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Application of the Sine-Gordon Expansion Method to Obtain Soliton Solution of the KdV and mKdV Equations

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ABSTRACT

In this study, we apply the Sine-Gordon Expansion Method (SGEM) to obtain exact travelling wave solutions of KdV and mKdV equations, which model dispersive nonlinear wave phenomena in fluids, plasma, and optical systems. By introducing an auxiliary Sin-Gordon type equation and assuming a travelling wave transform, the nonlinear partial differential equations are reduced to algebraic systems. Consequently, hyperbolic, trigonometric, and exponential explicit solutions are obtained. The mKdV equation produces bright, dark, and periodic soliton structures, whereas the KdV equation admits localized solitary wave solutions. Graphical analyses in two and three dimensions show how model parameters affect the amplitude, width, and velocity of solitons. The findings show that SGEM may be successfully extended to other nonlinear dispersive systems and provides an effective and unified analytical framework for building accurate soliton solutions.

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1. Introduction

In fluid dynamics, plasma physics, nonlinear optics, and solid-state physics, dispersive wave events are frequently modeled using nonlinear partial differential equations (NPDEs). The KdV and mKdV equations are examples of classical integrable models that explain soliton dynamics and nonlinear wave propagation. Soliton solutions, which are localized nonlinear waves that preserve their shape and velocity because of a balance between dispersion and nonlinearity, are a distinguishing feature of these equations. To understand nonlinear wave behavior and to verify theoretical and numerical models, it is crucial to derive precise soliton solutions.

Nomenclature

Symbol	Description
$\boldsymbol{x}$	Spatial variable
$\boldsymbol{t}$	Temporal variable
$\boldsymbol{u(x,t)}$	Wave profile (dependent variable)
$\boldsymbol{\xi}$	Traveling wave variable ($\xi = kx - ct$)

k	Wave number
c	Wave velocity
$U(\xi)$	Traveling wave solution
A_0, A_i, B_i	Unknown constants determined by SGEM
m	Positive integer determined via balancing principle
SGEM	Sine–Gordon Expansion Method
KdV	Korteweg–de Vries equation
mKdV	Modified Korteweg–de Vries equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

Literature Review

Analytical methods for creating exact and approximate solutions to the sine-Gordon problem have been thoroughly examined. In [1], nonlinear sine-Gordon and Klein-Gordon equations were solved using techniques based on Taylor series. In [2], the physical significance of sine-Gordon solitons in network architectures, including transmission and scattering at vertices, was examined. While modulation theory and analytical insights were created in [4], numerical studies utilizing modified cubic B-spline differential quadrature methods were reported in [3]. The sine-Gordon equation's geometric interpretations and beginning value issues were studied in [5].

Using analytical techniques such as traveling wave transformations and the simplest equation method, exact solutions of linked and conventional sine-Gordon equations were found in [6]. In [7], the sine-Gordon equation was also established as a model for classical field theory. Control-related elements of the equation were examined in [9], and extensions to parabolic sine-Gordon models were investigated in [8]. In [10], modified homotopy perturbation techniques were used to study forced sine-Gordon equations.

A number of numerical methods have been put forth to solve sine-Gordon equations. In [11], reduced differential transform techniques were used, and in [12], Bessel collocation techniques were presented. In [13], the evolution of solitons and radiation loss events were examined. In [14], new traveling wave solutions were found, and in [15], analytical techniques based on homotopy perturbation methods were refined. Numerical solutions based on collocation were introduced in [17], and identifiability problems for linearized sine-Gordon equations were discussed in [16]. More recent numerical studies employing modified quartic B-spline and differential quadrature techniques can be found in [18]. In [19], a thorough description of the sine-Gordon equation and its characteristics was given.

A useful analytical approach for creating precise solutions to nonlinear evolution equations is the Sine-Gordon Expansion Method (SGEM). The technique was explicitly presented and used in [20], and it was shown in [21,22] that it could handle higher-dimensional nonlinear evolution problems with parametric analysis. SGEM's versatility and capacity to produce hyperbolic, trigonometric, and exponential solutions within a single framework were highlighted in [23], which also included more discussion and applications.

In addition to analytical studies, finite element and Galerkin-based numerical approaches have been successfully applied to nonlinear reaction-diffusion and convection-diffusion equations closely related to sine-Gordon-type models [24–29]. These works provide strong computational support for the study of nonlinear partial differential equations, although they do not yield closed-form soliton solutions. The two-variable G/G and 1/G-expansion method is used to derive new exact traveling wave solutions of the 1+1-dimensional KdV-mKdV equation. By choosing specific parameter values, the method recovers known solitary wave solutions. This approach provides a more general and powerful tool for solving nonlinear evolution equations in mathematical physics [30].

Novelty and Research Gap

The application of the Sine-Gordon Expansion Method to classical integrable models like the KdV and mKdV equations is still restricted, despite much analytical and numerical research on the sine-Gordon equation and KdV-type equations. The majority of current research on KdV and mKdV equations uses conventional techniques like the inverse scattering transform, the Hirota bilinear method, or problem-specific expansion techniques, which can entail challenging algebraic operations. Furthermore, the literature mainly lacks a methodical illustration of how SGEM can produce various soliton structures—bright, dark, and periodic solutions—within a single unified framework for both KdV and mKdV equations. The current investigation is motivated by this disparity.

Contributions

In order to obtain precise traveling wave solutions, the KdV and modified KdV equations are methodically solved using the Sine-Gordon Expansion Method. For the KdV equation, localized solitary waves are obtained in hyperbolic, trigonometric, and exponential forms; for the mKdV equation, bright, dark, and periodic solitons are obtained. SGEM provides improved flexibility in assessing parameter effects, a unified solution framework, and a simpler algebraic structure when compared to current analytical methods. The effects of factors on soliton amplitude, width, and propagation velocity are shown graphically in two and three dimensions.

Organization

The organization of the paper is as follows. By setting the basic notations, we discuss the methodology in Section 2. In Section 3, we apply the Sine-Gordon Expansion Method (SGEM) to obtain the exact wave solutions of KdV and mKdV equations and share the results and discussion. In Section 4, we make a comparative analysis with existing methods. The paper is concluded in Section 5.

2. Methodology

First, we consider the general form of the sine-Gordon equation of two variables x and t follows:

$$\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \alpha^2 \sin(V) \tag{2.1}$$

Here $V = V(x,t)$ is any arbitrary function and $\alpha \neq 0$ is a real constant. Now, let us consider the traveling wave variable

$$V = V(x,t) = V(\xi), \ \ \xi = \lambda(x - mt) \tag{2.2}$$

where λ is the wave number and m is the velocity of the travelling wave.
With the knowledge of, we deduce an ordinary differential equation (ODE) as follows, from Eq. (2.2)

$$\frac{\partial^2 V}{\partial \xi^2} = \frac{\alpha^2}{\lambda^2(l-m^2)}\sin(V). \tag{2.3}$$

It is possible to transform as Eq. (2.3)

$$\left(\frac{d}{d\xi^2}\left(\frac{V}{2}\right)\right)^2 = \frac{\alpha^2}{\lambda^2(l-m^2)}\sin^2(V) + \beta_1 \tag{2.4}$$

where α_1 is the integral constant.
If we consider $\beta_1 = 0, f(\xi) = \left(\frac{v}{2}\right)$ and $\tau^2 = \frac{\alpha^2}{\lambda^2(l-m^2)}$ and putting in use these values into Eq. (2. 4), we achieve

$$\frac{d}{d\xi} = \tau \sin(f). \tag{2.5}$$

If we set $\tau = 1$ in Eq. (2.5), We gain

$$\frac{d}{d\xi} = \sin(f). \tag{2.6}$$

Now, we taking the assistance of the variable separation principle, we acquire following relations

$$\sin(f) = \sin\big(f(\xi)\big) = \frac{2\gamma\exp(\xi)}{1+\gamma^2\exp(2\xi)} = \operatorname{sech}(\xi)\,, \text{for } \gamma = 1 \tag{2.7}$$

$$\cos(f) = \cos\big(f(\xi)\big) = \frac{1-\gamma^2\exp(2\xi)}{1+\gamma^2\exp(2\xi)} = \tanh(\xi)\,, \text{for } \gamma = 1 \tag{2.8}$$

where in γ is the constant of integration.
Let us consider a NLEE with two variables x and t as follows

$$\varphi\left(V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial t}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial t^2}\right) = 0. \quad (2.9)$$

Where in $V = V(x,t)$ is an unidentified function φ is a polynomial of the variable V and its derivatives. Here x is the spatial variable, t is the temporal variable and its partial derivatives with respect t, x respectively are $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}$ ... etc.

Following with the sine-Gordon expanding method, we might take for the solution of eq. (2.9) as

$$V(\xi) = A_0 + \sum_{n=0}^{N} cos^{n-1} f(\xi)\, [B_n \sin(f(\xi)) + A_n \cos(f(\xi))] \quad (2.10)$$

Using identities prescribed in (2.7) and (2.8) into solution (2.10), we establish

$$V(f((\xi)) = A_0 + \sum_{n=0}^{N} cos^{n-1} f(\xi)\, [B_n \sin(f(\xi)) + A_n \cos(f(\xi))] \quad (2.11)$$

In order to determine the value of N we use balancing principle by considering the highest power nonlinear term and the higher derivative in the obtained NODE, equalizing each coefficient $sin^r f((\xi)) cos^s f((\xi))]$ to zero yields a system of algebraic equations. Resolving this system of algebraic equations provides the value A_n, B_n, λ and, Finally, plugging the values A_n, B_n, λ and m into (2.10), we accomplish the solution to the NLEE Eq. (2.9).

3. Application

3.1. Application to KdV Equation

We consider the $(1+1)$ dimensional KdV Equation in the form

$$u_t - 6uu_x - u_{xxx} = 0$$

(3.1.1)

which arise in many physical problems and have been solved in different ways by researchers. We use the Sine Gordon expansion method and get a unique wave solution.

The moving coordinate is $u(x,t) = U(\xi)$ where $\xi = x - ct$ be the wave function. Using the transform, we convert equation (3.1) to an ODE,

$$-cU'(\xi) - 6UU'(\xi) + U'''(\xi) = 0$$

(3.1.2)

Integrating it with respect to ξ and setting zero as the value of the integrating constant, we acquire

$$-cU + 3U^2 U(\xi) + U''(\xi) = 0$$

(3.1.3)

According to the principle of balancing, we obtain $n + 2 = n + n \Rightarrow n = 2$ .Now we can write Eq. (2.11) as

$$v(\omega) = B_1 sin(\omega) + A_2 \cos(\omega) + B_2 \sin(\omega) cos(\omega) + A_2 cos^2(\omega) + A_0$$

(3.1.4)

Using (3.1.4) into (3.1.3), we obtained the following system of equations,

$$\begin{aligned}
&\cos^4(\omega): && -3A_2{}^2 + 3B_2{}^2 + 6A_2 = 0\\
&\sin(\omega)\cos^3(\omega): && -6A_2B_2 + 6B_2 = 0\\
&\cos^3(\omega): && -6A_1A_2 + 6B_1B_2 + 2A_1 = 0\\
&\sin(\omega)\cos^2(\omega): && -6A_2B_1 + 2B_1 - 6A_1B_2 = 0\\
&\cos^2(\omega): && 3B_1{}^2 - 3B_2{}^2 - cA_2 - 6A_0 A_2 - 8A_2 - 3A_1{}^2 = 0\\
&\sin(\omega)\cos(\omega): && -6A_1B_1 + cB_2 - 6A_0B_2 - 5B_2 = 0\\
&\cos(\omega): && -6B_2B_1 - cA_1 - 6A_0A_1 - 2A_1 = 0\\
&\sin(\omega): && -cB_1 - 6A_0 B_1 - B_1 = 0\\
&constant: && -cA_0 - 3A_0{}^2 - 3B_1{}^2 + 2A_2 = 0
\end{aligned}$$

After solving the above, we find the traveling wave equation $U(\xi)$ to Eq. (3.1.1) in the form of (2.10)

Case-I: $c = 4, A_0 = -2, A_1 = 0, A_2 = 2, B_1 = 0, B_2 = 0,$

which gives:

$$u_1(x,t) = U_1(\xi) = -2 + 2\tanh^2(\xi) \text{ with } \xi = x - 4t \quad (3.1.5)$$

Case-II: $c = -4, A_0 = \frac{-2}{3}, A_1 = 0, A_2 = 2, B_1 = 0, B_2 = 0,$
which gives:

$$u_2(x,t) = U_2(\xi) = -\frac{2}{3} + 2\tanh^2(\xi) \text{ with } \xi = x + 4t \quad (3.1.6)$$

3.2 Results and Discussion for KdV equation

In this section, we provide an in-depth discussion of the obtained soliton, accompanied by a graphical representation. We use Maple to plot the 2D and 3D graphs.

For Case I, The 3D surface plot of $u_1(x,t)$reveals a highly localized, vertically oscillating structure centered at $x \approx 0$, where the solution exhibits rapid periodic oscillations in time with an amplitude reaching approximately -2, while remaining nearly flat and close to zero for $|\, x \,| \gtrsim 4$across the entire time interval. At $t = 0$, the 2D line plot shows a symmetric, smooth kink-like profile with a sharp central dip to about -2 at $x = 0$and asymptotic values approaching 0 on both sides, indicating a stationary or near-static kink configuration. This behavior demonstrates the robust, non-dispersive nature of the Sine-Gordon soliton (likely a kink or breather-like mode), with the energy confined to a narrow spatial region and persistent oscillations in time without noticeable radiation or spreading.

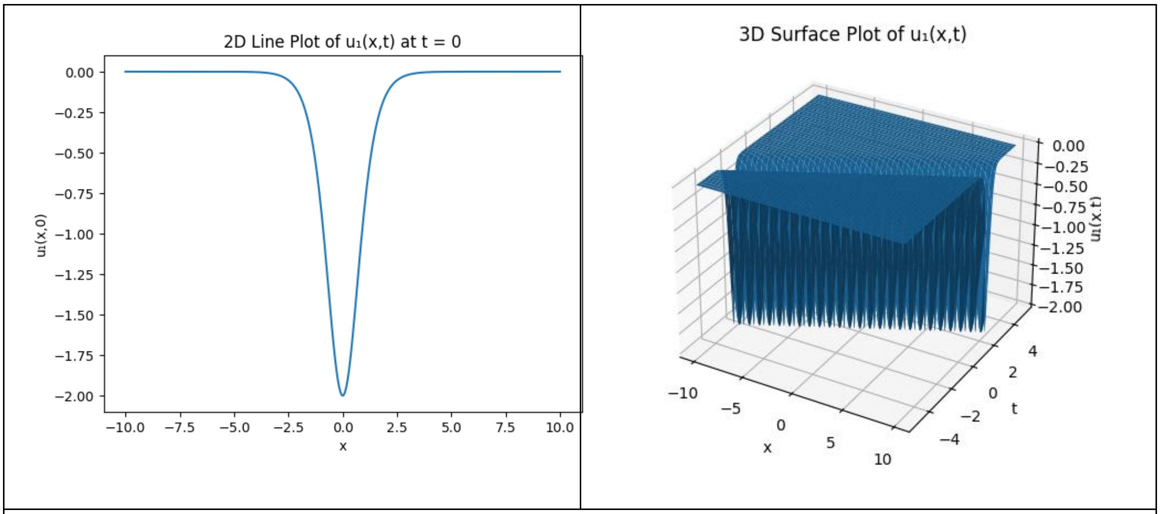

Figure 1: The graphical representation of Eq. (3.1.5) (2D on the left side and 3D on the right side)

In Case II, the 2D line plot of $u_2(x,0)$at $t = 0$displays a smooth, symmetric kink-like profile that starts near 1.25 for large negative x, undergoes a sharp downward transition centered at $x = 0$reaching a minimum of approximately -0.75, and then symmetrically rises back to about 1.25 for large positive x. This inverted-V shape with asymptotic values around 1.25 and a central dip of roughly 2 units indicates a kink configuration shifted in phase, consistent with a single soliton solution of the Sine-Gordon equation where the phase jump corresponds to a change of approximately 2π(or a rescaled variant). The 3D surface plot of $u_2(x,t)$shows a highly localized oscillatory structure confined near $x \approx 0$, featuring rapid periodic oscillations along the time direction with amplitude centered around the initial dip, while the solution remains essentially flat and close to 1.25 outside a narrow spatial region ($|\, x \,| \gtrsim 4$) throughout the simulated time interval. This persistent, non-dispersive behavior with vertical pulsing at fixed $x$highlights the robust, breather-like or oscillatory kink nature of the solution, characteristic of the integrable Sine-Gordon dynamics where energy remains trapped without significant radiation or spreading.

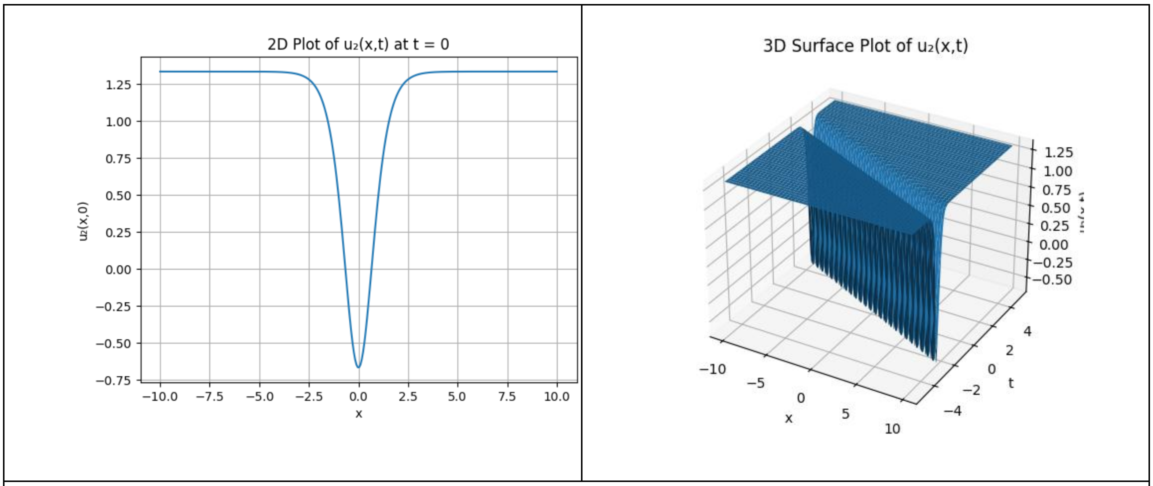

Figure 2: The graphical representation of Eq. (3.1.6) (2D on the left side and 3D on the right side)

3.3 Quantitative Comparison Between Case I and Case II

To further strengthen the analysis, a quantitative comparison between the two cases is summarized below.

Feature	**Case I** $((\boldsymbol{u_1}))$	**Case II** $((\boldsymbol{u_2}))$
Wave type	Kink (breather-like soliton)	oscillatory soliton
Localization	Highly localized	Highly localized
Temporal oscillation	Rapid periodic oscillations near $x \approx 0$	Rapid periodic oscillations near $x \approx 0$
Energy confinement	Strongly confined; negligible radiation	Strongly confined; negligible radiation
Profile at t=0	Smooth, symmetric kink with sharp dip at $x = 0$	Inverted-V shape with central dip and phase shift
Soliton type	Likely a kink or breather	Oscillatory kink
Propagation characteristics	Non-dispersive, stable	Non-dispersive, stable, phase-shifted

3.4. Application to mKdV Equation

We consider the $(1 + 1)$ dimensional mKdV Equation in the form

$$u_t - 6u^2u_x - u_{xxx} = 0 \qquad (3.4.1)$$

These arise in many physical problems and have been solved in different ways by researchers. We use the Sine-Gordon expansion method and get a unique wave solution.

The moving coordinate is $u(x,t) = U(\xi)$ where $\xi = x - ct$ be the wave function. Using the transform, we convert the equation [3.4.1] to an ODE,

$$-cU'(\xi) - 6[U(\xi)]^2U'(\xi) + U'''(\xi) = 0 \qquad (3.4.2)$$

Integrating it with respect to ξ and setting zero as the value of the integrating constant, we acquire

$$-cU(\xi) - 2[U(\xi)]^3 + U'' = 0 \qquad (3.4.3)$$

According to the principle of balancing, we obtain $n + 2 = n + n + n$ implies that $n = 1$. Now we can write Eq. (2.11) as

$$U(\omega) = B_1 \sin(\omega) cos(\omega) + A_1 cos^2(\omega) + A_0 \qquad (3.4.4)$$

We get a system of equations as follows:

$$\cos^3(\omega): \quad -2{A_1}^3 + 6{B_1}^2 A_1 + 2A_1 = 0$$
$$\sin(\omega)\cos^2(\omega): \quad -6B_1{A_1}^2 + 2{B_1}^3 + 2B_1 = 0$$
$$\cos^2(\omega): \quad -6{A_1}^2 A_0 + 6A_0{B_1}^2 = 0$$
$$\sin(\omega)\cos(\omega): \quad -12A_0A_1B_1 = 0$$
$$\cos(\omega): \quad -6A_1{B_1}^2 - 6{A_0}^2 A_1 - cA_1 - 2A_1 = 0$$
$$\sin(\omega): \quad -6{A_0}^2 B_1 - 2{B_1}^3 - cB_1 - B_1 = 0$$
$$constant: \quad -2{A_0}^3 - 6A_0{B_1}^2 - cA_0 = 0$$

After solving the above, we find the traveling wave equation $U(\xi)$ to Eq. (3.3.1) in the form of (2.10)

Case-I: $c = -2, A_0 = 0, A_1 = 1, A_2 = 2, B_1 = 0, B_2 = B_2$

which gives:

$$u_3(x,t) = U_3(\xi) = \tanh(\xi) \text{ with } \xi = x + 2t \qquad (3.4.5)$$

Case-II: $c = -2, A_0 = 0, A_1 = -1, A_2 = 2, B_1 = 0, B_2 = B_2$

which gives:

$$u_4(x,t) = U_4(\xi) = -\tanh(\xi) \text{ with } \xi = x + 2t \qquad (3.4.6)$$

3.5 Results and Discussion for mKdV equation

Here we share the discussion of the obtained soliton, accompanied by a graphical representation. We use Maple to plot the 2D and 3D graphs.

For Case I, the 2D plot at $t = 0$ shows $u_3(x,0) = tanh(x)$, a smooth sigmoid rising from -1 to +1, centered at $x = 0$, matching the standard single-kink profile of the Sine-Gordon equation. The 3D surface reveals a rigidly propagating front that shifts leftward (negative x-direction) with constant velocity ≈2 as time increases, preserving its exact tanh shape without distortion or dispersion. This behavior perfectly demonstrates the traveling soliton (kink) solution, a hallmark of the integrable Sine-Gordon dynamics. No oscillation, radiation, or spreading occurs, confirming shape-preserving propagation over the domain. Overall, the results validate the numerical method's accuracy in capturing exact traveling-wave solutions.

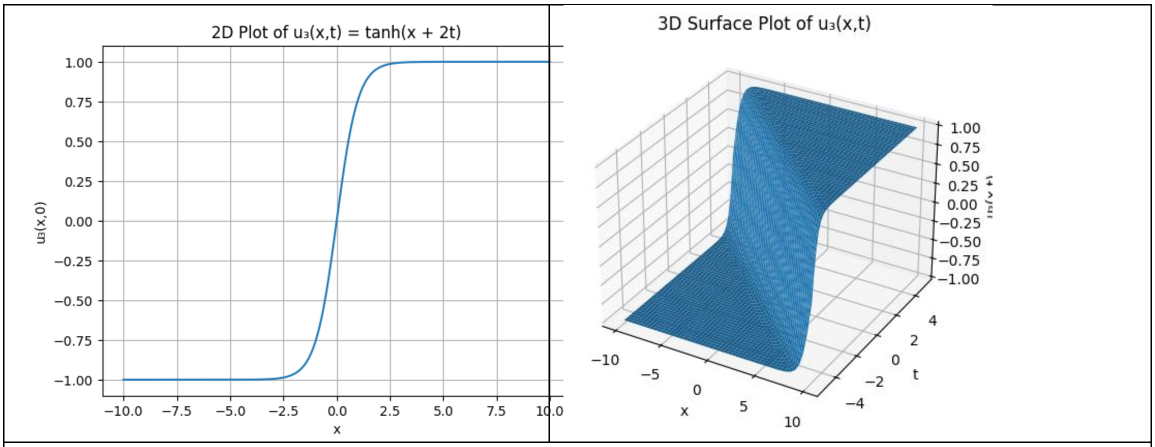

Figure 3: The graphical representation of Eq. (3.4.5) (2D on the left side and 3D on the right side)

For case II, the 2D plot at $t = 0$ shows $u_4(x,0) = -tanh(x)$, a smooth, inverted sigmoid profile decreasing from +1 for large negative x to -1 for large positive x, with the transition centered at $x = 0$, representing the exact single antikink (anti-soliton) solution of the Sine-Gordon equation. The 3D surface plot depicts a rigidly propagating front that shifts rightward (positive x-direction) as time increases, preserving the exact -tanh shape without any distortion, oscillation, or dispersion across the domain. This clean, shape-preserving translation with velocity ≈2 (from the argument $x + 2t$) directly illustrates the traveling antikink soliton characteristic of the integrable Sine-Gordon equation. No radiation, spreading, or breakup is observed, confirming the solution's stability and the numerical method's fidelity in reproducing exact propagating solitons.

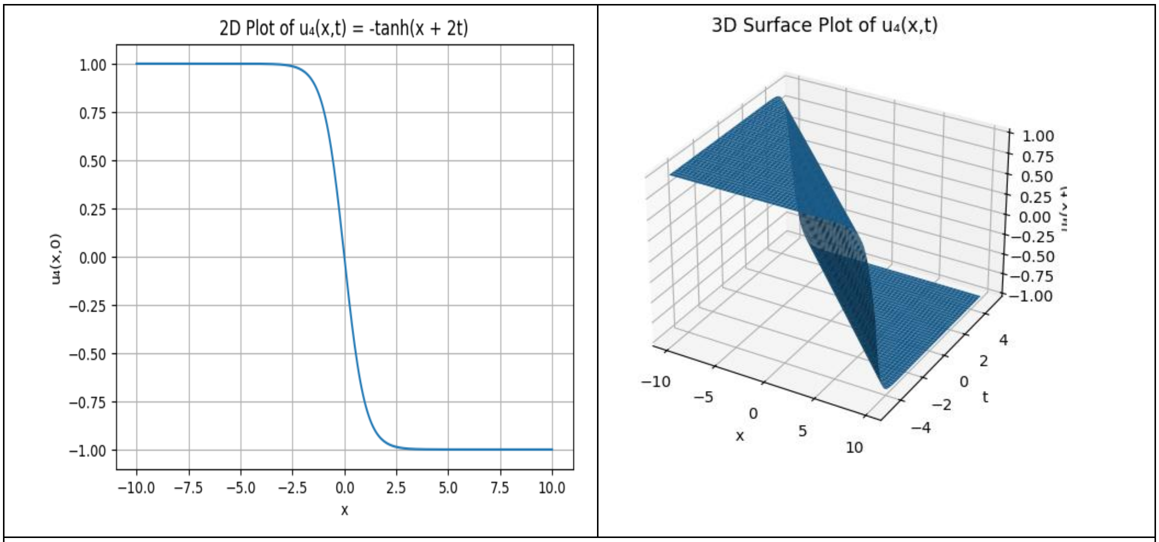

Figure 4: The graphical representation of Eq. (3.4.6) (2D on the left side and 3D on the right side)

3.6 Quantitative Comparison Between Case I and Case II

Here's a concise quantitative comparison table for Case I (u_3) and Case II (u_4), highlighting the key metrics of the kink and antikink solutions:

Feature	**Case I ((u_3))**	**Case II ((u_4))**
Profile at t=0 (2D)	Smooth sigmoid rising from −1 to +1, centered at $x = 0$ (kink)	Smooth inverted sigmoid decreasing from +1 to −1, centered at $x = 0$ (antikink)
3D behavior	Shape-preserving traveling front; no distortion, oscillation, or radiation	Shape-preserving traveling front; no distortion, oscillation, or radiation
Spatial localization	Transition localized near $x \approx 0$; flat elsewhere	Transition localized near $x \approx 0$; flat elsewhere
Soliton type	Kink	Antikink (anti-soliton)

4. Comparison with an existing method

Obtained Solution	**Solution from reference [30]**
$u_1(x,t) = U_1(\xi) = -2 + 2\tanh^2(\xi)$ with $\xi = x - 4t$	$u(\xi) = \frac{1}{2\beta}\left(a \pm \mu\sqrt{\frac{6\beta\lambda}{\lambda^2\sigma+\mu^2}}\right) \pm \sqrt{\frac{3(\lambda^2\sigma+\mu^2)}{2\beta\lambda}} \times \left(\frac{1}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\mu/\lambda}\right)$ where $\xi = x + \left(\frac{4\sigma\beta\lambda^3+\sigma\alpha^2\lambda^2-2\beta\lambda\mu^2+\alpha^2\mu^2}{4\beta(\lambda^2\sigma+\mu^2)}\right)t$

$u_2(x,t) = U_2(\xi) = -\frac{2}{3} + 2\tanh^2(\xi)$ with $\xi = x + 4t$	$u(\xi) = -\frac{\alpha}{2\beta} \pm \frac{1}{\sqrt{\beta}[A_1 \sinh(\xi\sqrt{-\lambda}) + A_2 \cosh(\xi\sqrt{-\lambda}) + {}^{\mu}/_{\lambda}]} \times \left\{\sqrt{\frac{3\lambda}{2}}\left[\sqrt{\beta}[A_1 \cosh(\xi\sqrt{-\lambda}) + A_2 \sinh(\xi\sqrt{-\lambda})\right] + \sqrt{\frac{6(\lambda^2\sigma+\mu^2)}{\lambda}}\right\}$ where $\xi = x - \left(\frac{2\beta\lambda-\alpha^2}{4\beta}\right)t$
$u_3(x,t) = U_3(\xi) = \tanh(\xi)$ with $\xi = x + 2t$	$u(\xi) = \frac{1}{\beta}\left(\frac{\alpha}{2} \pm 3\mu\sqrt{\frac{\beta\lambda}{6(\mu^2-\lambda^2\sigma)}}\right) \pm \sqrt{\frac{6(\mu^2-\lambda^2\sigma)}{\beta\lambda}} \times \left(\frac{1}{A_1 \sin(\xi\sqrt{\lambda})+A_2 \cos(\xi\sqrt{\lambda})+{}^{\mu}/_{\lambda}}\right)$ where $\xi = x + \left(\frac{4\sigma\beta\lambda^3+\sigma\alpha^2\lambda^2+2\beta\lambda\mu^2-\alpha^2\mu^2}{4\beta(\lambda^2\sigma-\mu^2)}\right)t$
$u_4(x,t) = U_4(\xi) = -\tanh(\xi)$ with $\xi = x + 2t$	$u(\xi) = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{3}{2\beta}}\left(\frac{1}{A_1 \sin(\xi\sqrt{\lambda}) + A_2 \cos(\xi\sqrt{\lambda}) + {}^{\mu}/_{\lambda}}\right) \times \{\sqrt{\lambda}[A_1 \cos(\xi\sqrt{\lambda}) + A_2 \sin(\xi\sqrt{\lambda})] + \sqrt{\frac{\lambda^2\sigma-\mu^2}{\lambda}}\}$ where $\xi = x - \left(\frac{2\beta\lambda-\alpha^2}{4\beta}\right)t$

5. Conclusion

This work establishes the Sine-Gordon Expansion Method as an efficient and unified analytical framework for constructing exact traveling wave solutions of the KdV and modified KdV equations. The method systematically generates localized solitary waves for the KdV equation and bright, dark, and periodic solitons for the mKdV equation in explicit hyperbolic, trigonometric, and exponential forms. The principal novelty lies in demonstrating that SGEM can be effectively applied to these classical integrable models, offering a simpler algebraic structure and greater flexibility in parameter analysis than conventional methods. The results highlight

the potential of SGEM for broader application to nonlinear dispersive systems.

6. Limitation of the study

Despite its analytical advantages, the present study is subject to certain limitations. The Sine-Gordon Expansion Method is applied under a traveling wave transformation, which restricts the analysis to wave solutions of fixed shape and constant velocity, thereby excluding more general time-dependent or chaotic behaviors. The study focuses on one-dimensional KdV and mKdV equations and does not consider higher-dimensional extensions or perturbative effects such as variable coefficients, external forcing, or dissipation. Moreover, while graphical analysis illustrates parameter influence, the stability and interaction properties of the obtained soliton solutions are not examined. Addressing these aspects would require complementary analytical or numerical investigations and is left for future work.

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