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ON GLOBAL EXISTENCE THEOREM OF CERTAIN VOLTERRA INTEGRAL EQUATION OF SECOND KIND

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ABSTRACT

The aim of the paper was to fabricate an alternative proof of a global existence theorem of certain type of Volterrea integral equation on the basis of the hypothesis. The new proof has been given by constructing suitable function space and using fixed point theorem. Relaxing some hypotheses in the same and using Bielecki's notion of norm another global existence theorem has been proposed and proved.

Key words: Global Existence Theorem, Certain Volterra Integral Equation

INTRODUCTION

Consider the Volterra integral equation of the second kind in R^n

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds \quad (1)$$

where the given function $f(t)$ and the kernel $k(t,s)$ are assumed to be continuous on the interval $[0,\infty)$ and the extended triangular region $0 \le s < t < \infty$, respectively. Moreover there exists $M > 0$ such that $\int_0^t |k(t,s)|ds \le M, t \ge 0$ and $g(t,x)$ is continuous on $[0,\infty) \times R^n$, $|g(t,x) - g(t,y)| \le L|x - y|, t \ge 0$ where $ML < 1$ is bounded on $[0,\infty)$.Where $|.|$ means supremum norm of vector and matrix.

By employing the constructing uniformly convergent sequence of functions (successive approximation) as in Corduneanu (1969), it can be shown (1) possesses a unique global continuous solution $x(t)$ of equation (1) on the interval $[0,\infty)$. On the other hand the present authors applied Banach contraction mapping principle to (1), by formulating suitable function space under the same set of hypotheses. The operator $T\phi(t) = f(t) + \int_0^t k(t,s)g(s,\phi(s))ds; t \ge 0$ takes B into B, where B is the space of all real valued continuous function on $[0,\infty)$. The operator T is not, in general, contracting unless the product $ML < 1$. Hence to apply the contraction principle to *T*, it is necessary to restrict either the highest possible mass of the kernel or the Lipschitz's constant in the second argument of $g(t,x)$. L and M may be interpreted otherwise. In second case, if the authors relax the hypotheses $ML < 1$ and define the norm of a function using Bielecki

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Idea in (Miller 1973), we can show, at the influence of Miller (1973), that (1) has unique continuous solution on $[0,\infty)$ (Strauss 1970).

Global existence theorem (1): Corduneanu (1969) described a global existence theorem under the following hypotheses. The present authors proved the same using contraction mapping theorem instead constructing sequence of functions.

Consider the Volterra integral equation $x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds$

Assume that (1) satisfies the following conditions:

(i) $f(t)$ is continuous and bounded on $[0,\infty)$.

(j) $k(t,s)$ is continuous on $0 \le s < t < \infty$ and there exists $M > 0$ such that

$\int_0^t |k(t,s)|ds \le M, t \ge 0$.

(k) $g(t,x)$ is continuous on $[0,\infty)\times R^n$ and $|g(t,x)-g(t,y)| \le L|x-y|, t \ge 0$ where $L < M^{-1}$ or $M < L^{-1} \Rightarrow ML < 1$.

(l) $g(t,0)$ is bounded on $[0,\infty)$, i.e. $|g(t,0)| < m_2 < \infty; t \in [0,\infty)$

then there exists a unique continuous bounded solution $x(t)$ of (1) on $[0,\infty)$.

Proof: Let B be the Banach space of bounded continuous functions on $[0,\infty)$ with supremum norm $|.|$ (Simmons 2004), for $\phi \in B$, $|\phi| = \sup\limits_{0\le t<\infty}|\phi(t)|$.

For each $\phi \in B$, we define the mapping $\phi \mapsto T\phi$ by

$T\phi(t) = f(t) + \int_0^t k(t,s)g(s,\phi(s))ds; t \ge 0$.

Now, $|T\phi(t)| \le |f(t)| + \int_0^t |k(t,s)||g(s,\phi(s))|ds; t \ge 0$.

Given, $|f(t)| \le m_1 < \infty$

$|T\phi(t)| \le m_1 + \int_0^t |k(t,s)||g(s,\phi(s))|ds - \int_0^t |k(t,s)||g(s,0)|ds + \int_0^t |k(t,s)||g(s,0)|ds$

$\le m_1 + \int_0^t |k(t,s)||g(s,\phi(s)) - g(s,0)|ds + \int_0^t |k(t,s)||g(s,0)|ds$

But $|g(t,0)| < m_2 < \infty; t \in [0,\infty)$ and $|g(s,\phi(s)) - g(s,0)| < L|\phi(s) - 0|$

$|T\phi(t)| \le m_1 + L\int_0^t |k(t,s)||\phi(s)|ds + m_2\int_0^t |k(t,s)|ds$

$\le m_1 + |\phi| L\int_0^t |k(t,s)|ds + m_2\int_0^t |k(t,s)|ds$ and $\int_0^t |k(t,s)|ds < M < \infty$

$\le m_1 + |\phi|LM + m_2 M < \infty$

Again $|T\phi(t+h) - T\phi(t)| \to 0$ as $h \to 0$

$$|T\phi(t+h) - T\phi(t)| =$$

$$\left| f(t+h) - f(t) + \int_0^{t+h} k(t+h,s)g(s,\phi(s))ds - \int_0^t k(t,s)g(s,\phi(s))ds \right|$$

$$\leq |f(t) - f(t+h)| + \underset{0\leq s\leq t+h}{Max}|g(s,\phi(s))| \left|\int_0^{t+h} k(t+h,s)ds - \int_0^t k(t,s)ds\right|$$

$$\leq |f(t) - f(t+h)| + \underset{0\leq s\leq t+h}{Max}|g(s,\phi(s))| \left|\int_0^t (k(t+h,s) - k(t,s))ds + \int_t^{t+h} k(t+h,s)ds\right|$$

$$\leq |f(t) - f(t+h)| + \underset{0\leq s\leq t+h}{Max}|g(s,\phi(s))| \left[\int_0^t |(k(t+h,s) - k(t,s))| ds + \int_t^{t+h} |k(t+h,s)| ds\right]$$

$f(t)$ and $k(t,s)$ are continuous on $t \in [0,\infty)$

≤ 0, as $h \to 0$.

Therefore, $T\phi$ is bounded and continuous. Hence, $T\phi : B \to B$.

For ϕ and $\psi \in B$

$$|T\phi(t) - T\psi(t)| \leq \left|\int_0^t k(t,s)g(s,\phi(s))ds - \int_0^t k(t,s)g(s,\psi(s))ds\right|$$

$$|T\phi(t) - T\psi(t)| \leq \int_0^t |k(t,s)| |g(s,\phi(s)) - g(s,\psi(s))| ds$$

$$|T\phi(t) - T\psi(t)| \leq L\int_0^t |k(t,s)| |\phi(s) - \psi(s)| ds$$

$$|T\phi - T\psi| \leq L|\phi - \psi| \int_0^t |k(t,s)| ds$$

$$|T\phi - T\psi| \leq ML|\phi - \psi| \quad \text{But } ML < 1$$

$T : B \mapsto B$ is a contraction mapping. By Banach contraction mapping theorem T has unique fixed point in B. Therefore (1) has a global continuous bounded solution under considered settings.

Global existence theorem (2): Here, relaxing the hypothesis (k) in the previous theorem and defining the norm of the function using Bielecki ideas in Sherwood *et al*. (1964], the authors have established the following global existence theorem:

Consider the Volterra integral equation $x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds$

Assume that (1) satisfies the following conditions

(i) $f(t)$ is continuous and bounded on $[0,\infty)$.

(j) $k(t,s)$ is continuous on $0 \leq s < t < \infty$ and there exists $M > 0$ such that

$\int_0^t |k(t,s)| ds \leq M, t \geq 0$.

(k) $g(t,x)$ is continuous on $[0,\infty)\times R^n$ and $|g(t,x)-g(t,y)|\le L|x-y|, t\ge 0$.

(l) $g(t,0)$ is bounded on $[0,\infty)$

then there exists a unique continuous bounded solution $x(t)$ of (1) on $[0,\infty)$.

Proof: Let B_λ be the Banach space of the bounded continuous function $[0,\infty)$ with the norm $|.|$, where for $\phi\in B_\lambda$, $|\phi| = \underset{0\le t<\infty}{Sup}|\phi(t)|e^{-\lambda t}<\infty, \lambda>0$ for each $\phi\in B_\lambda$ (Corduneanu 1973).

We define the mapping $\phi :\mapsto T\phi$ by

$$T\phi(t) = f(t)+\int_0^t k(t,s)g(s,\phi(s))ds; t\ge 0$$

$$|T\phi(t)| \le |f(t)| + \int_0^t |k(t,s)||g(s,\phi(s))|ds; t\ge 0$$

$$= |f(t)| + \int_0^t |k(t,s)||g(s,\phi(s))-g(s,0)|ds + \int_0^t |k(t,s)||g(s,0)|ds; t\ge 0$$

$$T\phi(t) \le |f(t)| + L\int_0^t |k(t,s)||\phi(s)-0|ds + \int_0^t |k(t,s)||g(s,0)|ds; t\ge 0$$

$$T\phi(t) \le |f(t)| + L\int_0^t |k(t,s)|\frac{|\phi(s)|}{e^{\lambda s}}e^{\lambda s}ds + \int_0^t |k(t,s)||g(s,0)|ds; t\ge 0$$

$$T\phi(t) \le |f(t)| + L|\phi|e^{\lambda t}\int_0^t |k(t,s)|ds + \int_0^t |k(t,s)||g(s,0)|ds; t\ge 0$$

$$T\phi(t)e^{-\lambda t} \le |f(t)|e^{-\lambda t} + L|\phi|\int_0^t |k(t,s)|ds + e^{-\lambda t}\int_0^t |k(t,s)||g(s,0)|ds; t\ge 0$$

Taking supremum on both sides for $0<t<\infty$

$|T\phi| \le m_1 + LM|\phi| + m_2 M$, where $|f(t)|\le m_1$ and $|g(t,0)|\le m_2$

Therefore $T\phi(t)$ is bounded on $0\le t<\infty$ and

$$|T\phi(t+h)-T\phi(t)| =$$

$$\left| f(t+h) - f(t) + \int_0^{t+h} k(t+h,s)g(s,\phi(s))ds - \int_0^t k(t,s)g(s,\phi(s))ds \right|$$

$$|T\phi(t+h)-T\phi(t)| =$$

$$\le |f(t+h)-f(t)| + \int_0^t |k(t+h,s)-k(t,s)||g(s,\phi(s))|ds + \int_t^{t+h} |k(t+h,s)||g(s,\phi(s))|ds$$

But, $Max\{|g(s,\phi(s))| \le m_3, 0\le s\le t\}; m_3>0.$

$Max\{|g(s,\phi(s))| \le m_4, t\le s\le t+h\}; m_4>0.$

$$Max\{|k(t+h,s)| \le m_5, t \le s \le t+h\}; m_5 > 0.$$

$$|f(t+h) - f(t)| \to 0 \text{ as } h \to 0 \text{ and}$$

$$|k(t+h,s) - k(t,s)| \to 0 \text{ as } h \to 0 \text{ therefore}$$

$$|T\phi(t+h) - T\phi(t)| \le m_4 m_5 h$$

$$\Rightarrow e^{-\lambda t}|T\phi(t+h) - T\phi(t)| \le m_4 m_5 h$$

$$\Rightarrow e^{-\lambda t}|T\phi(t+h) - T\phi(t)| \to 0 \text{ as } h \to 0 .$$

Therefore $T\phi$ is continuous and $T\phi \in B_\lambda$ which implies $T : B_\lambda \mapsto B_\lambda$

For any $\phi, \psi \in B_\lambda$

$$|T\phi(t) - T\psi(t)| \le L\int_0^t |k(t,s)||\phi(s) - \psi(s)| ds$$

$$\Rightarrow |T\phi(t) - T\psi(t)| \le L\int_0^t |k(t,s)| e^{-\lambda s}|\phi(s) - \psi(s)| e^{\lambda s} ds$$

$$\Rightarrow |T\phi(t) - T\psi(t)| \le LM|\phi - \psi| \int_0^t e^{\lambda s} ds$$

$$\Rightarrow |T\phi(t) - T\psi(t)| \le LM|\phi - \psi| \frac{1}{\lambda}(e^{\lambda t} - 1)$$

$$\Rightarrow |T\phi(t) - T\psi(t)| \le LM|\phi - \psi| \frac{1}{\lambda} e^{\lambda t}$$

$$\Rightarrow |T\phi - T\psi| \le LM|\phi - \psi| \frac{1}{\lambda}$$

Here, we choose λ large enough so that $LM/\lambda < 1 \Rightarrow LM < \lambda$. Therefore, for $LM < \lambda$,

$T : B_\lambda \mapsto B_\lambda$ is a contraction mapping. By Banach contraction mapping theorem T has unique fixed point in B_λ . Therefore (1) has a global continuous bounded solution under considered settings.

CONCLUSION

If λ is taken zero both the function spaces in the theorems (1) and (2) becomes identical and second theorem is no longer valid without the hypothesis (k) and $LM < 1$ as in the first theorem.

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