cap(M:R)$, where R ranges over all maximal regular left ideals of the Γ-ring M.

4.4 Lemma. An ideal R in a Γ-ring M is primitive if and only if $R = (M : Q)$, where Q is a maximal regular left ideal of M.

Proof. Let Q be a maximal regular left ideal M. Then M/Q is clearly a faithful, irreducible left $\Gamma\text{-}M/(M:Q)$-module. Hence $M/(M:Q)$ is primitive. So $(M:Q)$ is primitive. Thus R is primitive.

Conversely, let R be a primitive ideal of M and let P be a faithful, irreducible left $\Gamma\text{-}M/R$-module. Then P is a left ΓM-module and, infact, is an irreducible left ΓM-module. As a left $\Gamma\text{-}M/R$-module P is faithful, so its annihilator is the zero sub $\Gamma\text{-}M/R$-module. Thus the annihilator of P considering P as a left ΓM-module, is R. Then

$R = l(P) = \{x\in M \mid x\gamma M = R$, some $\gamma\in\Gamma\} = \{x\in M \mid x\gamma M = R\subseteq Q\} = \{x\in M \mid x\gamma M\subseteq Q\} = (M: Q)$.

Therefore $R = (M:Q)$, where Q is a maximal regular left ideal of M.

4.5 Theorem : If M is primitive, then $e\gamma M\gamma e$ and $(1-e)\gamma M\gamma(1-e)$ are primitive for some $\gamma\in\Gamma$.

Proof. Let P be a faithful, irreducible left ΓM-module. Then we can write

$$P = e\gamma P\oplus(1 - e)\gamma P.$$

Then clearly $e\gamma P$ is a left Γ-$e\gamma M\gamma e$-module. Since $e\gamma m\gamma e\gamma(1 - e)\gamma P=0$, if $e\gamma m\gamma e\gamma e\gamma P = 0$, then $e\gamma m\gamma e \in l(P)$. Hence $e\gamma P$ is faithful as a left Γ-$e\gamma M\gamma e$-module.

Now we let $e\gamma x \neq 0$ and $e\gamma y\in e\gamma P$. Then there exists $m\in M$ such that $m\gamma(e\gamma x) = e\gamma y$. But then $e\gamma m\gamma(e\gamma x) = e\gamma(e\gamma y)$. This implies that $e\gamma m\gamma(e\gamma e\gamma x) = (e\gamma e)\gamma y$.

Thus $(e\gamma m\gamma e)\gamma(e\gamma x) = e\gamma y$ and $e\gamma P$ is an irreducible left Γ-$e\gamma M\gamma e$ module, since any nonzero element generates all of $e\gamma P$. Hence $e\gamma M\gamma e$ has a faithful, irreducible left module $e\gamma P$. Therefore $e\gamma M\gamma e$ is primitive. The proof for $(1 - e)\,\gamma M\gamma(1 - e)$ is analogous.

4.6 Definition. Let P be a left ΓM-module and let $E(P)$ be the Γ-ring of all endomorphisms of the additive group of P with the obvious addition and multiplication. If $m\in M$, we define $T_m\in E(P)$ by $T_m\gamma p = m\gamma p$ for all $p\in P$ and $\gamma\in\Gamma$. The set

$$C_M(P) = \{\varphi\in E(P) \mid \varphi\gamma T_m = T_m\gamma\varphi \text{ for all } m\in M \text{ and } \gamma\in\Gamma\}$$

is a sub Γ-ring of $E(P)$ called the **commuting** Γ-ring of M on P.

4.7 Theorem (Schur's Lemma). If P is an irreducible left ΓM-module, then $C_M(P)$ is a division Γ-ring.

The proof is given in [5].

If M is a primitive Γ-ring and P is a faithful, irreducible left ΓM-module, then P can be regarded as a left Γ-vector space over the division Γ-ring $C_M(P)$.

4.8 Definition. A Γ-ring M is called a **dense Γ-ring** of linear Γ-transformations on P if, given any $v_1, v_2, \ldots, v_n \in P$ which are linearly Γ-independent over $C_M(P)$ and any $w_1, w_2, \ldots, w_n \in P$, there is an $m\in M$ such that $m\gamma v_i, = w_i$, $i = 1, 2, \ldots, n$ and some $\gamma\in\Gamma$. We sometimes say simply that M is dense on P.

4.9 Theorem (Jacobson Density Theorem). Let M be a Γ-ring and P be an irreducible left ΓM-module. Then, considering P as a left Γ-vector space over $C_M(P)$, M is a dense Γ-ring of linear Γ-tranformations on P.

Proof. It is sufficient to prove the following : **(*)** if V is a finite-dimensional sub Γ-space of P over $C_M(P)$ and if $p\in P$, $p\notin V$, then there is an $m\in M$ such that $m\gamma V = 0$ but $m\gamma p \neq 0$ for some $\gamma\in\Gamma$.

For suppose we can always find such an m. Since $m\gamma p \neq 0$, we can apply the statement to the 0 sub Γ-space and $m\gamma p$. Thus we can find $m_1\in M$ such that $m_1\gamma m\gamma p \neq 0$. Since $M\gamma m\gamma p \neq 0$, then we must have $M\gamma m\gamma p = P$. Thus, given any $p_1\in P$, we can find $s\in M$ such that $s\gamma m\gamma V = 0$ and $s\gamma m\gamma p = p_1$.

If we are given $v_1, v_2, \ldots, v_n \in P$, linearly Γ-independent over $C_M(P)$ and $w_1, w_2, \ldots, w_n \in P$, then we can, by virtue of the above argument, find $m_1, m_2, \ldots, m_n \in P$ such that

$$m_i\gamma v_j = \begin{cases} 0 \text{ if } j \neq i \\ w_i \text{ if } j = i \end{cases}$$

Let $m = m_1+ m_2+ \ldots + m_n$. Then for $i = 1, 2, \ldots, n$,

$$m\gamma v_i = (m_1+ m_2+ \ldots + m_n)\gamma v_i = m_1\gamma v_i + m_2\gamma v_i + \ldots + m_n\gamma v_i = m_i\gamma v_i = w_i.$$

Therefore $m\gamma v_i = w_i$. Thus M is dense on P.

We now establish **(*)** by induction on the demension of V.

First suppose that $[V: C_M(P)] = 0$, and let $p\in P$, $p \neq 0$. Then $V \neq 0$ and we choose an element $m\in M$ such that $m\gamma p \neq 0$; the existence of such an m is guaranteed by the irreduciblity of P.

Now suppose $[V: C_M(P)] > 0$ and let $V = V_0 \oplus C_M(P)\gamma w$, $w \neq 0$, $w\notin V_0$. Then $[V_0: C_M(P)] = [V: C_M(P)] - 1$ and we assume that the statement holds for V_0, that is, we assume that for any $p\in P$, $p\notin V_0$, then there is an $m\in M$ such that $m\gamma V_0=0$ and $m\gamma p \neq 0$ for some $\gamma\in\Gamma$.

Let $l(V_0)$ be the left annihilator of V_0 in M. Then if $l(V_0)\gamma p = 0$ for all $p\in P$, we must have $p\in V_0$. $l(V_0)$ is a left ideal of M and so $l(V_0)$ is a sub ΓM-module of P. Since $w\notin V_0$, $l(V_0)\gamma w \neq 0$, so we have $l(V_0)\gamma w = P$.

Suppose the desired result does not hold. Then there is a $p\in P$, $p\notin V$, such that $m\gamma p = 0$ whenever $m\gamma V = 0$. Define $T_m:P \to P$ by $T_m\gamma p = m\gamma p$. It is clear that T_m is well defined. Clearly $T_m\in E(P)$. If $x = m\gamma w$, $m\in l(V_0)$ and $m_1\in M$, then $m_1\gamma x = m_1\gamma(m\gamma w) = (m_1\gamma m)\gamma w$.

Thus $T_m\gamma(m_1\gamma x) = m\gamma(m_1\gamma x)= m_1\gamma(T_m\gamma x)$.

Therefore $T_m\in C_M(P)$. Hence if $m\in l(V_0)$, we have $m\gamma x = T_m\gamma(m\gamma w) = m\gamma(T_m\gamma w)$, so that $m\gamma x - m\gamma(T_m\gamma w) = 0$ So $m\gamma(x - T_m\gamma w) = 0$.Thus $x - T_m\gamma w\in V_0$ and so $x\in V_0 \oplus C_M(P)\gamma w = V$, a contradiction. Hence the theorem is proved.

4.10 Definition. Let M be a Γ-ring of linear Γ-tranformations on a left Γ-vector space over a division Γ-ring Δ. Then M is called **k-fold transitive** if given $i \leq k$ and any $v_1, v_2, \ldots, v_i \in V$ linearly Γ-independent over Δ, and any $w_1, w_2, \ldots, w_i \in V$, there is an $m\in M$ such that $m\gamma v_1 = w_1, m\gamma v_2 = w_2, \ldots, m\gamma v_i = w_i$.

In this terminology the Jacobson Density Theorem for Γ-rings say that under the hypothesis of that theorem, M is n-fold transitive for any finite n less than or equal to the dimension of P over $C_M(P)$. We have the following strong converse of the Density Theorem for Jacobsion Γ-rings.

4.11 Theorem. Let M be a two-fold transitive Γ-ring of linear Γ-transformations on a non trivial left Γ-vector space over a division Γ-ring Δ. Then V is an irreducible left ΓM-module, M is dense on V and $\Delta = C_M(V)$.

Proof. Let $v\in V$, $v \neq 0$. Since M is one-fold transitive, given any $w\in V$, then there is an element $m\in M$ such that $m\gamma v = w$ for some $\gamma\in\Gamma$. But this implies that V is an irreducible left ΓM-module. We consider the elements of Δ as linear Γ-transformations of V by identifying an element with the left translation by that element. Since $\delta\gamma(m\gamma v) = m\gamma(\delta\gamma v)$ for all $v\in V$, $m\in M$, $\delta\in\Delta$ and some $\gamma\in\Gamma$, then $\Delta\subset C_M(V)$.

Suppose we have $\varphi\in C_M(V)$, $\varphi\notin\Delta$. Let $v\in V$, $v \neq 0$. Suppose v and $(\varphi\gamma v)$ are not linearly Γ-independent, i. e., suppose we can find $\delta_1, \delta_2 \in\Delta$, not both zero, such that $\delta_1\gamma v + \delta_2\gamma(\varphi\gamma v)=0$. If $\delta_1=0$, then $\delta_2\gamma(\varphi\gamma v)=0$ and $\delta_2 \neq 0$. Hence $\varphi\gamma v = 0$ and since $\varphi \neq 0$ and V is irreducible, then $v = 0$, a contradiction. Thus $\delta_1\neq 0$. Similarly $\delta_2 \neq 0$.

Let $\delta = -\delta_1^{-1}\gamma\delta_2$ for some $\gamma\in\Gamma$ so that $v = \delta\gamma(\varphi\gamma v)$. Thus $v - \delta\gamma(\varphi\gamma v) = 0$. So $v - (\varphi\gamma\delta)\gamma v = 0$. Therefore $(1-\varphi\gamma\delta)\gamma v = 0$. If $(1-\varphi\gamma\delta)\in\Delta$, then $\varphi\in\Delta$, so $(1-\varphi\gamma\delta)\notin\Delta$ and hence $1-\varphi\gamma\delta \neq 0$. So as before $v = 0$, a contradiction.

Since v and $(\varphi\gamma v)$ are thus linearly Γ-independent, then there is an $m\in M$ such that $m\gamma v = 0$ and $m\gamma(\varphi\gamma v) = v$. However, since $\varphi\in C_M(V)$, then we have

$v = m\gamma(\varphi\gamma v) = \varphi\gamma(m\gamma v) = \varphi\gamma 0 = 0$, a contradiction. Therefore, we can not have $\varphi\in C_M(V) \setminus \Delta$. Hence $C_M(V) = \Delta$. Thus the theorem is proved.

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