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SOME FEATURES OF FUZZY COMPACTNESS

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ABSTRACT

In this paper several features of fuzzy compactness to established fuzzy analogues of well-known theorems on (usual) compact topological spaces have been described.

Key words: Fuzzy topology, Fuzzy compactness

INTRODUCTION

The concept of a fuzzy set was first introduced by Zadeh (1965) to provide a natural frame work for generalizing many of the concepts of general topology which has useful applications in various areas in mathematics. Chang (1968) developed the theory of fuzzy topological spaces and fuzzy compactness. The purpose of this paper was to study this concept in more detail and to obtain several other features.

Definition: Let X be a non-empty set and I is the closed unit interval [0, 1]. A fuzzy set in X is a function $u : X \rightarrow I$ which assigns to every element $x \in X$. u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. A member of I^X may also be a called fuzzy subset of X (Zadeh 1965).

Definition: The union and intersection of fuzzy sets are denoted by the symbols $\cup$ and $\cap$, respectively and defined by

$\cup u_i(x) = \max\{u_i(x) : i \in J \text{ and } x \in X\}$

$\cap u_i(x) = \min\{u_i(x) : i \in J \text{ and } x \in X\}$, where J is an index set (Zadeh 1965).

Definition: Let X be a non-empty set and $A \subseteq X$. Then the characteristic function $1_A(x) : X \rightarrow \{0,1\}$ defined by $1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$

Thus the workers can consider any subset of a set X as a fuzzy set whose range is {0, 1} (Zadeh 1965).

Definition: Let u and v be two fuzzy sets in X. Then we define

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(i) $u = v$ iff $u(x) = v(x)$ for all $x \in X$

(ii) $u \subseteq v$ iff $u(x) \leq v(x)$ for all $x \in X$

(iii) $\lambda = u \cup v$ iff $\lambda(x) = (u \cup v)(x) = \max[u(x), v(x)]$ for all $x \in X$

(iv) $\mu = u \cap v$ iff $\mu(x) = (u \cap v)(x) = \min[u(x), v(x)]$ for all $x \in X$

(v) $\gamma = u^c$ iff $\gamma(x) = 1 - u(x)$ for all $x \in X$ (Chang 1968).

Definition: Let $f : X \to Y$ be a mapping and u be a fuzzy set in X. Then the image of u, written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{u(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases} \quad \text{(Chang 1968).}$$

Definition: Let $f : X \to Y$ be a mapping and v be a fuzzy set in Y. Then the inverse of v, written $f^{-1}(v)$, is a fuzzy set in X whose membership function is given by $\left(f^{-1}(v)\right)(x) = v\left(f(x)\right)$ (Chang 1968).

De-Morgan's laws: De-Morgan's laws valid for fuzzy sets in X i.e. if u and v are any fuzzy sets in X, then

(i) $1 - (u \cup v) = (1 - u) \cap (1 - v)$

(ii) $1 - (u \cap v) = (1 - u) \cup (1 - v)$

For any fuzzy set in u in X, $u \cap (1 - u)$ need not be zero and $u \cup (1 - u)$ need not be one (Zadeh 1965).

Definition: Let X be a non-empty set and $t \subseteq I^X$ i.e. t is a collection of fuzzy sets in X. Then t is called a fuzzy topology on X if

(i) $0, 1 \in t$

(ii) if $u_i \in t$ for each $i \in J$, then $\bigcup_i u_i \in t$

(iii) if $u, v \in t$, then $u \cap v \in t$

The pair (X, t) is called a fuzzy topological space fts in short. Every member of t is called a t-open fuzzy set. A fuzzy set is t-closed iff its complements is t-open. In the sequel, when no confusion is likely to arise, the authors shall call a t-open (t-closed) fuzzy set simply an open closed) fuzzy set (Chang 1968).

Definition: Let (X, t) and (Y, s) be fuzzy topological spaces. A mapping $f : (X, t) \to (Y, s)$ is called an F- continuous iff the inverse of each s-open fuzzy set is t-open (Chang 1968).

Definition: Let (X, t) be an fts and $A \subseteq X$. Then the collection $t_A = \{u|_A : u \in t\} = \{u \cap A : u \in t\}$ is fuzzy topology on A, called the subspace fuzzy topology on A and the pair (A, t_A) is referred to as a fuzzy subspace of (X, t) (Mira 1981).

Definition: An fts (X, t) is said to be fuzzy Hausdorff or fuzzy-T_2 iff for all $x, y \in X$, $x \neq y$, there exist $u, v \in t$ such that $u(x) = 1$, $v(y) = 1$ and $u \cap v = 0$ (Gantner *et al.* 1978).

Distributive laws : Distributive laws remain valid for fuzzy sets in X i.e. if u, v and w are fuzzy sets in X, then

(i) $u \cup (v \cap w) = (u \cup v) \cap (u \cup w)$

(ii) $u \cap (v \cup w) = (u \cap v) \cup (u \cap w)$ (Zadeh 1965).

Definition: A family F of fuzzy sets is a cover of a fuzzy set u iff $u \subseteq \bigcup \{u_i : u_i \in F\}$. It is an open cover iff each member of F is an open fuzzy set. A subcover of F is a subfamily of F which also is a cover (Chang 1968).

Definition: An fts (X, t) is compact iff each open cover has a finite subcover (Chang 1968).

The ideas of the following theorems are taken from (Lipschutz 1965, Murdeshwar 1983 and Gaal 1964).

Theorem: Let F be a closed subset of a compact fts (X, t). Then 1_F is compact.

Proof: Let $M = \{u_i : i \in J\}$ be an open cover of 1_F i.e. $1_F \subseteq \bigcup_i u_i$. Then $1 = \left(\bigcup_i u_i\right) \cup 1_{F^c}$ that is $M^* = \{u_i\} \cup \{1_{F^c}\}$ is an open cover of 1. But 1_{F^c} is open, since 1_F is closed. So M^* is an open cover of 1. As (X, t) is compact; hence M^* has a finite subcover of 1, say $1 = u_{i_1} \cup u_{i_2} \cup ... \cup u_{i_n}$; $u_{i_k} \in M\ (1 \le k \le n)$. But 1_F and 1_{F^c} are disjoint; hence $1_F \subseteq u_{i_1} \cup u_{i_2} \cup ... \cup u_{i_n}$; $u_{i_k} \in M\ (1 \le k \le n)$. The authors have just shown that any open cover $M = \{u_i\}$ of 1_F contains a finite subcover i.e. 1_F is compact.

Theorem: For a fts (X, t), the following statements are equivalent :

(i) (X, t) is compact.

(ii) For each $\{F_i\}$ of closed subsets of (X, t); $\bigcap_i 1_{F_i} = 0$ implies $\{F_i\}$ contains a finite subfamily $\{F_{i_1}, F_{i_2}, ..., F_{i_n}\}$ with $1_{F_{i_1}} \cap 1_{F_{i_2}} \cap ... \cap 1_{F_{i_n}} = 0$.

Proof: (i) $\Rightarrow$ (ii) : Suppose $\bigcap_i 1_{F_i} = 0$. Then by De-Morgan's law, $1 = 0^c = \left(\bigcap_i 1_{F_i}\right)^c = \bigcup_i 1_{F_i^c}$. So $\{1_{F_i^c}\}$ is an open cover of (X, t), since each F_i is closed. As

(X,t) is compact, hence there exist $1_{F^c_{i_1}}$, $1_{F^c_{i_2}}$, …, $1_{F^c_{i_n}} \in \{1_{F^c_i}\}$ such that $1 = 1_{F^c_{i_1}} \cup 1_{F^c_{i_2}} \cup ... \cup 1_{F^c_{i_n}}$. Thus by De-Morgan's law, $0 = 1^c = \left(1_{F^c_{i_1}} \cup 1_{F^c_{i_2}} \cup ... \cup 1_{F^n_{i_n}}\right)^c = 1_{F_{i_1}} \cap 1_{F_{i_2}} \cap \ldots \cap 1_{F_{i_n}}$ and the authors have shown that (i) $\Longrightarrow$ (ii).

(ii) $\Rightarrow$ (i) : Let $\{u_i\}$ be an open cover of (X,t) i.e. $1 = \bigcup_i u_i$. By De-Morgan's law, $0 = \left(\bigcup_i u_i\right)^c = \bigcap_i u_i^c$. Since each u_i is open, then $\{u_i^c\}$ is a family of closed fuzzy sets and hence by above has a empty intersection.

Hence by hypothesis, there exist $u^c_{i_1}$, $u^c_{i_2}$, …, $u^c_{i_n} \in \{u_i^c\}$ such that $u^c_{i_1} \cap u^c_{i_2} \cap \ldots \cap u^c_{i_n} = 0$. Thus by De-Morgan's law, $1 = 0^c = \left(u^c_{i_1} \cap u^c_{i_2} \cap ... \cap u^c_{i_n}\right)^c = u_{i_1} \cup u_{i_2} \cup \ldots \cup u_{i_n}$.

Accordingly, (X,t) is compact and so (ii) $\Rightarrow$ (i).

Theorem: Let A be a subset of an fts (X,t). Then the following statements are equivalent :

(i) 1_A is compact with respect to t.

(ii) 1_A is compact with respect to the subspace fuzzy topology t_{1_A}.

Proof : (i) $\Rightarrow$ (ii) : Let $\{u_i : i \in J\}$ be a t_{1_A} - open cover of 1_A. Then by definition of subspsce fuzzy topology, there exists $v_i \in t$ such that $u_i = 1_A \cap v_i \subseteq v_i$. Hence $1_A \subseteq \bigcup_i u_i \subseteq \bigcup_i v_i$ and hence $\{v_i\}$ is t-open cover of 1_A. By (i), 1_A is compact, so $\{v_i\}$ contains a finite subcover, say $v_{i_k} \in \{v_i\}\ (1 \le k \le n)$ such that $1_A \subseteq v_{i_1} \cup v_{i_2} \cup ... \cup v_{i_n}$.

But, then

$1_A \subseteq 1_A \cap (v_{i_1} \cup v_{i_2} \cup ... \cup v_{i_n}) = (1_A \cap v_{i_1}) \cup (1_A \cap v_{i_2}) \cup ... \cup (1_A \cap v_{i_n}) = u_{i_1} \cup u_{i_2} \cup ... \cup u_{i_n}$. Thus $\{u_i\}$ contains a finite subcover $\{u_{i_1}, u_{i_2}, ..., u_{i_n}\}$ and $(1_A, t_{1_A})$ is compact.

(ii) $\Rightarrow$ (i) : Let $\{v_i : i \in J\}$ be a t – open cover of 1_A. Set $u_i = 1_A \cap v_i$, then $1_A \subseteq \bigcup_i v_i$ implies that $1_A \subseteq 1_A \cap \left(\bigcup_i v_i\right) = \bigcup_i (1_A \cap v_i) = \bigcup_i u_i$. But $u_i \in t_{1_A}$, so $\{u_i\}$ is t_{1_A} - open cover of 1_A. As 1_A is t_{1_A} - compact, thus $\{u_i\}$ contains a finite subcover $\{u_{i_1}, u_{i_2}, ..., u_{i_n}\}$.

Accordingly, $1_A \subseteq u_{i_1} \cup u_{i_2} \cup ... \cup u_{i_n} = (1_A \cap v_{i_1}) \cup (1_A \cap v_{i_2}) \cup ... \cup (1_A \cap v_{i_n})$ = $1_A \cap (v_{i_1} \cup v_{i_2} \cup ... \cup v_{i_n}) \subseteq v_{i_1} \cup v_{i_2} \cup ... \cup v_{i_n}$. Thus $\{v_i\}$ contains a finite subcover $\{v_{i_1}, v_{i_2}, ..., v_{i_n}\}$ and therefore 1_A is compact with respect to t.

Theorem: Let (X, t) and (Y, s) be two fts and $f : (X, t) \to (Y, s)$ be an onto, continuous function. Then (X, t) is compact iff (Y, s) is compact.

The necessary part of this theorem has already been proved by (Chang 1968).

Proof: Suppose (X, t) is compact. Let M = $\{u_i : i \in J\}$ be an open cover of (Y, s) with $\bigcup_{i \in J} u_i = 1$. Since f is continuous, so $f^{-1}(u_i) \in t$. As (X, t) is compact, the authors have for each $x \in X$,

$\bigcup_{i \in J} f^{-1}(u_i)(x) = 1$. Thus it was seen that H = $\{f^{-1}(u_i) : i \in J\}$ is an open cover of (X, t). Hence there exists $u_{i_k} \in M$ such that $\bigcup_{i \in J} f^{-1}(u_{i_k}) = 1$. Again, let u be any fuzzy set in Y. Since f is onto, then for any $y \in Y$, the authors have $f\left(f^{-1}(u)\right)(y)$ = $\sup\left\{f^{-1}(u)(z) : z \in f^{-1}(y), f^{-1}(y) \neq \phi\right\}$ = $\sup\{u\left(f(z)\right) : f(z) = y\}$ = sup{u(y)} = u(y) i.e. $f\left(f^{-1}(u)\right) = u$. This is true for any fuzzy set in Y. Hence 1 = f(1) = $f\left(\bigcup_{i \in J} f^{-1}(u_{i_k})\right)$ = $\bigcup_{i \in J}\left(f(f^{-1}(u_{i_k})\right)$ = $\bigcup_{i \in J}(u_{i_k})$. Thus (Y, s) is compact.

Conversely, suppose (Y, s) is compact. Let W = $\{v_j : j \in J\}$ be an open cover of (X, t) with $\bigcup_{j \in J} v_j = 1$. Since f is onto, so $\left\{f(v_j) : j \in J\right\}$ is an open cover of (Y, s). As (Y, s) is compact, then for each $y \in Y$, we have $\bigcup_{j \in J} f(v_j)(y) = 1$. Hence, there exists $f(v_{j_k}) \in \{f(v_j) : j \in J\}$ such that $\bigcup_{j \in J} f(v_{j_k}) = 1$. Again, let v be any fuzzy set in X. Since f is onto and continuous, then for any $x \in X$, we have $f^{-1}\left(f(v)\right)(x)$ = f(v) (f (x)) = $v\left(f^{-1}(f(x))\right)$ = v(x) i.e. $f^{-1}\left(f(v)\right) = v$. This is true for any fuzzy set in X.

Hence $f^{-1}(1)$ = $f^{-1}\left(\bigcup_{j \in J} f(v_{j_k})\right)$ = $\bigcup_{j \in J}\left(f^{-1}(f(v_{j_k})\right)$ = $\bigcup_{j \in J}(v_{j_k})$. Thus (X, t) is compact.

Theorem: Let (X, t) be an fts and $\{Y_S\} \subseteq X$, where $\{Y_s\}$ be a finite family. If each Y_s is compact, then $\bigcup Y_s$ is a compact subspace of (X, t).

Proof : Let $\{u_i : i \in J\}$ be an open cover of $\bigcup Y_s$. Then $\{u_i : i \in J\}$ is an open cover of Y_s for each $s \in S$. Since Y_s is compact, then $\{u_i : i \in J\}$ contains a finite subcover, say $\{u_{i_k} : i_k \in J\}$ $(1 \leq k \leq n)$ which is a cover of Y_s. The union of these families is a finite subcover of $\bigcup Y_s$. Thus $\bigcup Y_s$ is compact.

Theorem: Let (X,t) be a fuzzy Hausdorff space and A be a compact subset of (X,t). Suppose $x \in A^c$, then there exist $u, v \in t$ such that $u(x) = 1$, $A \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Since $x \notin A$ $(x \in A^c)$, then clearly $x \neq y$. As (X,t) is fuzzy Hausdorff, then there exist $u_y, v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Hence $A \subseteq \bigcup\{v_y : y \in A\}$ i.e. $\{v_y : y \in A\}$ is an open cover of A. Since A is compact, then there exist $v_{y_1}, v_{y_2}, \ldots, v_{y_n} \in \{v_y\}$ such that $A \subseteq v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots \cap u_{y_n}$. Thus see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $u, v \in t$. Furthermore, $A \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, since each $u_{y_i}(x) = 1$ individually.

Finally, the authors claim that $u \cap v = 0$. We observe that $u_{y_i} \cap v_{y_i} = 0$ implies that $u \cap v_{y_i} = 0$, by distributive law, the authors observed that $u \cap v =$ $u \cap (v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}) = (u \cap v_{y_1}) \cup (u \cap v_{y_2}) \cup \ldots (u \cap v_{y_n}) = 0.$

Theorem: Let (X,t) be a fuzzy Hausdorff space and A, B be disjoint compact subsets of (X,t). Then there exist $u, v \in t$ such that $A \subseteq u^{-1}(0, 1]$, $B \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Then $y \notin B$, as A and B are disjoint. Since B is compact, then by previous theorem, there exist $u_y, v_y \in t$ such that $u_y(y) = 1$, $B \subseteq v_y^{-1}(0, 1]$ and $u_y \cap v_y = 0$. Since $u_y(y) = 1$, then we see that $\{u_y : y \in A\}$ is an open cover of A. As A is compact, then there exist $u_{y_1}, u_{y_2}, \ldots, u_{y_n} \in \{u_y\}$ such that $A \subseteq u_{y_1} \cup u_{y_2} \cup \ldots \cup u_{y_n}$. Furthermore, $B \subseteq v_{y_1} \cap v_{y_2} \cap \ldots \cap v_{y_n}$, as $B \subseteq v_{y_i}$ for each i. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots \cap v_{y_n}$. Thus the authors noted that $A \subseteq u^{-1}(0, 1]$ and $B \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $u, v \in t$.

Finally, the authors had to show that $u \cap v = 0$. First, they observe that $u_{y_i} \cap v_{y_i} = 0$ for each i, implies that

$u_{y_i} \cap v = 0$, by distributive law, it is seen that $u \cap v = (u_{y_1} \cup u_{y_2} \cup \ldots \cup u_{y_n}) \cap v$

$= (u_{y_1} \cap v) \cup (u_{y_2} \cap v) \cup \ldots \cup (u_{y_n} \cap v) = 0.$

Theorem: Let A be a compact subset of a fuzzy Hausdorff space (X,t). Then A is closed.

Proof: Let $x \in A^c$. The authors have to show that there exists $u \in t$ such that $u(x) = 1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Now, let $y \in A$, then there exist $u_y, v_y \in t$ such that $u_y(x)=1$, $v_y(y)=1$ and $u_y \cap v_y = 0$. Thus it is seen that $A \subseteq \bigcup\{v_y : y \in A\}$ i.e. $\{v_y : y \in A\}$ is an open cover of A. Since A is compact, so it has a finite subcover, say $v_{y_1}, v_{y_2}, \ldots, v_{y_n} \in \{v_y\}$ such that $A \subseteq v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}$. Again, let $u = u_{y_1} \cap u_{y_2} \cap \ldots \cap u_{y_n}$ and $v = v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}$. Hence the authors observed that u(x) = 1, as $u_{y_k}(x)=1$ for each k and $u \cap (v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}) = 0$. For each $z \in A$, it is clear that $\bigcup\{v_{y_k}\}(z)=1$ $(1 \le k \le n)$. Thus $u(z)=0$ and hence $u \subseteq A^p$. Therefore, A^c is open and so A is closed.

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