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SEDIMENTATION OF INDIVIDUAL MOLECULES OF HEAVY ELEMENTS IN INITIAL PROTOPLANETS

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ABSTRACT

Segregation problem of individual molecules of heavy elements, has been reinvestigated determining the polytropic structure of a protoplanet. In this study the sedimentation time of different silicate grains having radii lying between 5×10^{-3}cm and 1 cm was calculated. Settling times of the grains that came out through calculation could be found to be shorter than that of the time obtained by other authors.

Key words: Segregation, Grain, Gravitational field, Protoplanet, Polytrope

INTRODUCTION

One of the two current lines of thought, available in the literature, in the formation of the planetary system is disk instability model. The essence of the model is that an instability formed under right condition in a protoplanetary disk can lead to the creation of self-gravitating clumps of gas and dust. Such clumps can contract to form giant gaseous protoplanets (Boss 1997, 1998). The basic idea of this model was proposed earlier by Kuiper (1951), Urey (1966) and Cameron (1978) by considering the formation of giant planets in the solar nebula. This idea believes in the existence of a set of identical protoplanets, identical in mass, radius and composition, which subsequently formed planets by the segregation of heavy elements in the protoplanets followed by the removal of the correct amount of hydrogen and helium from all the bodies except from Jupiter and Saturn (Nelson *et al.* 2000, Rice *et al.* 2003, Pickett and Lim 2004). The segregation process was first investigated by McCrea and Williams (1965) who concluded that individual molecules of heavy elements and normal interstellar grains would both take such a long time to settle to the centre that the segregation could not occur by this process. The time of fall for such objects as calculated by the investigation was in fact so large ($> 10^9$ yrs) that this aspect of their work has not been reinvestigated. In their calculation, McCrea and Williams (1965) assumed a constant density model of a protoplanet without any physics behind it. It is well-known that every known astronomical object is centrally condensed. Therefore, the segregation time obtained by the investigation may be somewhat different from the real picture.

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In this communication the authors intend to reinvestigate the segregation problem of individual molecules of heavy elements, carried out by McCrea and Williams (1965), determining the polytropic structure of a protoplanet assuming that density distribution inside the protoplanet is given by a simple polytropic law.

THE MODEL USED

As in McCrea and Williams (1965), the present study started with an isolated non-rotating glob of gas with initial mass M = 10^{30}g and initial radius $R = 3\times10^{12}$ cm. The glob is in reality a protoplanet and authors take it to be isolated so that it does not collide with another one in the terrestrial planet region and the external influences on it cannot be included. In fact, solar radiation can affect the internal temperature profile (Helled *et al.* 2008). Following Helled and Schubert (2008) and DeCampli and Cameron (1979), the present authors assumed that the initial protoplanet is in a state of quasi-static equilibrium and is an object of solar composition in which ideal gas law holds. If the protoplanet is assumed to behave as a polytrope of index n, then its internal density distribution is given by

$$P = K\,\rho^{1+\frac{1}{n}} \quad , \tag{1}$$

where, P is the pressure, ρ is the density and K is the polytropic constant. The structure of the protoplanet is then given by the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n \tag{2}$$

subject to the boundary condition $\phi = 1$ and $\frac{d\phi}{d\xi} = 0$ at $\xi = 0$ (centre),

where $\xi = \xi_1\frac{r}{R}$, ξ_1 being the first zero of ϕ, r is the central distance and ϕ is related to the thermodynamic variables P, ρ and T through

$$\phi = \left(\frac{\rho}{\rho c}\right)^{\frac{1}{n}} = \left(\frac{P}{Pc}\right)^{\frac{1}{1+n}} = \frac{T}{Tc} \tag{3}$$

In Eq. (3), ρ_c, P_c and Tc represent the central values and are given by

$$\rho_c = a_n\frac{3M}{4\pi R^3},\, P_c = bn\frac{GM^2}{R^4} \text{ and } T_c = c_n\frac{GM}{R}$$

where, G is the universal gravitational constant and an, bn are numerical constants having different values for different n and are available in Menzel *et al.* (1963) whereas C_n is given by

$$c_n = \frac{4\pi\mu H}{3k}\frac{b_n}{a_n}. \tag{4}$$

Since the composition of gas in the initial protoplanet is assumed to be solar, so it will consist mainly of hydrogen, helium and heavy elements mostly in the form of grains. This is expected, as the temperature of a protoplanet is fairly low in quasi-static equilibrium. The authors followed silicate grains only as grains composed of organic materials will mostly be evaporated before they get to the core region, while water ice grains will completely be evaporated (Helled *et al.* 2008). Let a grain, assumed to be spherical, start moving from very near to the surface of the protoplanet towards its centre through the ambient gas. The gas will offer resistance to the motion of the grain. Then the equation of motion of the grain at depth $R - r$ below the surface of the protoplanet is given by

$$\frac{d}{dt}(m_g \frac{dr}{dt}) = -\frac{GM(r)\, m_g}{r^2} - F_{res}, \tag{5}$$

where m_g is the mass of the grain, G the gravitational constant, R the radius of the protoplanet, F_{res} the resistive force and $M(r)$ is the mass interior to a radius r.

A large amount of work has been carried out on the treatment of gas drag in modeling a protoplanet (e.g., Williams and Crampin 1971, Zhou and Lin 2007, Paul *et al*. 2011). As in McCrea and Williams (1965), if we consider individual grain, then the mean free path of each of the grains considered can be found to be larger than their radii. Therefore, Epstein drag is applicable and is given by

$$F_{res} = -\frac{4}{3}\pi\rho W r_g^2 \frac{dr}{dt} \quad \text{(Baines and Welliams 1965)}, \tag{6}$$

where W is the mean thermal velocity of the gas and r_g is the radius of the grain.

CALCULATION

Structure Determination: It is mentioned earlier that Eq. (2) directly determines the distribution of thermodynamic variables. Therefore, to determine the distribution, authors need to solve Eq. (2) which on non-dimensionalisation with $\xi = \xi_1 x$ gives

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2 \frac{d\phi}{dx}\right) = -\xi_1 \phi^n, \tag{7}$$

which is subject to the boundary conditions

$$\phi = 1 \text{ and } \frac{d\phi}{dx} = 0 \text{ at } x = 0 \text{ (centre).}$$

But to solve the equation, besides the value of ξ_1, parameter n has to be specified. The appropriate value of n for a protoplanet is unknown. However, since an initial protoplanet is expected to be less centrally condensed, so n is likely to be small (Paul *et al*. 2011). In the present calculation authors have considered four different values of *n*, namely 0, .5, 1 and 1.5. Also thermodynamic variables are related to central values for different *n*. These central values along with ξ_1 for different *n* are shown in Table 1.

Table 1. Some important quantities for the polytrope for some values of the polytropic index *n*.

n	ξ_1	ρ_c	P_c	T_c
0	2.4494	8.84×10^{-9}	98.37	296.67
0.5	2.7528	1.62×10^{-8}	157.53	258.76
1.0	3.1416	2.91×10^{-8}	323.61	296.67
1.5	3.6538	5.30×10^{-8}	634.65	319.51

Inserting the value of ξ_1 for the corresponding value of *n*, the authors have solved Eq. (7) in each case by the 4th order Runge-Kutta method, where required initial values have been obtained by series solution of Eq. (7) near the singular point. The distribution of ϕ is shown in Fig. 1. With the distribution of ϕ, authors have determined the distribution of thermodynamic variables using Eq. (3). Since to show all the results in tabular form would be space consuming and not physically instructive, so the results are shown in diagrammatic form through Figs. 2 and 3. It is mentioned that $n = 0$ refers to a model of constant density (Fig. 2). The figure for temperature distribution is not included as its distribution (T/T_c) can be given by the Fig. 1.

Now, the remaining term is the mass distribution. The mass distribution equation is given by

$$dM(r) = 4\pi r^2 \rho dr \tag{8}$$

In terms of the transformations $M(x) = Mq(x)$ and $r = xR$ and with the help of Eq. (3), Eq. (8) becomes

$$q(x) = \frac{4\pi R^3 \rho_c}{M} \int_0^x x^2 \phi^n dx\,, \tag{9}$$

which is subject to the boundary conditions

$$q(x) = 1 \text{ at } x = 1 \text{ and } q(x) = 0 \text{ at } x = 0\,.$$

Eq. (9), with the help of Eq. (7), can be reduced to the form

$$q(x) = \frac{4\pi R^3 \rho_c}{M\xi_1^2}\left(-x^2 \frac{d\theta}{dx}\right) \tag{10}$$

The mass distribution has been determined by solving Eq. (10) and is shown in Fig. 4. The results of present calculation for thermodynamic and physical variables are found to be consistent with those of Paul *et al.* (2011).

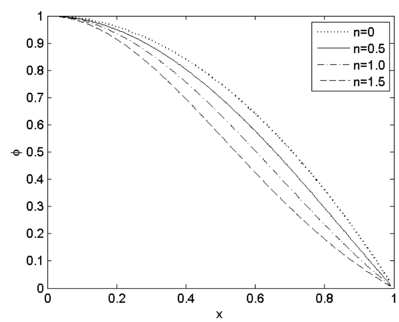

Fig. 1. The distribution of φ for different *n*.

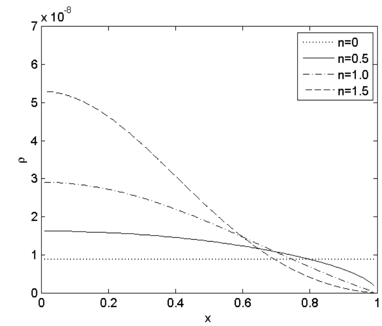

Fig. 2. Density distribution inside a protoplanet for different values of *n*.

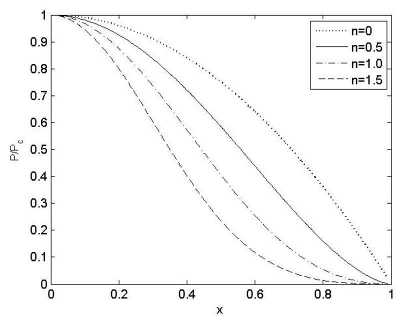

Fig. 3. Distribution of pressure in a protoplanet for different values of *n*.

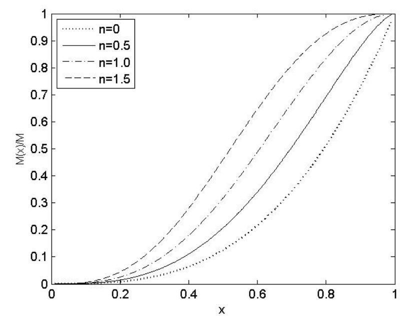

Fig. 4. Distribution of mass in a protoplanet for different values of *n*.

Calculation of sedimentation time : With the help of Eq. (6), Eq. (5) can be written as

$$mg\frac{d^2r}{dt^2} = -\frac{GM(r)m_g}{r^2} + \frac{4}{3}\pi W\rho r_g^2\frac{dr}{dt} \tag{11}$$

Since the grain is moving in a resisting medium, then the acceleration term can be neglected because the body moving in that resting medium will quickly gain a velocity close to its terminal velocity and will travel at such a velocity. It is noted that McCrea and Williams (1965) also used the same technique. If this simplification is made, then Eq. (11) is reduced to the form

$$\frac{dr}{dt} = \frac{3GM(r)\,m_g}{4\pi\rho W\,r_g^2 r^2}\,. \tag{12}$$

The mass m_g of the grain and its mean thermal velocity are given by

$$m_g = \frac{4}{3}\pi\,r_g^3\,\rho_g \tag{13}$$

and

$$W = \sqrt{\frac{8kT}{\pi H}} \tag{14}$$

respectively, where ρ_g is the density of the grain material, H is the mass of a hydrogen atom, T is the temperature and k is the Boltzmann constant.

From the equation of state of an ideal gas, it is seen that

$$\rho \frac{\mu H}{k}\frac{P}{T} \tag{15}$$

where, the symbol μ represents mean molecular weight of the standard composition.

Using Eqs. (13) - (15) in Eq. (12), the authors noted

$$\frac{dr}{dt} = \frac{G\rho_g r_g M(r)\sqrt{\pi k T}}{\mu P r^2 \sqrt{8H}} \tag{16}$$

Along with the used dimensionless variables if the author further introduce the dimensionless variables p, θ, and τ defined by $P = P_C p$, $T = T_C\theta$ and $t = 10^7\tau$, Eq. (16) becomes

$$\frac{dx}{d\tau} = \alpha \frac{q\sqrt{\theta}}{p\,x^2}, \tag{17}$$

where,

$$\alpha = 10^7 \times \sqrt{\frac{\pi T_c \Re}{8}} \times \frac{GM\, r_g \rho_g}{\mu\, P_c R^3}. \tag{18}$$

Thus the time of fall of the grain of constant mass from the surface to the centre of the protoplanet is given by

$$\tau = \int_0^1 F(x, p, q, \theta)dx \tag{19}$$

where,

$$F(x, p, q, \theta) = \frac{1}{\alpha}\frac{p\,x^2}{q\sqrt{\theta}}. \tag{20}$$

The evaluation of the integral in Eq. (19) includes a number of parameters. Central values for different prescribed values of n are available in Table 1. For grain radii, we have considered the values ranging from 5×10^{-3}cm to 1cm. In this study, the researchers used μ = 2.2, as is appropriate for molecular hydrogen, $\rho_g = 2.8\text{cm}^{-3}$ which is appropriate for silicate grains (Helled *et al.* 2008) and all other values involved in the problem have been assumed to have their standard values. Also the evaluation of the integral depends on the values of p, q and θ at different x for each n. But the values of p, q and θ at different x for each value of n have already been calculated. Now, the evaluation of the integral in Eq. (19) can easily be performed numerically. The authors evaluated the integral by Simpson's one-third rule for all prescribed values of the grain radii ($5 \times 10^{-3}\text{cm} \le r_g \le$ lcm) for each n. To avoid the singularity, integration has been performed between limits x = 0.01 and *x* = 0.99. The results of this calculation for only the grains having radii 5×10^{-3}, 10^{-2}, 10^{-1} and 1cm are tabulated (Table 2). For all other grains the time range can be found to be lying between 2.78×10^6 and 6.00×10^8 yrs.

Table 2. Grain sedimentation time.

Polytropic index, *n*	r_g = 1cm Time (yrs)	$r_g = 10^{-1}$cm Time (yrs)	$r_g = 10^{-2}$cm Time (yrs)	$r_g = 5 \times 10^{-3}$cm Time (yrs)
0	2.92×10^6	2.92×10^7	2.92×10^8	5.84×10^8
0.5	2.78×10^6	2.78×10^7	2.78×10^8	5.56×10^8
1.0	2.90×10^6	2.90×10^7	2.90×10^8	5.81×10^8
1.5	3.00×10^6	3.00×10^7	3.00×10^8	6.00×10^8

CONCLUSION

Authors reinvestigated the segregation problem of individual grains investigated by McCrea and Williams (1965) determining the polytropic structure of a protoplanet. The segregation time found by McCrea and Williams (1965) was in fact very large ($>10^9$ yrs). The reason behind the fact may be the assumption of constant density model of the protoplanet. The results of present calculation for settling of individual grain yield shorter time in each case than that of the time obtained by the investigation of McCrea and Williams (1965). In fact, the initial contraction time of protoplanets formed via disk instability is so short and of the order of $\sim10^5$ years within which segregation must have to take place (e.g., Boss 1998, Helled *et al*. 2008, Helled and Bodenheimer 2010). Therefore, though the present results differ from the results obtained by McCrea and Williams (1965), this study draws the same conclusion like them that segregation is impossible by this process.

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