CHAOTIC BEHAVIOR OF DYNAMICAL SYSTEMS OF HOMEOMORPHISM ON UNIT INTERVAL

PAYER AHMED
Department of Mathematics, Jaganath University, Dhaka, Bangladesh
AND
MD. SHAHIDUL ISLAM*
Department of Mathematics, University of Dhaka, Dhaka-1000, Bangladesh

ABSTRACT

In this article, we studied that no homeomorphism on unit interval into itself is chaotic in the sense of R.L. Devaney. We also studied the behavior of orbits of points in the dynamical system defined by homeomorphism on the unit interval.

INTRODUCTION

Chaotic dynamical systems have received a great deal of attention in recent years. Chaotic phenomena can be found in nearly all branches of non-linear modeling of dynamical systems. Chaos explains how very small changes in the initial configuration of a system model may lead to great discrepancies overtime, called the "butterfly effect". This phenomenon accounts for our inability to make accurate prediction of weather. For example, despite enormous computing power and loads of data, chaos reviews linear iteration and explains both linear and nonlinear iteration with fixed points, cycles, and orbits through both graphical iteration and orbit diagrams. A deeper look at chaotic behavior and its unpredictability is investigating a phenomenon known as sensitive dependence on initial conditions.(1) If we look at the logistic iteration rule $x \rightarrow 4x(1-x)$ with two nearby seeds 0.5 and 0.5001, the orbit of first seed 0.5 is close to that of seed 0.5001, for first 13 or 50 iterations, they move away very differently.

We studied mathematical theory concerning chaotic dynamical systems. Our attention is restricted to continuous maps on one-dimensional space. Typical chaotic maps on one-dimensional space are unimodal maps such as the tent map and logistic map on unit interval and the two-sided shift map of the Cantor set. The tent map, logistic map and two-sided shift map of the Cantor sets are chaotic. But it is obvious that the tent map and the logistic map on unit interval are not homoeomorphism of the unit interval into itself but the two-sided shift map of the Cantor set is a homoeomorphism of the Cantor set into itself. In this paper, we show that no homoeomorphism on unit interval into itself is chaotic. As the unit interval [0,1] is compact metric space and has infinitely many points, we use the theorem of J. Banks et al (2) instead of using Devaney's independent chaotic three conditions. If metric space X is a finite set but not the unit interval and the case where metric space X is homeomorphic to the Cantor set, there exists a lot of chaotic homeomorphisms on metric spaces.

We describe the behavior of orbits of point in the dynamical system defined by homeomorphism on the unit interval. (3) Taking into account all the cases, we find a lot of non-chaotic homeomorphisms on [0, 1].

PRELIMINARIES

Definition 2.1. Let *X* be a topological space and *f* be a continuous map of X into X. The pair (X, f) is called a **topological dynamical system**. Given *x* in *X*, the subset $\{f^n(x)\}_{n=1}^{\infty}$ of X is called the **orbit** of X under X and is denoted by $J(x)$. Sometimes we need the expression $x_n = f^n(x)$ for *x* in X and $x_0 = x$ is called the seed of the orbit. The function f^n is called the *n*-th iteration of f. A function f is called a **homeomorphism** if it satisfies the conditions (i) f is bijective (ii) f and f^{-1} are continuous.

* Author for Correspondence

Let $f : X \to X$ and $g : Y \to Y$ be two functions. We say that f and g are **conjugate** if there exits a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The map h is called conjugacy. We note here that the logistic map λ and the tent map τ on [0, 1] are topologically conjugate by conjugacy $h(x) = \sin^2 \frac{\pi}{2} x$ on [0, 1].

Notation 2.2. We denote by **R** the set of all real numbers, **Z** the set of all integers, **N** the set of all positive integers and ***I*** the unit interval. For $k \in$ **N**, the sets of periodic points for a continuous map $f : X \to X$ is denoted as follows: $P_k(f) = \{x \in X : f_k(x) = x\}$ and

$Q_k(f) = \{x \in X : f_k(x) = x,\ f^i(x) \neq x \quad \text{for} \quad i = 1,2,\ldots,k-1\}$ and

$Per(f) = \bigcup_{k=1}^{\infty} P_k(f)$. Namely, $P_k(f)$ is the set of all *k*-periodic points, $Q_k(f)$ is the set of those *k*-periodic points whose prime period is *k* and $Per(f)$ is the set of all periodic points. Obviously, $Per(f)$ and $\{Q_k(f)\}_{k=1}^{\infty}$ are family of mutually disjoint subsets of *X*.

2.3. Examples of topological Dynamical Systems

Example 2.3.1. Let *X* = [0, 1] and the λ be the tent map which is defined by $\lambda(x) = 4x(1 - x)$.

Example 2.3.2. Let *X* = [0, 1] and τ be the tent map which is defined by $\tau(x) = 1 - |1 - 2x|$.

Example 2.3.3. Let *X* = [0, 1] and f be a continuous map of X into itself.

Example 2.3.4. Let *X* = **R** and f be continuous map of X into itself.

Example 2.3.5. Let *X* = **N** and f be the map of X into itself defined by $f(n) = n + 1$, for $\forall\ n \in$ **N**.

Example 2.3.6. Let *X* = {1,2,3,...,*n*} and f be the map of X into itself defined by $f(k) = k + 1$ for $1 \le k \le n - 1$ and $f(n) = 1$.

Example 2.3.7. Let $X = \{e^{2\pi i x} : 0 \le x \le 1\}$ be the unit circle in the complex plane and f_θ be the map defined by $f_\theta(e^{2\pi i x}) = e^{2\pi i (x+\theta)}$. The map f_θ is called the θ-rotation on the unit circle.

In the above examples, we considered the usual metrics, that is, $d(x, y) = |x - y|$ in [0, 1] and **R** (Examples 2.3.3 and 2.3.4),

$d(i, j) = \begin{cases} 1 & if \quad i \neq j \\ 0 & if \quad i = j \end{cases}$ in **N** and {1,2,3.......,*n*} (Examples 2.3.5 and 2.3.6)

and $d\left(e^{2\pi i x}, e^{2\pi i y}\right) = |e^{2\pi i x} - e^{2\pi i y}|$ in the unit circle (Example 2.3.7).

The metric *d* induces a canonical topology $O(d)$ in *X*. The family $O(d)$ is defined by the open balls $U(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$. Namely, $O(d)$ is the family of those subsets U which satisfies the following conditions: for any *x* in U there exits $\varepsilon > 0$ such that $U(x, \varepsilon) \subset U$. The infinite product of two points is called the Cantor set and in this paper we use two kinds of Cantor sets with matrices, namely $\Sigma_+ = \{(x_n)_{n \in N} : x_n \in \{0,1\}\}$ with metric d_+, where $d_+(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$ for $x = (x_n)_{n \in N}$ and $y = (y_n)_{n \in N}$ in Σ_+ and $\Sigma = \{(x_n)_{n \in Z} : x_n \in \{0,1\}\}$ with metric *d*, where $d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n} + \sum_{n=1}^{\infty} \frac{|x_{-n} - y_{-n}|}{2^n}$ for $x = (x_n)_{n \in Z}$ and $y = (y_n)_{n \in Z}$ in Σ.

The following is a typical dynamical system on the Cantor set.

Example 2.3.8. Let X = Σ_+ and S_+ be the map of X into itself defined by $y = S_+(x)$ where $x = (x_n)_{n \in N} \in \Sigma_+$, and $y = (y_n)_{n \in N} \in \Sigma_+$ and $y_n = x_{n+1}$ for all n in **N**.

Example 2.3.9. (**Chaotic dynamical systems**)

(**Tent map**) $t : [0, 1] \to [0, 1]$, $t(x) = 1 - |1 - 2x|$.

(**Logistic map**) $l : [0, 1] \to [0, 1]$, $l(x) = 4x(1 - x)$.

(**Shift map**) $\Sigma = \Pi_{n \in N}\{0, 1\}$, $S_+ : (x_0 x_1 x_2 \ldots\ldots) \to (x_1 x_2 \ldots\ldots)$.

Definition 2.4. (Devaney) Let X be a metric space. A continuous map $f : X \to X$ is said to be **chaotic** on X if f satisfies the following properties [1] :

(C-1) Periodic points of f are dense in X.

(C-2) f is one-sided topologically transitive.

(C-3) f has sensitive dependence on initial conditions.

Using notation, conditions (C-1) and (C-2) can be re-written as follows:

(C-1) $Per(f)$ is dense in X.

(C-2) For any pair of non-empty point sets U and V in X there exists $k \in$ **N** such that $f^k(U) \cap V \neq \phi$.

(C-3) There exists $\delta > 0$ which satisfies for any $x \in X$ and any neighborhood N_x of x there exist $y \in N_x$ and $k \in$ **N** such that $d\left(f^k(x), f^k(y)\right) \geq \delta$.

The Conditions (C-1) and (C-2) are topological properties but the Condition (C-3) is not a topological property but metric one. However, if X is a compact metric space, the Condition (C-3) becomes as a topological property.

Theorem 2.5. (J. Banks et al. [2]) Let X be compact and has infinitely many points. If $f : X \to X$ is transitive and has dense periodic points, then f has sensitive dependence on initial conditions. Namely, Conditions (C-1) and (C-2) implies Condition (C-3)

REMARK: If X is a subset of **R** and finite, then there exists a homeomorphism of X which satisfies Conditions (C-1) and (C-2) but does not satisfy Condition (C-3). Namely the above result does not hold and is shown in Example 3.1.3.

In order to describe the behavior of orbits of given points on the unit interval, we need the following lemma.

Lemma 2.6. Let f be continuous map on **I** = [0, 1] with $f(0) = 0$ and $f(1) = 1$. If $f(x_0) > x_0$ (resp. $f(x_0) < x_0$) for some $x_0 \in$ **I**, then there exist x_1 and x_2 in **I** such that

(i) $x_1 < x_0 < x_2$ (ii) $f(x_1) = x_1$ and $f(x_2) = x_2$, (iii) $f(x) > x$ for all $x \in (x_1, x_2)$ (resp. $f(x) < x$).

Proof: It is sufficient here to note only that $x_1 = \max\{x \in I : f(x) = x \text{ and } x < x_0\}$ and $x_2 = \min\{x \in I : f(x) = x \text{ and } x > x_0\}$.

Now we confirm the following theorem and the proof is given according to the cases may be considered, in which we show that the map does not satisfy Condition (C-1) or (C-2). In order to describe the behavior of orbits in the unit interval ***I***, we use the following lemma.

Lemma 2.7. Let U be a bounded open set in the real line with usual topology. Then U is of the form $U = \bigcup_{i \in A} (x_i, y_i)$, where A is at most countable set and $\{(x_i, y_i) : i \in A\}$ is family of mutually disjoint open intervals.

Proof: Since the set of all rational numbers in *U* is countable, that is, $Q \subset U$ is countable, we can write, $Q \subset U = \{r_i\}_{i=1}^{\infty}$. For each r_i, there exists $\varepsilon > 0$ such that $(r_i - \varepsilon, r_i + \varepsilon) \subset U$. Let $a_i = \min\{a : (a, x) \subset U\}$, $b_i = \max\{b : (x, b) \subset U\}$. Then $\bigcup_{i=1}^{\infty}(a_i, b_i) \subset U$. Next we show that $U \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$.

Let $x \in U$. Since *U* is open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$. Since *Q* is dense, there exists $r_i \in Q \subset U$ such that $r_i \in (x - \varepsilon, x + \varepsilon)$. Thus $(x - \varepsilon, x + \varepsilon) \subset (a_i, b_i)$, that is, $U \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$.

Now we show that if $(a_i, b_i) \subset (a_j, b_j) \neq K$ then $(a_i, b_i) = (a_j, b_j)$. Suppose that there exists $z \in (a_i, b_i) \subset (a_j, b_j)$ and let $a = \min(a_i, b_i)$, and $b = \max(a_j, b_j)$.

Then $(a_i, b_i) \cup (a_j, b_j) = (a, b)$. Since $r_i \in (a_i, b_i) \subset (a, b) \subset U$, $r_j \in (a_j, b_j) \subset (a, b) \subset U$ and (a_i, b_i) is maximal, we have $(a_i, b_i) = (a, b)$ and $(a_j, b_j) = (a, b)$.

Therefore, if $(a_i, b_i) \neq (a_j, b_j)$ then $(a_i, b_i) \subset (a_j, b_j) = K$. This means, $(a_i, b_i) = (a_j, b_j)$ or $(a_i, b_i) \subset (a_j, b_j) \neq K$. Now we define a finite or infinite sequence $\{k_i\}$ as follows:

$$k_1 = 1$$

$$k_2 = \min\{i > 1 : (a_i, b_i) \subset (a_1, b_1) = \phi\}.$$

By the mathematical induction,

$$k_n = \min\{i > k_{n-1} : (a_i, b_i) \subset (a_k, b_k) = \phi \text{ for } k = k_1, k_2, \ldots.., k_{n-1}\}.$$

Now we put $A = \{k_n\}$. Then *A* is finite or countable and $U = \bigcup_{i \in A}(a_i, b_i)$.

MAIN RESULTS

3.1. Existence of chaotic homeomorphisms on the unit interval

Theorem 3.1.1. Let *f* be a homeomorphism of [0, 1]. Then *f* is not chaotic.

Proof: Case (A) [*f* is increasing, that is, $f(0) = 0$ and $f(1) = 1$]

Case (A-1) [$f(x) = x$ for all $x \in I$]

Let *U* and *V* be two non-empty disjoint open sets in ***I***. Then $f^k(U) \subset V = U \subset V = \phi$ for any $k \in$ **N**. Therefore, *f* is not topologically transitive. Hence *f* does not satisfy (C-2).

Case (A-2) [There exists $x_0 \in$ ***I*** such that $f(x_0) \neq x_0$]

First we suppose that $f(x_0) > x_0$. Then by Lemma 1.1, there exist x_1 and x_2 in ***I*** such that (i) $x_0 \in (x_1, x_2)$, (ii) $f(x_1) = x_1$, $f(x_2) = x_2$, (iii) $f(x) > x$ for all $x \in (x_1, x_2)$. In this case we have $\lim_{n \to \infty} f^n(x) = x_2$ for all $x \in (x_1, x_2)$. Namely, $x \notin P_k(f)$ for all $k \in$ **N**. Thus $x \notin \bigcup_{k=1}^{\infty} P_k(f)$, so that $(x_1, x_2) \cap Per(f) = \phi$. Therefore, $Per(f)$ is not dense in ***I***. Hence *f* does not satisfy (C-1). In the case where $f(x_0) < x_0$, it can be proved in same way as the above case that *f* does not satisfy (C-1).

Case (B) [*f* is decreasing, that is, $f(0) = 1$]

We note that $P_1(f) = Q_1(f) = \{x_0\}$ for some $x_0 \in (0, 1)$.

Case (B-1) [$f^2(x) = x$ for all $x \in$ ***I***] Let $x_1 < x_0$ and $y_1 = f(x_1)$. Then $y_1 > x_0$ and $f(y_1) = f^2(x_1) = x_1$. Put $U = (0, x_1)$ and $V = (x_1, x_0)$. Then $f(U) = (y_1, 1)$ and $f^2(U) = f(f(U)) = f(y_1, 1) = (0, x_1) = U$. Thus we have

$$f^k(U) \cap V = \begin{cases} (y_1, 1) \cap (x_1, x_0) = \phi \\ (0, x_1) \cap (x_1, x_0) = \phi. \end{cases}$$

Therefore, f is not topologically transitive. Hence f does not satisfy condition (C-2).

Case (B-2) [There exists $x_0 \in I$ such that $f^2(x_0) \neq x_0$] We suppose that $f^2(x_0) > x_0$. Then by Lemma 2.6, there exist x_1 and x_2 in ***I*** such that (i) $x_0 \in (x_1, x_2)$, (ii) $f^2(x_1) = x_1$, $f^2(x_2) = x_2$, (iii) $f^2(x) > x$ for all $x \in (x_1, x_2)$. In this case we have that $\lim_{n \to \infty} f^{2n}(x) = x_2$ for all $x \in (x_1, x_0)$. Namely, $x \notin P_k(f)$ for all $k \in$ **N**. Thus $x \notin \bigcup_{k=1}^{\infty} P_k(f)$. Thus $(x_1, x_2) \cap Per(f) = \phi$. Therefore $Per(f)$ is not dense in ***I***. Hence f does not satisfy Condition (C-1). In the case where $f^2(x_0) < x_0$, it can be proved in the same way as the above case that f does not satisfy Condition (C-1).

It goes without saying that no homeomorphisms of [*a*, *b*] are one-sided topologically transitive for all *a* and *b* with $a < b$.

In the above theorem, we have proved that any homeomorphism of the unit interval does not satisfy the Condition (C-1) or (C-2). There are a lot of homeomorphisms which are chaotic on metric spaces. The following examples are of them, one of which is the case where metric space *X* is a finite set but not the unit interval and the other is the case where *X* is homeomorphic to the Cantor set.

Example 3.1.2. Let $X = \{x_0, x_1, ..., x_{n-1}\}$ and f be the map on *X* defined by $f(x_i) = x_{i+1}$ for $0 \leq i \leq n-2$ and $f(x_{n-1}) = x_0$. Then f is chaotic and homeomorphism of *X* into itself.

Example 3.1.3. Let $X = \Pi_{n \in Z}\{0,1\}$, where ***Z*** is the set of all integers and f be the two sided shift of *X*. Namely, for $x = (x_k)_{k \in Z}$ and $y = f(x) = (y_k)_{k \in Z}$, we have $y_k = y_{k+1} (k \in Z)$. Then f is chaotic and homeomorphism of *X* into itself.

3.2. Orbits of Points in dynamical systems defined by homeomorphism on unit interval.

In the preceding sub-section we have shown that any homoeomorphism of the unit interval is not chaotic. However, our interest is to know the behavior of orbits of given points in the dynamical systems. Therefore, in this sub- section, we describe the behavior of orbits for those homeomorphisms completely. In the following, we show the behavior of orbits of points defined by homeomorphisms in unit interval ***I*** = [0, 1] according to the cases in Theorem 3.1.1

Observation 3.2.1. Let f is a homeomorphism of ***I*** = [0, 1]. Then the orbits of points in unit interval ***I*** as follows:

Case (A) [f is increasing, that is, $f(0) = 0$ and $f(1) = 1$]

Case (A-1) [$f(x) = x$ for all $x \in I$] $\lim_{n \to \infty} f^n(x) = x$ for all $x \in$ ***I***.

Case (A-2) [There exists $x_0 \in (0, 1)$ such that $f(x_0) \neq x_0$] Put $U = I - P_1(f)$. Since $P_1(f)$ is a closed set containing {0, 1}, the set *U* is non-empty and open. Thus, by Lemma 2.7, *U* is a union of mutually disjoint open sets, that is, $U = \bigcup_{i \in A} (x_i, y_i)$, where $(x_i, y_i) \cap (x_j, y_j) = \emptyset$ for $i \neq j$. Now it follows that $f(x) > x$ for all $x \in (x_i, y_i)$ or $f(x) < x$ for all $x \in (x_i, y_i)$ for each open interval. Then for $x \in (x_i, x_{i+1})$ we have $\lim_{n \to \infty} f^n(x) = x_{i+1}$ (resp. x_i) if $f(x) > x$ (resp. $f(x) < x$). We note that in the case where $P_1(f)$ is finite, we have $\{x_i\}_{i \in A} = P_1(f)$ and $\{x_i\}_{i \in A}$ can be arranged as follows: $\{x_i\}_{i \in A} = \{x_i\}_{i=1}^{n}$ where $0 = x_1 < x_2 < ..... < x_n = 1$.

Case (B) [f is decreasing, that is, $f(0) = 1$ and $f(1) = 0$]

Case (B-1) [$f^2(x) = x$ for all $x \in$ ***I***]

Since f is a decreasing homeomorphism, $P_1(f)$ consists of only one point in (0, 1). Let p be the fixed point. Then we have $f([0,p])=[p,1]$ and $f^2(x)=x$ for all $x\in$ ***I***. Thus we have $\lim_{n\to\infty} f^{2n}(x)=x$ and $\lim_{n\to\infty} f^{2n+1}(x)=f(x)$ for all $x\in$ ***I***.

Case (B-2) [There exists $x_0\in$ (0, 1) such that $f^2(x_0)\neq x_0$]

In the above case, we put $P_1(f)=\{p\}$. Since $P_2(f)$ is closed set which contains {0, 1}, the set $1-P_2(f)$ is non-empty and open. Moreover, we can see that there exists a bijective correspondence by the map f between $P_2(f)\cap(0,p)(=Q_2(f)\cap(0,p))$ and $P_2(f)\cap(p,1)(=Q_2(f)\cap(p,1))$. Now we put $U_0=(1-P_2(f))C(0,p)$ and $U_1=(1-P_2(f))C(p,1)$.

Then U_0 and U_1 are non-empty open sets having following properties:

$$U_0=\bigcup_{i\in A}(x_i,y_i),\ U_1=\bigcup_{i\in A}(z_i,w_i),\quad f(x_i)=w_i,\quad f(y_i)=z_i.$$

Now it follows that (A) $f^2(x)>x$ for all $x\in(x_i,y_i)$ or (B) $f^2(x)<x$ for all $x\in(x_i,y_i)$ for each open interval.

Thus in Case (A) (resp. Case (B)) we have, for $x\in(x_i,y_i)$, $f^{2n}(x)\in(x_i,y_i)$ and $\lim_{n\to\infty} f^{2n}(x)=y_i$ (resp. x_i), $f^{2n+1}(x)\in(z_i,w_i)$ and $\lim_{n\to\infty} f^{2n+1}(x)=w_i$ (resp. z_i). This completes the observation.

3.3. Classification of homeomorphism of *I* = [0, 1] with corresponding examples

We classify the homeomorphism of the unit interval and try to give examples of all possible classes (if any). By this classification and Theorem 3.1.1, we find a lot of non-chaotic homeomorphisms of the unit interval.

Class 3.3.1. [$f(x)=x$ for all $x\in$ ***I***]

$$f(x)=x \text{ for } x\in I.$$

Class 2. [There exists $x_0\in(0,1)$ such that $f(x_0)\neq x_0$]

Class 2.1. [$f(x)>x$ for all $x\in$ (0, 1)]

Class 2.2.

$$f(x)=\sqrt[n]{x}\ (n=2,3,4....)\quad \text{for}\quad x\in I.$$

$$[f(x)<x\quad \text{for all } x\in(0,1)].$$

$$f(x)=x^n\ (n=2,3,4...)\ \text{for}\quad x\in I.$$

Class 2.3. [There exist finitely many $x\in$ (0, 1) such that $f(x)=x$]

$$f(x)=\frac{1}{k\pi}\sin(k\pi x)+x\ (k\in N) \text{ for } x\in \boldsymbol{I}.$$

Class 2.4. [There exist countable many $x\in$ (0, 1) such that $f(x)=x$]

$$f(x)=\frac{1}{7}x^2\sin\left(\frac{2\pi}{x}\right)+x \text{ for } x\in \boldsymbol{I}.$$

Class 2.5. [There exist uncountable many $x\in$ (0, 1) such that $f(x)=x$].

$$f(x)=\begin{cases} x, & 0\le x\le \frac{1}{2}\\ \sqrt{\left(\frac{1}{2}\right)^2-(x-1)^2}+\frac{1}{2}, & \frac{1}{2}\le x\le 1.\end{cases}$$

Class 3. [$f^2(x)=x$ for all $x\in$ ***I***]

Class 3.1. [$f(x)=1-x$ for all $x \in \boldsymbol{I}$]

$$f(x)=1-x \text{ for } x \in \boldsymbol{I}.$$

Class 3.2. [$f(x)>1-x$ for all $x \in (0, 1)$]

$$f(x)=\sqrt[n]{1-x^n} \quad (n = 2,3,4,...) \text{ for } x \in \boldsymbol{I}.$$

Class 3.3. [$f(x)<1-x$ for all $x \in (0, 1)$]

$$f(x)=\left(1-\sqrt[n]{x}\right)^n \quad (n = 2,3,4,...) \text{ for } x \in \boldsymbol{I}.$$

Class 4. [There exists $x_0 \in (0, 1)$ such that $f^2(x_0) \neq x_0$]

Class 4.1. [$f^2(x) > x$ for all $x \in (0, 1)$]

There is no example in this class since $P_1(f)$ has only one point.

Class 4.2. [$f^2(x) < x$ for all $x \in (0, 1)$]

There is no example in this class because of the same reason as the above case.

Class 4.3. [There exist finitely many $x \in (0, 1)$ such that $f^2(x) = x$]

$$f(x)=\frac{1}{k\pi}\sin(k\pi(1-x))+1-x \text{ for } x \in \boldsymbol{I}.$$

Class 4.4. [There exist countable many $x \in (0, 1)$ such that $f^2(x) = x$]

$$f(x)=\frac{1}{7}(1-x)^2\sin\left(\frac{2\pi}{1-x}\right)+1-x \text{ for } x \in \boldsymbol{I}.$$

Class 4.5. [There exist uncountable many $x \in (0, 1)$ such that $f^2(x) = x$]

CONCLUSION

We have confirmed that there does not exist any chaotic homeomorphism on the unit interval into itself and if metric space X is a finite set but not the unit interval or if X is homeomorphic to the Cantor set, there exist a lot of chaotic homeomorphisms. As a result it is shown that there exists a chaotic homeomorphism of X into itself if and only if X is homeomorphic to the Cantor set. [(4)] Here we have described the behavior of orbits of given points on the unit interval which are defined by homeomorphisms. The behavior of orbits of given probability density functions instead of given points is shown in.[(5)]

ACKNOWLEDGEMENT

The authors would like to thank to the referee for reading the article carefully and comment to improve the article.

REFERENCES

1. R.L. Devaney, An Introduction to Chaotic Dynamical Systems, Addison-Wesley, Redwood City, 1989.
2. J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's Definition of Chaos, *Amer. Math. Monthly*, 99, 332-334, 1992.
3. P. Ahmed, Chaotic Homeomorphisms in One Dimensional Dynamics, *Proc.*, *Iwate Univ.*, *Japan*, 1- 4, *May* 2001.
4. S. Kawamura, Chaotic Maps on a Measure Space and the Behavior of the Orbit of State, *Tokyo J. Math.*, 2001.
5. P. Ahmed, Analysis of Chaotic Dynamical Systems, *Ph.D. Thesis*, *Yamagata University*, *Yamagata*, *Japan*, *March* 2002.