CERTAIN FEATURES OF FUZZY CONTRA-CONTINUOUS FUNCTIONS

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ABSTRACT

We deal with fuzzy topological spaces, fuzzy compact space, fuzzy S-closed space, fuzzy graph, fuzzy continuous functions and fuzzy LC-continuous functions. In this paper, we introduce the concepts of fuzzy contra-continuities and explore properties and relationships of such types of functions.

Keywords: fuzzy contra-continuity, fuzzy S-closed space, fuzzy graph.
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1. INTRODUCTION

C L. Chang[1] defined fuzzy topological space in 1968 by using fuzzy sets introduced by Zadeh. In 1976, Thompson[2] has introduced the notion of S-closed spaces via Levine's semi-open sets[3]. In 1981 Azad[4] has introduced some weaker forms of fuzzy continuity in fuzzy topological spaces. He introduced fuzzy semi-continuous functions, semi-open functions, semi-closed functions, almost continuous functions and weakly continuous functions in fuzzy topological spaces. Using Azad's notion of fuzzy sets Abdulla and Bin Shahna[5] have introduced fuzzy δ-open, fuzzy δ-closed, fuzzy pre-open, fuzzy pre-closed sets and have made preliminary study of fuzzy strong semi-continuous and fuzzy pre-continuous functions in their papers. In 1989, Ganster and Reilly[6] introduced the notion of LC-continuous functions via the concepts of locally closed sets. In 1996, Dontchev[7] studied a stronger form of LC-continuity called contra-continuity and proved that contra-continuous images of strongly S-closed spaces are compact as well as that contra-continuous, ρ-continuous images of S-closed spaces are also compact. In 2006, Ekici and Kerre[8] studied the notion of fuzzy contra-ρ-continuous functions. In our study, we introduce several types of fuzzy contra-continuities, the notion of fuzzy contra-semi-continuous functions, fuzzy contra-pre-continuous functions and investigate of some of their properties.

2. Preliminaries

In this section, we recall some definitions and some results, which will be useful in our investigations.

2.1 Definition: Let X be a non-empty set and I = [0, 1]. A fuzzy set in X is a function $\lambda: X \to I$ which assigns to each element $x \in X$, a degree or grade of membership $\lambda(x) \in I$. Fuzzy sets in X will be denoted by Greek letters as $\alpha, \beta, \lambda, \mu, \eta,$ etc.

2.2 Definition[1]: Let X be a non-empty set and t be a collection of fuzzy sets in X. Then t is called a fuzzy topology in X if.

(i) $0, 1 \in t$

(ii) $\alpha, \beta \in t \Rightarrow \alpha \cap \beta \in t$ and (iii) $\alpha_i \in t \Rightarrow \psi\ \alpha_i \in t$.

Then the pair (X, t) is called a fuzzy topological space (in brief fts). Every member of t is called a fuzzy open set.

2.3 Definition: Let λ be a fuzzy set in fts (X,t). Then the closure of λ is denoted by cl(λ), is given by cl(λ) = $\cap\{\mu : \lambda \subseteq \mu$ and $\mu \in t^c\}$ and the interior of λ is denoted by int(λ), is given by int(λ) = $\cup\{\mu : \mu \subseteq \lambda$ and $\mu \in t\}$.

2.4 Definition[9] : A fuzzy singleton in X is a fuzzy set in X which is zero everywhere, except at one point, say x, where it takes value, say r, with $r \in (0, 1]$ i.e. $0 < r \leq 1$. We denote it by x_r, where the point x is called its support and r its value. Also $x_r \in \alpha$ if and only if $r \leq \alpha(x)$.

2.5. Definition : A fuzzy singleton x_r is called quasi-coincident (in short q-coincident) with a fuzzy set a in X, denoted $x_r\, qa$ iff $r + a(x) > 1$. Similarly, a fuzzy set a in X is called q-coincident with a fuzzy set β in X. denoted $\alpha q \beta$ iff $\alpha(x) + \beta(x) > 1$, for some $x \in X$.

2.6. Definition [4] : Let (X, t) and (Y,s) be two fuzzy topological spaces and let f: X $\to$ Y be function between them. Then the function g: X $\to$ X $\times$ Y defined by $g(x_r) = (x_r, f(x_r))$ is called the fuzzy graph of f.

2.7. Definition [4] : A fuzzy set α in an fts (X,t) is called fuzzy semi-open if $a \subseteq$ cl(int(α)). The complement of a fuzzy semi-open set is said to be fuzzy semi-closed.

2.8. Definition [5]: A fuzzy set α in fts (X,t) is called fuzzy pre-open iff $\alpha \subseteq$ int(cl(α)). The complement of a fuzzy pre-open set is said to be fuzzy pre-closed.

2.9. Definition [10,11]: A fuzzy set α in its fts (X,t) is called fuzzy semi-pre-open or p-open iff $\alpha \subseteq$ cl(int(cl(α)). The complement of a fuzzy semi-pre-open set is said to be fuzzy semi-pre-closed or fuzzy ρ-closed.

2.10. Definition [8]: Let μ be a fuzzy set in fts (X,t). The fuzzy ρ-closure and ρ-interior of μ, denoted by ρ-cl(μ) and ρ-int (μ) are defined by $\Lambda\{\lambda : \mu \subseteq \lambda, \lambda\ is \rho$ -closed} and $\vee\{\lambda : \mu \supseteq \lambda, \lambda$ is ρ- open} respectively.

2.11. Definition [8]: Let X and Y be fuzzy topological spaces. A function f : X $\to$ Y is said to be fuzzy contra-ρ-continuous if for each fuzzy singleton $x_r \in X$ and each fuzzy closed set μ in Y containing $f(x_r)$, there exists a fuzzy ρ-open set λ in X containing x_r such that $\int(\lambda) \subseteq \mu$

3. Fuzzy contra-semi-continuous functions

In this section, we introduce several types of fuzzy contra-continuous functions and characterize the fuzzy contra-semi-continuous functions in particular.

3.1. Definition: Let X and Y be fuzzy topological spaces. A function $f : X \to Y$ is said to be fuzzy contra-continuous if for each fuzzy singleton $x_r \in \times$ and each fuzzy closed set μ in Y containing $f(x_r)$, there exists a fuzzy open set λ in X containing x_r such that $f(\lambda) \subseteq \mu$.

3.2. Definition: Let X and Y be fuzzy topological spaces. A function $f : X \to Y$ is said to be fuzzy contra-semi-continuous if for each fuzzy singleton $x_r \in X$ and each fuzzy closed set μ in Y containing $f(x_r)$, there exists a fuzzy semi-open set λ in X containing x_r such that $f(\lambda) \subseteq \mu$.

3.3. Definition: Let X and Y be fuzzy topological spaces. A function $f : X \to Y$ is said to be fuzzy contra-pre-continuous if for each fuzzy singleton $x_r \in X$ and each fuzzy closed set μ in Y containing $f(x_r)$, there exists a fuzzy pre-open set λ in X containing x_r such that $f(\lambda) \subseteq \mu$.

3.4. Theorem : Let (X,t) and (Y, s) be fuzzy topological spaces and let $f : X \to Y$ be a function. Then the following statements are equivalent.

(1) f is fuzzy contra-semi-continuous function.

(2) For every fuzzy closed μ in Y, $f^{1}(\mu)$ is fuzzy semi-open in X,

(3) For every fuzzy open set λ in Y, $f^{1}(\lambda)$ is fuzzy semi-closed in X.

Proof : (1) $\Leftrightarrow$ (2): Let α be fuzzy closed set in Y and let $x_r \in f^{1}(\alpha)$. Since $f(x_r) \in \alpha$, by (1), there exists a fuzzy semi-open set $\mu\, x_r$ in X containing x_r such that $f(\mu\, x_r) \subseteq \alpha \Rightarrow \mu\, x \subseteq f^{1}(\alpha)$. Therefore, $f^{1}(\alpha)$ is fuzzy semi-open, which proves (2).

Conversely, let $x_r \in X$ and μ be a fuzzy closed set in Y containing $f(x_r)$. Then by (2), f^{1} (μ) is fuzzy semi-open. Put $\lambda = f^{1}$ (μ). Then λ is a fuzzy open set in X containing x_r and hence $f(\lambda) \subseteq \mu$. This shows that f is fuzzy contra-semi-continuous function.

(2) $\Leftrightarrow$ (3) Let λ is a fuzzy open set in Y. Put $\mu = \lambda^c$. Then μ is a fuzzy closed set in Y. Then by (2), $f^{1}(\mu)$ is fuzzy semi-open. Now, $f^{1}(\mu) = f^{1}(\lambda^c) \Rightarrow f^{1}(\mu) = (f^{1}(\lambda))^c$

$\Rightarrow f^{1}(\mu))^c =$. This implies that $(f^{1}(\lambda)$ is fuzzy semi-closed, which is (3). The converse is similar.

3.5 Theorem: Let (X, t) and $Y, s)$ be fuzzy topological spaces and let $f : X \to Y$ be a function. Then the following statements are equivalent:

(1) For any fuzzy closed set μ in Y and for any $x_r \in X$, $f(x_r)q\mu$ implies that $x_r\, q$ s-int $(f^{1}(\mu))$.

(2) For any fuzzy closed set μ in Y and for any $x_r \in X$, if $f(x_r)q\mu$, there exists a fuzzy semi-open set λ such that $x_r q\lambda$ and $f(\lambda) \subseteq \mu$..

Proof : Suppose (1) is true. Let μ be fuzzy closed set in Y and let $f(x_r)q\mu$, for any $x_r \in X$. Then by (1), we have $x_r q$ s-int$(f^{1}(\mu))$. Put $\lambda =$ s-int $(f^{1}(\mu))$, then $f(\lambda) = f($s-int$(f^{1}(\mu)))$ $\Rightarrow f(\lambda) \subseteq f(f^{1}(\mu)) \subseteq \mu$, which proves (2).

Conversely, suppose that (2) is true. Let μ be fuzzy closed set in Y and let $f(x_r)q\ \mu$, for any $x^r \in X$. Then by (2), there exists a fuzzy semi-open set λ such that $x_r q\ \lambda$ and f(λ) $\subseteq \mu$. This implies that $\mu \supseteq f\,(f^{-1}(\mu)),\supseteq f\,(s\text{-int}\,(f^{-1}(\mu)) = f\,(\lambda) \Rightarrow \lambda$ = s-int ($f^{-1}(\mu)$). Hence, x_r q s-int ($f^{-1}(\mu)$), which proves (1).

3.6. Definition: Let (X,t) and (Y,s) be two fuzzy topological spaces. A function f: $X \rightarrow$Y is called a fuzzy s-irresolute if the inverse image of each fuzzy semi-open if the direct image of each fuzzy semi-open set is fuzzy semi-open.

3.7. Definition: Let (X,t) and (Y,s) be two fuzzy topological spaces. A function f: $X \rightarrow Y$ is called a fuzzy semi-open if the direct image of each fuzzy semi-open set is fuzzy semi-open.

3.8. Theorem: Let (X,t), (Y,s) and (Z,u) be fuzzy topological spaces and let f: $X \rightarrow$Y and g: $Y \rightarrow Z$ be functions. If f is fuzzy s-irresolute and g is fuzzy contra-semi-continuous, then g ^{o} f is fuzzy contra-semi-continuous function.

Proof: Let μ be a fuzzy closed set in Z and let (g ^{o} f) (x_r) $\in\mu$, for every fuzzy singleton x^r in X. Then, we have g(f(x_r)) $\in\mu$. Since g is fuzzy contra-semi-continuous, there exists a fuzzy semi-open set λ containing $f(x_r)$ such that g (λ) $\subseteq \mu$. Again, since f is fuzzy s-irresolute, there exists a fuzzy semi-open set η containing x_r such that $f(\eta) \subseteq\lambda$. Hence, we have (g ^{o} f) (η) = g(f(η))$\subseteq$ g (λ) $\Rightarrow$ (g ^{o} f) (η) $\subseteq \mu$. This shows that g ^{o} f is fuzzy contra-semi-continuous function. This completes the proof of the theorem,

3.9. Theorem: Let (X,t), (Y,s) and (Z,u) be fuzzy topological spaces. If f: $X \rightarrow Y$ is a surjective fuzzy semi-open function and g: $Y \rightarrow Z$ is a function such that g ^{o} f is fuzzy contra-semi-continuous, then g is fuzzy contra-semi-continuous.

Proof: Let μ be a fuzzy closed set in Z let (g ^{o} f)(x_r) $\in\mu$, for every fuzzy singleton x_r X. Then, we have g(f(x_r)) $\in\mu$. Since g ^{o} f is fuzzy contra-semi-continuous, there exists a fuzzy semi-open set λ in X containing x_r such that g(f(λ)) $\subseteq \mu$. Again since f is a surjective fuzzy semi-open, f(λ) is a semi-open set in Y containing f(x_r) such g(f(λ)) $\subseteq \mu$. This shows that g is fuzzy contra-semi-continuous function. This completes the proof of the theorem.

3.10. Theorem: Let (X,t) (Y,s) and (Z,u) be fuzzy topological spaces and let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ be functions. If f is fuzzy continuous and g is fuzzy contra-semi-continuous, then g ^{o} f is fuzzy contra-semi-continuous function.

Proof: Let μ be a fuzzy closed set in Z and let g ^{o} f)(x_r) $\in\mu$, for every fuzzy singleton x_r in X. Then, we we have g(f(x_r)) $\in\mu$. Since g is fuzzy contra-semi-continuous, there exists a fuzzy open set λ in Y containing f(x_r) such that g(λ) $\subseteq \mu$. Again, since f is a fuzzy continuous, there exists a fuzzy open set η in X containing x_r such that $\subseteq \int^{-1}(\lambda) \Rightarrow \int(\eta) \subseteq \lambda$. Now, we have (g ^{o} f) (η) = g (f(η))$\subseteq$ g (λ) .$\subseteq \mu$, so that g ^{o} f is fuzzy contra-semi-continuous.

3.11. Theorem: Let f: $X \rightarrow Y$ be a function and let g: $X \rightarrow X \times Y$ be the fuzzy graph of f, defined by $g(x_r) = (x_r, f\,(x_r))$ for every $x_r \in X$. If g is fuzzy contra-semi-continuous, then f is fuzzy contra-semi-continuous.

Proof: Let μ be a fuzzy closed set in Y containing $f(x_r)$ for every $x_r \in X$. Then $X\times \mu$ is a fuzzy closed set in $X \times Y$ and $f^{-1}(\mu) = g^{-1}(X \times \mu)$. Since g is fuzzy contra-semi-continuous, then there exists a fuzzy semi-open set λ in X containing x_r such that $g(\lambda) \subseteq X \times \mu$. This implies that $\lambda \subseteq g^{-1}(X \times \mu)$. Thus, we have $\lambda \subseteq f^{-1}(\mu) \Rightarrow \int(\lambda) \subseteq \mu$. It follows that *f* is fuzzy contra-semi-continuous function.

4. Properties of fuzzy contra-continuous functions

In this section, we investigate the properties and preservation theorems of fuzzy contra-semi-continuous function.

4.1. Definition: Let (X, t) be a fuzzy topological space. Then the fuzzy topological space (X,t) is said to be fuzzy s-compact if every fuzzy semi-open cover of X has a finite subcover.

4.2. Definition: An fts (X,t) is said to be fuzzy strongly s-closed if every fuzzy semi-closed cover of X has finite subcover.

4.3. Definition: An fts (X, t) is said to be fuzzy strong countably s-closed if every fuzzy countable semi-closed cover of X has a finite subcover.

4.4. Definition: An fts (X, t) is said to be fuzzy countably s-compact if every fuzzy countable semi-open cover of X has a finite subcover.

4.5. Definition: An fts (X, t) is said to be fuzzy s-Lindelöf if every fuzzy semi-open cover of X has a finite countable subcover.

4.6. Definition: An fts (X, t) is said to be fuzzy strongly s-Lindelof if every fuzzy semi-closed cover of X has a finite countable subcover.

4.7. Theorem: Let (X, t) be a fuzzy s-compact space. If $f: X\rightarrow Y$ is a surjective fuzzy contra-semi-continuous, then the image of *f* is fuzzy strongly s-closed space.

Proof: Let $\{\mu_i : i \in I\}$ be any fuzzy closed cover of Y. Since *f* is fuzzy contra-semi-continuous, there exists a fuzzy semi-open set $\{f^{-1}(\mu_i) : i \in I\}$ which is a fuzzy semi-open cover of *X*. Again, since (X,t) is a fuzzy s-compact space, there exists a finite subset I_0 of I such that $X = V\{f^{-1}(\mu_i) : i \in I_0\}$. It follows that $f(X) = V\{\mu_i : i \in I_0\}$. Since *f* is surjective, then we have $Y = V\{\mu_i) : i \in I_0\}$ and therefore the image if *f* is fuzzy strongly s-closed. This competes the proof of the theorem.

4.8 Theorem: Let (X,t) be a fuzzy countably s-compact space. If $f : X \rightarrow Y$ is a surjective fuzzy contra-semi-continuous function, then the image of *f* is fuzzy strong countably s-closed space.

The proof of this theorem can be obtained following the proof of Theorem ([4.7]).

4.9. Theorem: Let (X, t) be a fuzzy s-Lindelof space. If $f : X \rightarrow Y$ is surjective fuzzy contra-semi-continuous, then the image of f is fuzzy strongly s-Lindelof space.

The proof is similar to that of Theorem ([4.7]).

4.10. Definition[12] : A fts (X, t) is called fuzzy connected if *X* is not the union of two disjoint non-empty fuzzy open sets.

4.11. Definition : A fts (X, t) is called fuzzy s-connected if *X* is not the union of two disjoint non-empty fuzzy semi-open sets.

4.12. Theorem: Let (X, t)- and (Y, s) be two fuzzy topological spaces. If $f : X \to Y$ is a subjective fuzzy contra-semi-continuous function and X is fuzzy s-connected, then Y is fuzzy connected.

Proof: Suppose Y is not a fuzzy connected space. Then there exists non-empty disjoint fuzzy open sets μ_1 and μ_2 such that $Y = \mu_1 \vee \mu_2$. Therefore, μ_1 and μ_2 are fuzzy clopen in Y. Since f is fuzzy contra-semi-continuous and onto, then $f^{-1}(\mu_1)$ and $f^{-1}(\mu_2)$ are fuzzy semi-open in X. Moreover, $f^{-1}(\mu_1)$ and $f^{-1}(\mu_2)$ are non-empty disjoint and $X = f^{-1}(\mu_1) \vee f^{-1}(\mu_2)$. This shows that X is not fuzzy s-connected, which contradicts our assumption. Therefore, Y is fuzzy connected.

4.13. Definition: Let (X, t) be a fuzzy topological space. An fts (X, t) is called a fuzzy s-ultra-connected if every pair of non-empty fuzzy semi-closed subsets of X intersects.

4.14. Definition[(8)]: Let (X, t) be a fuzzy topological space. An fts (X, t) is called fuzzy hyper-connected if every fuzzy open set is dense.

4.15. Theorem: Let (X, t) and (Y, s) be two fuzzy topological spaces. If $f : X \to Y$ is a surjective fuzzy contra-continuous function and X is fuzzy s-ultra-connected, then Y is fuzzy hyper-connected.

Proof: Suppose Y is not a fuzzy hyper-connected space. Then there exists a fuzzy open set μ such that μ is not dense in Y. Therefore, there exists non-empty fuzzy semi-open subsets μ_1 and μ_2 in Y. Since f is fuzzy contra-semi-continuous, then by

Theorem ([3.4]) we can write $\lambda_1 = f^{-1}(\mu_1)$ and $\lambda_2 = f^{-1}(\mu_2)$ are disjoint non-empty fuzzy semi-closed sets in X, which contradicts the fact that λ_1 and λ_2 interesect i.e. X is fuzzy s-ultra-connected. Therefore, we conclude that Y is fuzzy hyper-connected.

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