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A New Concept on Fuzzy Soft $\mathcal{R}_1$ Spaces

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ABSTRACT

The primary goal of this article is to expose the four concepts of fuzzy soft $\mathcal{R}_1$ spaces that underlie the comprehension of fuzzy topological spaces. We then go into some new theories and the implications of such spaces. Additionally, authors like speculating on the relationships between these ideas and offering fresh perspectives on certain traits.

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1. Introduction

Undoubtedly, the more generalized version of ordinary set theory, such as fuzzy set theory, was initially presented in 1965 by the American mathematician L. A. Zadeh [1]. This type of set is used in Boolean algebra [2-4], computer science [5-6], and health science [7-8], as well as in other specific subjects that benefit from it; additionally, it is frequently utilized to create many of our home appliances [9-11]. The idea behind this set can clearly convey the degree of inclusion and ambiguity of each part in it, but it is too intricate to show which elements are not members. In order to shed light on these constraints, K. T. Atanassov [12] first proposed the concept of intuitionistic fuzzy sets in 1983. This concept gives us a clear understanding of the grade of inclusion and exclusion of any element and may be used to demonstrate the ambiguity and uncertainties in supplemental situations. Nevertheless, this category of theory has the limitation of being unable to provide an approximate description of an object. Therefore, in 1999, Russian mathematician Dmitri

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Molodtsov [13] introduced the idea of soft set theory with a set of parameters that can handle any ambiguity that occurs in our lives in order to overcome this challenge. Following the development of soft set theory's tenets, numerous academics [14-18] worked tirelessly to expand and enhance this discipline. For instance, Maji *et al.* [19] defined a fuzzy soft set as the combination of a soft set and a fuzzy set in 2001, and Maji and Roy [20] also mentioned the use of soft sets in a decision-making problem in 2002. Thereafter, this theory helps to develop soft topological structures, and in the next decade, several scholars [21-25] extended it to fuzzy soft topology. After the widespread acceptance of fuzzy soft topology, Salem Al-Wosabi and Amani Al-Salemi [26] introduced and studied some new properties of fuzzy soft R_0 and R_1 spaces by using quasi-coincident relation for fuzzy soft points in 2018, and at the same time Kandil *et al.* [27] defined the notion of fuzzy soft regularity axioms (particularly FSR_i; $i = 0, 1, 2, 3$ axioms) and the notion of fuzzy soft hereditary property. Consequently, Alias B. Khalaf and Qumri H. Hamko [28] introduced some soft separation axioms called soft R_0 and soft R_1 axioms in soft topological spaces, which are defined over an initial universe with a fixed set of parameters in 2022.

In this study, we established a relationship between the four viewpoints of soft $\mathcal{R}_1$ space in fuzzy soft topological conception and attempted to refine a number of associated theorems. Lastly, we try to depict some theorems from various perspectives for these kinds of structures and offer novel concepts for some features, such as "Good Extension," "Hereditary," and "Topological" properties. The remaining of this paper is accomplished in the following way: Section two reveals some key notations, fundamental operations (union, intersection, and complement), and the mappings of fuzzy soft sets. We mentioned the four conceptions of $FS - \mathcal{R}_1$ spaces in section three and built up an implication among them, and section four discusses three main principles of $FS - \mathcal{R}_1$ spaces as "Good Extension," "Hereditary," and "Topological" principles. Finally, some conclusions and our further study are referred to in section 5.

2. Notations and some fundamental operations on fuzzy soft sets

This section discusses several fundamental concepts and simple operations related to soft sets, fuzzy soft sets (a hybrid of fuzzy and soft sets), fuzzy soft topologies, fuzzy soft mappings, and so on. We make use of the symbol $\mathcal{D}$ as a universe of discourse, $P(\mathcal{D})$ as a power set of $\mathcal{D}$, $\mathcal{E}$ as a set of parameters and $\Sigma, \beta \sqsubseteq \mathcal{E}$. Here σ is the mapping from the parameters set Σ to the power set of $\mathcal{D}$, which refers to the soft set, and if ρ maps from the parameters set Σ to the fuzzy set μ in $\mathcal{C}(\mathcal{D})$, then the ordered pair $(\rho, \Sigma) = \{(\eta, \mu): \eta \in \Sigma\}$ is known as the fuzzy soft set in $\mathcal{D}$. In this situation, we only display the symbol (ρ, Σ) as a fuzzy soft set instead of $\{(\eta, \mu): \eta \in \Sigma\}$. Also, for the fuzzy soft topological space, soft topological space, and fuzzy topological space, the writers indicated the terms $(\rho_\Sigma, \ \delta)$, (σ_Σ, τ), and $(\mathcal{D}, t)$ respectively.

Definition 2.1. [13] Over a universe of discourse $\mathcal{D}$ a pair (σ, Σ) is called soft set (SS). Where $\sigma: \Sigma \longrightarrow P(\mathcal{D})$ and $\Sigma \sqsubseteq \mathcal{E}$ is the set of parameters. We meant it by σ_Σ.

Definition 2.2. [17] A null SS (σ, Σ) over $\mathcal{D}$ is used for $\emptyset_{\mathcal{D}}$ and represented as $\emptyset_{\mathcal{D}}(\eta) = \emptyset$ and an absolute SS over $\mathcal{D}$ is used for $\sigma_{\mathcal{D}}$ and represented as $\sigma_{\mathcal{D}}(\eta) = \mathcal{D}$ for every $\eta \in \mathcal{E}$.

Definition 2.3. [24] Suppose that $\mathcal{D}$ is universal set and $\mathcal{E}$ is the parameter's set. Then the mapping ρ_Σ from the parameter set Σ to all the fuzzy set in $\mathcal{C}(\mathcal{D})$, where $\Sigma \sqsubseteq \mathcal{E}$, that implies $\rho_\Sigma: \Sigma \longrightarrow \mathcal{C}(\mathcal{D})$ is called the fuzzy soft set in $\mathcal{D}$ and we denoted it by ρ_Σ. So by the set theoretical assumption we represent an ordinary soft set (SS) $\sigma_\Sigma(\eta)$ over a universe of discourse $\mathcal{D}$ as a fuzzy soft set (FSS) by a characteristics function (κ) in the following way:

$$\rho_\Sigma(\eta)(d) = \kappa_{\sigma_\Sigma(\eta)}(d) = \begin{cases} 1, & \text{if } d \in \sigma_\Sigma(\eta); \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4. [21] A *FSS* ρ_Σ is referred to a fuzzy soft subset of the *FSS* γ_β if $\rho_\Sigma(\eta) \preccurlyeq \gamma_\beta(\eta)$, for every $\eta \in \mathcal{E}$ and we denoted it by $\rho_\Sigma \sqsubseteq \gamma_\beta$.

Definition 2.5. [21] The union of two *FSSs* over $\mathcal{D}$ is signified by $\hbar_\alpha = \rho_\Sigma \sqcup \gamma_\beta$ and prescribed as $\hbar_\alpha(\eta) = \rho_\Sigma(\eta) \curlyvee \gamma_\beta(\eta)$ for each $\eta \in \mathcal{E}$, where $\alpha = \Sigma \sqcup \beta$. In the same manner, the intersection is signified by $\hbar_\alpha = \rho_\Sigma \overline{\sqcap} \gamma_\beta$ and prescribed as $\hbar_\alpha(\eta) = \rho_\Sigma(\eta) \curlywedge \gamma_\beta(\eta)$ for each $\eta \in \mathcal{E}$, where $\alpha = \Sigma \overline{\sqcap} \beta$.

Definition 2.6. [21] A null *FSS* $\rho_\mathcal{E}$ over $\mathcal{D}$ is used for $0_\mathcal{E}$ and represented as $\rho_\mathcal{E}(\mathcal{E}) = 0_\mathcal{D}$, and an absolute *FSS* over $\mathcal{D}$ is used for $1_\mathcal{E}$ and represented as $\rho_\mathcal{E}(\mathcal{E}) = 1_\mathcal{D}$ for every $d \in \mathcal{D}$. Note that, we write the complement of *FSS* $\rho_\mathcal{E}$ is $\rho_\mathcal{E}^c(\eta) = 1 - \rho_\mathcal{E}(\eta)$ for each $\eta \in \mathcal{E}$.

Definition 2.7. [25] Consider $FSS(\mathcal{D}_\mathcal{E})$ is the collection of all *FSS* over $\mathcal{D}$, $\mathcal{E}$ is the collection of all parameters and $\delta \sqsubseteq FSS(\mathcal{D}_\mathcal{E})$. If δ meets the following criteria, it is referred to as the fuzzy soft topology *(FST)* on $\mathcal{D}$:

(i) $0_\mathcal{E}, 1_\mathcal{E} \in \delta$,
(ii) δ is closed under finite intersection of members in δ, and
(iii) δ is the arbitrary union of members in δ.

The ordered pair $(\rho_\mathcal{E}, \delta)$ is mentioned as a fuzzy soft topological space *(FSTS)*. The members of δ are known as fuzzy soft open sets *(FSOS)* while the fuzzy soft closed sets *(FSCS)* are the complement of members of δ with regard to $\rho_\mathcal{E}$.

Definition 2.8. [21] Suppose that $\omega_1: \mathcal{D} \to \mathcal{Q}$ and $\omega_2: \mathcal{E} \to \mathcal{F}$ are two mappings, where $\mathcal{E}$ and $\mathcal{F}$ are parameter sets for the universal sets $\mathcal{D}$ and $\mathcal{Q}$, respectively. Then this ordered pair (ω_1, ω_2) is called a fuzzy soft mapping (*FSM*) from $(\mathcal{D}, \mathcal{E})$ into $(\mathcal{Q}, \mathcal{F})$ and denoted by $(\omega_1, \omega_2): (\mathcal{D}, \mathcal{E}) \longrightarrow (\mathcal{Q}, \mathcal{F})$.

Definition 2.9. [21] If ρ_Σ and γ_β are the two *FSSs* over the universal sets $\mathcal{D}$ and $\mathcal{Q}$, respectively and the ordered pair (ω_1, ω_2) is a *FSM* from $(\mathcal{D}, \mathcal{E})$ into $(\mathcal{Q}, \mathcal{F})$. Then we represented the image and the pre-image of ρ_Σ and γ_β by $(\omega_1, \omega_2)(\rho_\Sigma)$ and $(\omega_1, \omega_2)^{-1}(\gamma_\beta)$ respectively, and prescribed as the following way:

(a) $$(\omega_1, \omega_2)(\rho_\Sigma)\mathfrak{f}(q) = \begin{cases} \vee\omega_1(d) = q \vee \omega_2(\eta) = \mathfrak{f}\rho_\Sigma(\eta)(d), & \text{if } \omega_1^{-1}(q) \neq \emptyset, \omega_2^{-1}(\mathfrak{f}) \neq \emptyset; \\ 0, & \text{otherwise} \end{cases}$$

For all $\mathfrak{f} \in \mathcal{F}$, for all $q \in \mathcal{Q}$.

(b) $(\omega_1, \omega_2)^{-1}(\gamma_\beta)(\eta)(d) = \gamma_\beta\big(\omega_2(\eta)\big)\big(\omega_1(d)\big)$, for all $\eta \in \mathcal{E}$, for all $d \in \mathcal{D}$.

3. Definitions and properties of fuzzy soft $\mathcal{R}_1$ spaces

First of all, we mention $\mathcal{R}_1$ space for the fuzzy topological space, and then we bring up the four notions of FS $\mathcal{R}_1$ space, build up a relationship among them, and boom, several hypotheses interact with them from a new angle. Note that the grade of inclusion of any element in the *FSS* is $1_{\mathcal{D}}$ and the grade of exclusion of any element in the *FSS* is $0_{\mathcal{D}}$. Also, the authors take into account a number $\pi_{\mathcal{D}}$ that exists between $0_{\mathcal{D}}$ and $1_{\mathcal{D}}$.

Definition 3.1. [29] A fuzzy topological space $(\mathcal{D}, t)$ is called $\mathcal{R}_1$ space if $\forall\, d_1, d_2 \in \mathcal{D},\ d_1 \neq d_2$, whenever there is an FSOS $\mathcal{W} \in t$ with either $\mathcal{W}(d_1) = 1$ and $\mathcal{W}(d_2) = 0$ or $\mathcal{W}(d_1) = 0$ and $\mathcal{W}(d_2) = 1$, then there exist $\mathcal{U}, \mathcal{V} \in t$ such that $d_1 \in \mathcal{U}, d_2 \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = 0$.

Definition 3.2. If $(\rho_{\mathcal{E}},\ \delta)$ is a *FSTS* over a universe of discourse $\mathcal{D}$, then such space is called:

(a) $FS-\mathcal{R}_1(i)$: $\forall\, d_1, d_2 \in \mathcal{D},\ d_1 \neq d_2$, *and for all* $\eta \in \Sigma \sqsubseteq \mathcal{E}$, *whenever* $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, *then then there exist FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ *such that* $\rho_{1\Sigma}(\eta)(d_1) = 1_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \rho_{2\Sigma}(\eta)(d_2) = 1_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}$, *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

(b) $FS-\mathcal{R}_1(ii)$: $\forall\, d_1, d_2 \in \mathcal{D},\ d_1 \neq d_2$, *and for all* $\eta \in \Sigma \sqsubseteq \mathcal{E}$, *whenever* $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, *then then there exist FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ *such that* $\rho_{1\Sigma}(\eta)(d_1) = \pi_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \rho_{2\Sigma}(\eta)(d_2) = \pi_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}$, *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

(c) $FS-\mathcal{R}_1(iii)$: $\forall\, d_1, d_2 \in \mathcal{D},\ d_1 \neq d_2$, *and for all* $\eta \in \Sigma \sqsubseteq \mathcal{E}$, *whenever* $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, *then then there exist FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ *such that* $\rho_{1\Sigma}(\eta)(d_1) > \rho_{2\Sigma}(\eta)(d_1)$; $\rho_{2\Sigma}(\eta)(d_2) > \rho_{1\Sigma}(\eta)(d_2)$ *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

(d) $FS-\mathcal{R}_1(iv)$: $\forall\, d_1, d_2 \in \mathcal{D},\ d_1 \neq d_2$, *and for all* $\eta \in \Sigma \sqsubseteq \mathcal{E}$, *whenever* $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, *then then there exist FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ *such that* $\rho_{1\Sigma}(\eta)(d_1) \neq \rho_{2\Sigma}(\eta)(d_1)$ *and* $\rho_{2\Sigma}(\eta)(d_2) \neq \rho_{1\Sigma}(\eta)(d_2)$ *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

Theorem 3.2. If $(\rho_\Sigma,\ \delta)$ is a fuzzy soft topological space (FSTS), $\mathcal{E}$ is the set of all parameters and $\Sigma \sqsubseteq \mathcal{E}$. Then these four notions are related in the following way.

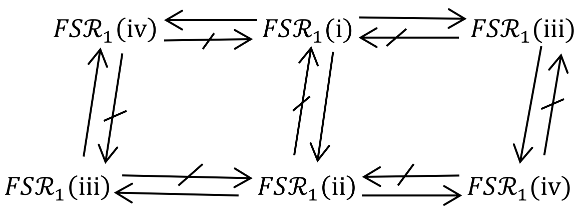

Figure: Sketch of four notions of $FS-\mathcal{R}_1$ space

Proof: Suppose that, $(\rho_\Sigma,\ \delta)$ is a $FS-\mathcal{R}_1(i)$. So we have for every pair of $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, then then there exist *FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ such that $\rho_{1\Sigma}(\eta)(d_1) = 1_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \rho_{2\Sigma}(\eta)(d_2) = 1_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}$, *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) = \pi_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \\ \rho_{2\Sigma}(\eta)(d_2) = \pi_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}, \text{as } 0_{\mathcal{D}} < \pi_{\mathcal{D}} < 1_{\mathcal{D}} \\ \quad\quad \textit{and } \rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{1}$$

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) > \rho_{2\Sigma}(\eta)(d_1); \\ \rho_{2\Sigma}(\eta)(d_2) > \rho_{1\Sigma}(\eta)(d_2) \\ \quad and\ \rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{2}$$

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) \neq \rho_{2\Sigma}(\eta)(d_1); \\ \rho_{2\Sigma}(\eta)(d_2) \neq \rho_{1\Sigma}(\eta)(d_2) \\ \quad and\ \rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{3}$$

Hence from (1), (2) and (3) the authors observe that $FS-\mathcal{R}_1(i) \Longrightarrow FS-\mathcal{R}_1(ii) \Longrightarrow FS-\mathcal{R}_1(iii) \Longrightarrow FS-\mathcal{R}_1(iv)$. Again suppose that $(\rho_\Sigma,\ \delta)$ is a $FS-\mathcal{R}_1(i)$. So it provides that, for each $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, then then there exist *FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ such that $\rho_{1\Sigma}(\eta)(d_1) = 1_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \rho_{2\Sigma}(\eta)(d_2) = 1_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}$, *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) > \rho_{2\Sigma}(\eta)(d_1); \\ \rho_{2\Sigma}(\eta)(d_2) > \rho_{1\Sigma}(\eta)(d_2) \\ \quad and\ \rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{4}$$

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) \neq \rho_{2\Sigma}(\eta)(d_1); \\ \rho_{2\Sigma}(\eta)(d_2) \neq \rho_{1\Sigma}(\eta)(d_2) \\ \quad and\ \rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{5}$$

Hence the equations (4) and (5) represent that $FS-\mathcal{R}_1(i) \Longrightarrow FS-\mathcal{R}_1(iii)$, and $FS-\mathcal{R}_1(i) \Longrightarrow FS-\mathcal{R}_1(iv)$.

Finally if we consider $(\rho_\Sigma,\ \delta)$ is $FS-\mathcal{R}_1(i)$, then the equation (1) tell us for each $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, then then there exist *FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ such that $\rho_{1\Sigma}(\eta)(d_1) = \pi_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \rho_{2\Sigma}(\eta)(d_2) = \pi_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}$, and $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$.

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) \neq \rho_{2\Sigma}(\eta)(d_1); \\ \rho_{2\Sigma}(\eta)(d_2) \neq \rho_{1\Sigma}(\eta)(d_2) \\ \quad and\ \rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{6}$$

From (6) we see that $FS-\mathcal{R}_1(ii) \Longrightarrow FS-\mathcal{R}_1(iv)$.

But the converse relationship does not always possible, and it is shown by the following counter examples:

Example 3.2.1. Let $\mathcal{D} = \{d_1,\ d_2\}, \mathcal{E} = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ a set of parameters, $\Sigma = \{\eta_1, \eta_2\} \subset \mathcal{E}$, and δ be a fuzzy soft topology on a universal set $\mathcal{D}$ generated by $\delta = \{0_{\mathcal{D}}, 1_{\mathcal{D}}, \rho_{1\Sigma}, \rho_{2\Sigma}, \rho_{3\Sigma}\}$ where $\rho_{3\Sigma} = \Big\{\eta_1 =$

$\left\{\frac{0.3}{d_1},\frac{0.5}{d_2}\right\}, \eta_2 = \left\{\frac{0.4}{d_1},\frac{0.8}{d_2}\right\}\Bigg\}, \rho_{1\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0}{d_1},\frac{0.5}{d_2}\right\}, \eta_2 = \left\{\frac{0}{d_1},\frac{0.7}{d_2}\right\}\Bigg\}, \rho_{2\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0.5}{d_1},\frac{0}{d_2}\right\}, \eta_2 = \left\{\frac{0.7}{d_1},\frac{0}{d_2}\right\}\Bigg\}$. Here we see that for any pair of $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever $\exists$ *a FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, then then there exist *FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ such that $\rho_{1\Sigma}(\eta)(d_1) \neq \rho_{2\Sigma}(\eta)(d_1)$, $\rho_{2\Sigma}(\eta)(d_2) \neq \rho_{1\Sigma}(\eta)(d_2)$, and $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$. Hence we observe that (ρ_Σ, δ) is $FS-\mathcal{R}_1(iv)$ but not $FS-\mathcal{R}_1(i), FS-\mathcal{R}_1(ii)$, and $FS-\mathcal{R}_1(iii)$. Therefore $FS-\mathcal{R}_1(iv) \nRightarrow FS-\mathcal{R}_1(i), FS-\mathcal{R}_1(iv) \nRightarrow FS-\mathcal{R}_1(ii)$, and $FS-\mathcal{R}_1(iv) \nRightarrow FS-\mathcal{R}_1(iii)$.

Example 3.2.2. Let $\mathcal{D} = \{d_1, d_2\}, \mathcal{E} = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ a set of parameters, $\Sigma = \{\eta_1, \eta_2\} \subset \mathcal{E}$, and δ be a fuzzy soft topology on a universal set $\mathcal{D}$ generated by $\delta = \{0_{\mathcal{D}}, 1_{\mathcal{D}}, \rho_{1\Sigma}, \rho_{2\Sigma}, \rho_{3\Sigma}\}$, where $\rho_{3\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0.3}{d_1},\frac{0.5}{d_2}\right\}, \eta_2 = \left\{\frac{0.4}{d_1},\frac{0.8}{d_2}\right\}\Bigg\}, \rho_{1\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0.7}{d_1},\frac{0}{d_2}\right\}, \eta_2 = \left\{\frac{0.7}{d_1},\frac{0}{d_2}\right\}\Bigg\}, \rho_{2\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0}{d_1},\frac{0.5}{d_2}\right\}, \eta_2 = \left\{\frac{0}{d_1},\frac{0.5}{d_2}\right\}\Bigg\}$. Here we have for all $\eta \in \Sigma$, $\rho_{1\Sigma}(\eta)(d_1) > \rho_{2\Sigma}(\eta)(d_1)$ and $\rho_{2\Sigma}(\eta)(d_2) > \rho_{1\Sigma}(\eta)(d_2)$, and $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_{\mathcal{D}}$. Hence we see that (ρ_Σ, δ) is $FS-\mathcal{R}_1(iii)$ but not $FS-\mathcal{R}_1(i)$, and $FS-\mathcal{R}_1(ii)$. Henceforth $FS-\mathcal{R}_1(iii) \nRightarrow FS-\mathcal{R}_1(i)$, and $FS-\mathcal{R}_1(iii) \nRightarrow FS-\mathcal{R}_1(ii)$.

Finally, if we consider $\rho_{3\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0.3}{d_1},\frac{0.5}{d_2}\right\}, \eta_2 = \left\{\frac{0.4}{d_1},\frac{0.8}{d_2}\right\}\Bigg\}, \rho_{1\Sigma} = \Bigg\{\eta_1 = \left\{\frac{\pi_{\mathcal{D}}}{d_1},\frac{0_{\mathcal{D}}}{d_2}\right\}, \eta_2 = \left\{\frac{\pi_{\mathcal{D}}}{d_1},\frac{0_{\mathcal{D}}}{d_2}\right\}\Bigg\}$ and $\rho_{2\Sigma} = \Bigg\{\eta_1 = \left\{\frac{0_{\mathcal{D}}}{d_1},\frac{\pi_{\mathcal{D}}}{d_2}\right\}, \eta_2 = \left\{\frac{0_{\mathcal{D}}}{d_1},\frac{\pi_{\mathcal{D}}}{d_2}\right\}\Bigg\}$ as $0_{\mathcal{D}} < \pi_{\mathcal{D}} < 1_{\mathcal{D}}$ then we have (ρ_Σ, δ) is $FS-\mathcal{R}_1(ii)$ but not $FS-\mathcal{R}_1(i)$.

4. Extended idea on some fundamental properties

The authors attempt to explore several innovative thoughts and improve some hypotheses related to them on three key attributes in the following:

Definition 4.1. If (σ_Σ, τ) is a soft topological space, $\delta = \{1_{\sigma_\Sigma} : \sigma_\Sigma \in \tau\}$, and $1_{\sigma_\Sigma} = \rho_{1\Sigma}$. Then we consider (ρ_Σ, δ) as the corresponding *FSTS* of (σ_Σ, τ). Let $\mathcal{FP}$ be the fuzzy soft topological counterpart of a property $\mathcal{P}$ of soft topological spaces. Then $\mathcal{FP}$ is referred to as a "good extension" of $\mathcal{P}$ if and only if the assertion (σ_Σ, τ) contains $\mathcal{P}$. This is true for all *STSs* (σ_Σ, τ).

Theorem 4.1. If (σ_Σ, τ) is a soft $\mathcal{R}_1$ space and if (ρ_Σ, δ) is $FS-\mathcal{R}_1(j)$ spaces for $j = i, ii, iii, iv$. Then (σ_Σ, τ) will be $FS-\mathcal{R}_1(j)$ spaces if and only if $FS-\mathcal{R}_1(j)$ will be also a soft $\mathcal{R}_1$ space.

Proof: Let (σ_Σ, τ) be a soft $\mathcal{R}_1$ ($S\mathcal{R}_1$) space. It will be proved that such space is $FS-\mathcal{R}_1(j)$ spaces. Since (σ_Σ, τ) is soft $\mathcal{R}_1$ space, this provides for each $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever there exist a soft open sets (SOS) $\sigma_{3\Sigma} \in \tau$ with $\sigma_{3\Sigma}(d_1) \neq \sigma_{3\Sigma}(d_2)$, then $\exists$ soft open sets (SOS's) $\sigma_{1\Sigma}, \sigma_{2\Sigma} \in \tau$ such that $d_1 \in \sigma_{1\Sigma}, d_2 \notin \sigma_{1\Sigma}; d_1 \notin \sigma_{2\Sigma}, d_2 \in \sigma_{2\Sigma}$ and $\sigma_{1\Sigma} \sqcap \sigma_{2\Sigma} = \emptyset_{\mathcal{D}}$ Then by a characteristics function 1_{σ_Σ} we have

$$\Rightarrow \begin{cases} 1_{\sigma_\Sigma}(\eta)(d_1) = 1_{\mathcal{D}}, \quad 1_{\sigma_\Sigma}(\eta)(d_2) = 0_{\mathcal{D}} \text{ and} \\ 1_{\sigma_\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}, 1_{\sigma_\Sigma}(\eta)(d_2) = 1_{\mathcal{D}} \end{cases}$$

Let $1_{\sigma_\Sigma} = (\rho_{1\Sigma}, \rho_{2\Sigma})$. So this gives

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) = 1_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \\ \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_2) = 1_{\mathcal{D}} \\ \text{and } \rho_{1\Sigma} \mathrel{\overline{\sqcap}} \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{1}$$

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) = \pi_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}} \\ \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_2) = \pi_{\mathcal{D}} \\ \text{and } \rho_{1\Sigma} \mathrel{\overline{\sqcap}} \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{2}$$

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) > \rho_{2\Sigma}(\eta)(d_1) \\ \rho_{2\Sigma}(\eta)(d_2) > \rho_{1\Sigma}(\eta)(d_2) \\ \textit{and } \rho_{1\Sigma} \mathrel{\overline{\sqcap}} \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{3}$$

$$\Rightarrow \begin{cases} \rho_{1\Sigma}(\eta)(d_1) \neq \rho_{2\Sigma}(\eta)(d_1) \\ \rho_{2\Sigma}(\eta)(d_2) \neq \rho_{1\Sigma}(\eta)(d_2) \\ \textit{and } \rho_{1\Sigma} \mathrel{\overline{\sqcap}} \rho_{2\Sigma} = 0_{\mathcal{D}}. \end{cases} \tag{4}$$

The equations (1), (2), (3), and (4) demonstrate that soft $\mathcal{R}_1$ space is $FS - \mathcal{R}_1(i), FS - \mathcal{R}_1(ii), FS - \mathcal{R}_1(iii)$ and $FS - \mathcal{R}_1(iv)$ spaces. That implies soft $\mathcal{R}_1$ space is $FS - \mathcal{R}_1(j)$ spaces, where $j = i, ii, iii, iv$.

Conversely, we assume that $(\rho_\Sigma, \ \delta)$ is $FS - \mathcal{R}_1(j)$ spaces. We prove that $(\rho_\Sigma, \ \delta)$ is a soft $\mathcal{R}_1$ space, and for this it will be proved only for $j = i$. Since $(\rho_\Sigma, \ \delta)$ is a $FS - \mathcal{R}_1(i)$. Then we get for each $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever there exists a *FSOS* $\rho_{3\Sigma} \in \delta$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, then there exist *FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta$ such that $\rho_{1\Sigma}(\eta)(d_1) = 1_{\mathcal{D}}, \rho_{1\Sigma}(\eta)(d_2) = 0_{\mathcal{D}}; \rho_{2\Sigma}(\eta)(d_2) = 1_{\mathcal{D}}, \rho_{2\Sigma}(\eta)(d_1) = 0_{\mathcal{D}}$, *and* $\rho_{1\Sigma} \mathrel{\overline{\sqcap}} \rho_{2\Sigma} = 0_{\mathcal{D}}$.

$$\Rightarrow \begin{cases} \rho_{1\Sigma}^{-1}(\eta)(1_{\mathcal{D}}) = \{d_1\}, \rho_{1\Sigma}^{-1}(\eta)(0_{\mathcal{D}}) = \{d_2\} \quad \text{and} \\ \rho_{2\Sigma}^{-1}(\eta)(0_{\mathcal{D}}) = \{d_1\}, \rho_{2\Sigma}^{-1}(\eta)(1_{\mathcal{D}}) = \{d_2\} \end{cases}$$

Let $\rho_{1\Sigma}^{-1}(1_{\mathcal{D}}) = \sigma_{1\Sigma}$ and $\rho_{2\Sigma}^{-1}(1_{\mathcal{D}}) = \sigma_{2\Sigma}$. Therefore $\sigma_{1\Sigma}(\eta) = \{d_1\}$ and $\sigma_{2\Sigma}(\eta) = \{d_2\}$. Therefore, if for each $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2$, and for all $\eta \in \Sigma$, whenever there exist a soft open sets (SOS) $\sigma_{3\Sigma} \in \tau$ with $\sigma_{3\Sigma}(d_1) \neq \sigma_{3\Sigma}(d_2)$, then $\exists$ soft open sets (SOS's) $\sigma_{1\Sigma}, \sigma_{2\Sigma} \in \tau$ such that $d_1 \in \sigma_{1\Sigma}, d_2 \notin \sigma_{1\Sigma}; d_1 \notin \sigma_{2\Sigma}, d_2 \in \sigma_{2\Sigma}$ and $\sigma_{1\Sigma} \mathrel{\overline{\sqcap}} \sigma_{2\Sigma} = \emptyset_{\mathcal{D}}$. Hence $FS - \mathcal{R}_1(i)$ is $S\mathcal{R}_1$. In similar manner, $FS - \mathcal{R}_1(ii), FS - \mathcal{R}_1(iii)$ and $FS - \mathcal{R}_1(iv)$ imply $S\mathcal{R}_1$ space.

Definition 4.2. If $(\rho_\Sigma, \ \delta)$ is a *FSTS* and $\gamma_\Sigma \sqsubseteq \rho_\Sigma$ then the expression $\delta_{\gamma\Sigma} = \{\gamma_\Sigma \mathrel{\overline{\sqcap}} \hbar_\Sigma | \hbar_\Sigma \in \delta\}$ is called fuzzy soft subspace topology and $(\gamma_\Sigma, \delta_{\gamma\Sigma})$ is referred as a fuzzy soft subspace of $(\rho_\Sigma, \ \delta)$. If we mention the property $'P'$ in such a way that it holds both for the fuzzy soft topological space (*FSTS*) and the subspace of its, then such property is called 'hereditary'.

Theorem 4.2. If $(\gamma_\Sigma, \delta_{\gamma\Sigma})$ is a subspace of a FSTS (ρ_Σ, δ), then it is $\mathrm{FS} - \mathcal{R}_1(\mathrm{j})$ implies $(\gamma_\Sigma, \delta_{\gamma\Sigma})$ is also $\mathrm{FS} - \mathcal{R}_1(\mathrm{j})$ for $\mathrm{j} = \mathrm{i}, \mathrm{ii}, \mathrm{iii}, \mathrm{iv}$.

Proof: We prove this, solely for $j = i$. Suppose that (ρ_Σ, δ) is $FS - \mathcal{R}_1(i)$, it will be shown that $(\gamma_\Sigma, \delta_{\gamma\Sigma})$ is $FS - \mathcal{R}_1(i)$. Let $d_1, d_2 \in \mathcal{D}$ with $d_1 \neq d_2$, and for all $\eta \in \Sigma$ whenever if $\gamma_{3\Sigma}(\eta)(d_1) \neq \gamma_{3\Sigma}(\eta)(d_2)$, then $\gamma_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}, \gamma_{1\Sigma}(\eta)(d_2) = 0_\mathcal{D}$ and $\gamma_{2\Sigma}(\eta)(d_1) = 0_\mathcal{D}, \gamma_{2\Sigma}(\eta)(d_2) = 1_\mathcal{D}$. Again since (ρ_Σ, δ) is $FS - \mathcal{R}_1(i)$, hence for every $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2,$ and for every $\eta \in \Sigma$, whenever if there exists a *FSOS* $\hbar_{3\Sigma} \in \delta$ *with* $\hbar_{3\Sigma}(\eta)(d_1) \neq \hbar_{3\Sigma}(\eta)(d_2)$, then there exist FSOSs $\hbar_{1\Sigma}, \hbar_{2\Sigma} \in \delta$ such that $\hbar_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}, \hbar_{1\Sigma}(\eta)(d_2) = 0_\mathcal{D}; \hbar_{2\Sigma}(\eta)(d_1) = 0_\mathcal{D}, \hbar_{2\Sigma}(\eta)(d_2) = 1_\mathcal{D}$, and $\hbar_{1\Sigma} \sqcap \hbar_{2\Sigma} = 0_\mathcal{D}$. Since $\gamma_{3\Sigma}(\eta)(d_1) \neq \gamma_{3\Sigma}(\eta)(d_1)$ and $\hbar_{3\Sigma}(\eta)(d_1) \neq \hbar_{3\Sigma}(\eta)(d_2)$, it can be written as $(\gamma_{3\Sigma} \sqcap \hbar_{3\Sigma})(\eta)(d_1) \neq (\gamma_{3\Sigma} \sqcap \hbar_{3\Sigma})(\eta)(d_2)$. Again since, $\gamma_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}$ and $\hbar_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}$, hence from we have, $(\gamma_{1\Sigma} \sqcap \hbar_{1\Sigma})(\eta)(d_1) = 1_\mathcal{D}$. Similarly $(\gamma_{1\Sigma} \sqcap \hbar_{1\Sigma})(\eta)(d_2) = 0_\mathcal{D}$, and $(\gamma_{2\Sigma} \sqcap \hbar_{2\Sigma})(\eta)(d_1) = 0_\mathcal{D}, (\gamma_{2\Sigma} \sqcap \hbar_{2\Sigma})(\eta)(d_2) = 1_\mathcal{D}$. So we observe that, whenever if there exists a *FSOS* $(\gamma_{3\Sigma} \sqcap \hbar_{3\Sigma}) \in \delta_{\gamma\Sigma}$ *with* $(\gamma_{3\Sigma} \sqcap \hbar_{3\Sigma})(\eta)(d_1) \neq (\gamma_{3\Sigma} \sqcap \hbar_{3\Sigma})(\eta)(d_2)$ then there exist FSOSs $(\gamma_{1\Sigma} \sqcap \hbar_{1\Sigma}), (\gamma_{2\Sigma} \sqcap \hbar_{2\Sigma}) \in \delta_{\gamma\Sigma}$ such that $(\gamma_{1\Sigma} \sqcap \hbar_{1\Sigma})(\eta)(d_1) = 1_\mathcal{D}$, $(\gamma_{1\Sigma} \sqcap \hbar_{1\Sigma})(\eta)(d_2) = 0_\mathcal{D}$; $(\gamma_{2\Sigma} \sqcap \hbar_{2\Sigma})(\eta)(d_1) = 0_\mathcal{D}, (\gamma_{2\Sigma} \sqcap \hbar_{2\Sigma})(\eta)(d_2) = 1_\mathcal{D}$, and $(\gamma_{1\Sigma} \sqcap \hbar_{1\Sigma}) \sqcap (\gamma_{2\Sigma} \sqcap \hbar_{2\Sigma}) = (\gamma_{1\Sigma} \sqcap \gamma_{2\Sigma}) \sqcap (\hbar_{1\Sigma} \sqcap \hbar_{2\Sigma}) = (0_\mathcal{D} \sqcap 0_\mathcal{D}) = 0_\mathcal{D}$. Hence $(\gamma_\Sigma, \delta_{\gamma\Sigma})$ is $FS - \mathcal{R}_1(i)$. For $j = ii, iii,$ and iv can be proved in similar way.

Theorem 4.3. Let (ρ_Σ, δ_1) and (γ_β, δ_2) be two fuzzy soft topological spaces (FSTS's), over the universal sets $\mathcal{D}$ and $\mathcal{Q}$, respectively. If $(\omega_1, \omega_2): (\rho_\Sigma, \delta_1) \longrightarrow (\gamma_\beta, \delta_2)$ is a fuzzy soft one-one, onto, and continuous map, then (ρ_Σ, δ_1) is $\mathrm{FS} - \mathcal{R}_1(\mathrm{j}) \Leftrightarrow (\gamma_\beta, \delta_2)$ is $\mathrm{FS} - \mathcal{R}_1(\mathrm{j})$ for $\mathrm{j} = \mathrm{i}, \mathrm{ii}, \mathrm{iii}, \mathrm{iv}$.

Proof: To begin with it will be proved just for $j = i$. Suppose that (ρ_Σ, δ_1) is $FS - \mathcal{R}_1(i)$. We prove that (γ_β, δ_2) is also $FS - \mathcal{R}_1(i)$. Consider that $q_1, q_2 \in \mathcal{Q}$ with $q_1 \neq q_2$. Since (ω_1, ω_2) is onto, then there exist $d_1, d_2 \in \mathcal{D}$ with $d_1 \neq d_2$, such that $\omega_1(d_1) = q_1, \omega_1(d_2) = q_2$ and $\omega_2(\eta) = \alpha$ for all parameters $\eta \in \Sigma$ and for all $\alpha \in \beta$. Hence $d_1 \neq d_2$ as ω_1 is one-one. Again since (ρ_Σ, δ_1) is $FS - \mathcal{R}_1(i)$, this gives for each $d_1, d_2 \in \mathcal{D}, d_1 \neq d_2,$ and for each $\eta \in \Sigma$, whenever if there exists a *FSOS* $\rho_{3\Sigma} \in \delta_1$ *with* $\rho_{3\Sigma}(\eta)(d_1) \neq \rho_{3\Sigma}(\eta)(d_2)$, then then there exist *FSOSs* $\rho_{1\Sigma}, \rho_{2\Sigma} \in \delta_1$ such that $\rho_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}, \rho_{1\Sigma}(\eta)(d_2) = 0_\mathcal{D}; \rho_{2\Sigma}(\eta)(d_2) = 1_\mathcal{D}, \rho_{2\Sigma}(\eta)(d_1) = 0_\mathcal{D},$ *and* $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_\mathcal{D}$. Now we have, whenever if there exists a FSOS $(\omega_1, \omega_2)(\rho_{3\Sigma}) \in \delta_2$ with $(\omega_1, \omega_2)(\rho_{3\Sigma})(\alpha)(q_1) \neq (\omega_1, \omega_2)(\rho_{3\Sigma})(\alpha)(q_2)$, then there exist FSOS's $(\omega_1, \omega_2)(\rho_{1\Sigma}), (\omega_1, \omega_2)(\rho_{2\Sigma}) \in \delta_2$ such that $(\omega_1, \omega_2)(\rho_{1\Sigma})(\alpha)(q_1) = \{\vee \omega_1(d_1) = q_1 \vee \omega_2(\eta) = \alpha \rho_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}$ as $\rho_{1\Sigma}(\eta)(d_1) = 1_\mathcal{D}$. Also $(\omega_1, \omega_2)(\rho_{1\Sigma})(\alpha)(q_2) = \{\vee\ \omega_1(d_2) = q_2 \vee \omega_2(\eta) = \alpha \rho_{1\Sigma}(\eta)(d_2) = 0_\mathcal{D}$ as $\rho_{1\Sigma}(\eta)(d_2) = 0_\mathcal{D}$. In the same way, we find $(\omega_1, \omega_2)(\rho_{2\Sigma})(\alpha)(q_1) = 0_\mathcal{D}$; $(\omega_1, \omega_2)(\rho_{2\Sigma})(\alpha)(q_2) = 1_\mathcal{D}$, and $(\omega_1, \omega_2)(\rho_{1\Sigma}) \sqcap (\omega_1, \omega_2)(\rho_{2\Sigma}) = (\omega_1, \omega_2)(\rho_{1\Sigma} \sqcap \rho_{2\Sigma}) = 0_\mathcal{D}$ as $\rho_{1\Sigma} \sqcap \rho_{2\Sigma} = 0_\mathcal{D}$. Which proves that (γ_β, δ_2) is $FS - \mathcal{R}_1(i)$.

Conversely suppose that (γ_β, δ_2) is $FS - \mathcal{R}_1(i)$. It will be proved that (ρ_Σ, δ_1) is $FS - \mathcal{R}_1(i)$. Let $d_1, d_2 \in \mathcal{D}$ with $d_1 \neq d_2$. This implies that $\omega_1(d_1) \neq \omega_1(d_2)$ as ω_1 is one-one. Put $\omega_1(d_1) = q_1$ and $\omega_1(d_2) = q_2$. Then $q_1 \neq q_2$. Since (γ_β, δ_2) is $FS - \mathcal{R}_1(i)$, whenever if there exists a *FSOS* $\gamma_{3\beta} \in \delta_2$ with $\gamma_{3\beta}(\alpha)(q_1) \neq \gamma_{3\beta}(\alpha)(q_2)$, then there exist *FSOSs* $\gamma_{1\beta}, \gamma_{2\beta} \in \delta_2$ such that $\gamma_{1\beta}(\alpha)(q_1) = 1_\mathcal{D}, \gamma_{1\beta}(\alpha)(q_2) = 0_\mathcal{D}$; $\gamma_{2\beta}(\alpha)(q_1) = 0_\mathcal{D}, \gamma_{2\beta}(\alpha)(q_2) = 1_\mathcal{D}$, and $\gamma_{1\beta} \sqcap \gamma_{2\beta} = 0_\mathcal{D}$. Now whenever if there exists a FSOS

$(\omega_1, \omega_2)^{-1}(\gamma_{3\beta}) \in \delta_1$ with $(\omega_1, \omega_2)^{-1}(\gamma_{3\beta})(\eta)(d_1) \neq (\omega_1, \omega_2)^{-1}(\gamma_{3\beta})(\eta)(d_2)$, then there exist FSOSs $(\omega_1, \omega_2)^{-1}(\gamma_{1\beta})$, $(\omega_1, \omega_2)^{-1}(\gamma_{2\beta}) \in \delta_1$ such that $(\omega_1, \omega_2)^{-1}(\gamma_{1\beta})(\eta)(d_1) = \gamma_{1\beta}(\omega_2(\eta))(\omega_1(d_1)) = \gamma_{1\beta}(\alpha)(q_1) = 1_{\mathcal{D}}$ and $(\omega_1, \omega_2)^{-1}(\gamma_{1\beta})(\eta)(d_2) = \gamma_{1\beta}(\omega_2(\eta))(\omega_1(d_2)) = \gamma_{1\beta}(\alpha)(q_2) = 0_{\mathcal{D}}$. And from the similar procedure, $(\omega_1, \omega_2)^{-1}(\gamma_{2\beta})(\eta)(d_1) = 0_{\mathcal{D}}, (\omega_1, \omega_2)^{-1}(\gamma_{2\beta})(\eta)(d_2) = 1_{\mathcal{D}}$. Also we can have the third condition in this way that, $(\omega_1, \omega_2)^{-1}(\gamma_{1\beta}) \,\bar{\sqcap}\, (\omega_1, \omega_2)^{-1}(\gamma_{2\beta}) = (\omega_1, \omega_2)^{-1}(\gamma_{1\beta} \,\bar{\sqcap}\, \gamma_{2\beta}) = 0_{\mathcal{D}}$ as $\gamma_{1\beta} \,\bar{\sqcap}\, \gamma_{2\beta} = 0_{\mathcal{D}}$. Hence (ρ_Σ, δ_1) is $FS - \mathcal{R}_1(i)$. In the same way, it will be held for $j = ii, iii, iv$.

5. Conclusions

This context comes up with four different categories of notions of fuzzy soft separations axioms (particularly $FS - \mathcal{R}_1$ axioms) from the new approaches and establishes a firm relationship among such notions. Next, we enhanced the innovation of different types of theorems and basic principles. For instance, consider these concepts are true for the soft topology and proved such properties in the fuzzy soft topology and vice versa, which followed the “good extension” properties. Such property has been shown in theorem 4.1. Furthermore, we showed that all these notions satisfied the “soft hereditary” property. To clarify this, we referred to theorem 4.2, which establishes that if these concepts hold for a fuzzy soft topological space, then they are also preserved for every subspace of it. Consequently, to illustrate the ‘topological’ property, we have mentioned theorem 4.3, and this theorem explains that if these inferences are true for any two fuzzy soft topological spaces, then these spaces hold for the continuous mapping of such spaces. In further studies, we have a plan to extend and explore these ideas as well as develop related theorems for intuitionistic fuzzy soft (*IFS*) $\mathcal{R}_1$ spaces.

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