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STRUCTURE OF ENDOMORPHISM SEMIGROUP OF ENDOMAPPINGS OF SOME PARTICULAR ROOTED TREES

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ABSTRACT

For an endomapping of a finite set of points lying on some rooted particular trees, endomorphism and endomorphism semigroup were studied. The main aim of this paper was to obtain the structure of the semigroup of all endomorphisms of the endomappings represented by directed graphs on particular types of rooted trees.

Key words: Endomapping, Endomorphism, Endomorphism semigroup, Wreath product, Directed graph

INTRODUCTION

Very impressive ideas and techniques in semigroups have been studied in the publication *Algebraic Theory of Semigroups* in 1961 by Clifford and Preston. For being interested in semigroups a number of authors developed this part of mathematics as an important research tool. Some excellent results on inverse semigroups have been presented to reinforce the study of this area by Petrich 1984. A class of additive commutative semigroups of special elements having a unique expression was studied and some characterization properties were found by Majumdar and Hossain (2008). Also, the structure of endomorphism semigroup was obtained by these authors. A feature of the automorphism groups of special semigroup was studied in consequences of earlier work (Hossain 2010). From the interest of getting structures of the endomorphism semigroup of an endomapping of a finite set, a study was made on the semigroup of endomappings directed by graphs of some trees consisting of a chain or chains of equal lengths (Majumdar 2011a).

To determine the structure for endomorphism semigroup of endomappings of some particular rooted trees (almost general case of Majumdar 2011a), considered the semigroup *End f* for that class of endomappings f such that, for each $x \in X$, there exists a positive integer r_x with the property that $f^{r_x+1}(x) = f^{r_x}(x)$. The technique of structure-determination consists of :

(i) Representing f by a directed graph $G(f)$ with vertices the points of X and edges $x \to f(x)$, and

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(ii) determining the structure of the semigroup $End(G(f))$ of those transformations T of this directed graph $G(f)$ such that $T(f(x)) = f(T(x))$ i.e., $T(x \to f(x)) = (T(x) \to T(f(x)))$.

Since *T* maps vertices onto vertices and edges onto corresponding edges, *T* is called an endomorphism of the digraph of *f*. If *g* is the endomapping of *X* induced by *T*, the map $g \to T$ is an isomorphism of $End\, f$ into the endomorphism semigroup of $G(f)$. The structure of the transformation semigroup *End*(*G*(*X*)) for a class of endomappings f of some rooted trees is determined through the isomorphism $End\ f \cong End\ (G(X))$.

NECESSARY PRELIMINARIES

Consider an endomapping $f : X \to X$ of a finite non-empty set *X*. Under the composition of maps the collection of all endomappings of *X*, denoted by *E*(*X*), is a semigroup called the *full transformation semigroup* on *X*. If the number of elements of *X* is *n*, one may also write F_n for $E(X)$. A map $g : X \to X$ is called an *endomorphism of f* if $g f = f g$ i.e., if g belongs to the centraliser of f in *E*(*X*). The centraliser of *f*, $C(f) = \{g \in E(X) \mid g f = f g\}$, is a *transformation semigroup* on X called the *endomorphism semigroup of* f and it is denoted by *End f*. A semigroup *S* is called a *transformation semigroup* on a nonempty set *X*, and is written (S, X) if there is a map $S \times X \to X$ given by $(s, x) \to x$ such that $(s_1 s_2)(x) = s_1(s_2(x))$. If S is a monoid, then $1(x) = x$, for each $x \in X$. For transformation semigroups S_1 and S_2 on disjoint non-empty sets X_1, X_2, the direct product $S_1 \times S_2$ is a transformation semigroup on $X_1 \cup X_2$ with action given by $(s_1, s_2)(x_1) = s_1(x_1)$ and $(s_1, s_2)(x_2) = s_2(x_2)$. For two non-empty sets X_1, X_2, the *wreath product* $S_1 \wr S_2$ is a transformation semigroup on $X_1 \times X_2$ and consists of maps $^{-} : X_1, X_2 \to X_1, X_2$ given by $\theta(x_1, x_2) = (s_{1,x_2}(x_1), s_2(x_2))$, s_{1,x_2} being an element of S_1 determined by x_2.

To determine the structure of the transformation semigroup *End*(*G*(*X*)) the author needs some results (Majumdar 2011a,b, Meldrum 1995) about the direct product and wreath product of transformation semigroups. The author recalls these in the following way.

The wreath product has a description in terms of direct product which makes the sense that wreath product is associative and is distributive over direct product.

Theorem 1: $(S_1 \wr S_2, X \times X_2) \cong ((\underset{x_2 \in X_2}{\times} S_{1,x_2}) \times S_2,\ (\underset{x_2 \in X_2}{\cup} X_{1,x_2}) \times X_2)$ where each $x_2 \in X_2$, $S_{1,x_2} \cong S_1$ and $\left|X_{1,x_2}\right| = \left|X_1\right|$.

Theorem 2: $((S_1 \wr S_2) \wr S_3,\ (X_1 \times X_2) \times X_3) \cong (S_1 \wr (S_2 \wr S_3),\ X_1 \times (X_2 \times X_3))$.

Theorem 3: $(S_1 \wr (S_2 \times S_3),\ X_1 \times (X_2 \cup X_3)) \cong ((S_1 \wr S_2) \times (S_1 \wr S_3),$

$(X_1 \times X_2) \cup (X_1 \times X_3))$.

Remarks: (i) If $S_2 = \{1_{X_2}\}$, then $(S_1 \times S_2,\ X_1 \cup X_2)$ may be identified with (S_1, X_1) and $(S_1 \text{s}\ S_2,\ X_1 \times X_2)$ with $(\prod_{x_2 \in X_2} S_{1,\, x_2},\ \bigcup_{x_2 \in X_2} X_{1,\, x_2})$ where $\left|X_{1,\, x_2}\right| = \left|X_1\right|$.

(ii) If $S_1 = \{1_{X_1}\}$, then both $(S_1 \times S_2,\ X_1 \cup X_2)$ and $(S_1 \text{s}\ S_2,\ X_1 \times X_2)$ may be identified with $(S_2,\ X_2)$.

(iii) If $X_1 = X_2 = X$, then $(S_1 \text{s}\ S_2,\ X \times X)$ may be identified with $(\prod_{x \in X} S_{1,\, x}) \times S_2$, $\bigcup_{x \in X} X_{1,\, x} \cup X)$. As semigroups, $S_1 \text{s}\ S_2 \cong (\prod_{x \in X} S_{1, x}) \times S_2$.

STRUCTURE OF THE ENDOMORPHISM SEMIGROUP *End f*

Author knows about *End f* through representation of *f* by $G(f)$, the directed graph of *f* consisting of a single chain given by Fig. 1:

In this case, *End f* is the semigroup $E(m) = \{\pm_0, \pm_1, \cdots\cdots, \pm_m\}$, where σ_0 is the identity element and $\{\sigma_1, \cdots\cdots, \sigma_m\}$ is a cyclic semigroup generated by σ_1 with $\sigma_1^m = \sigma_m$ as the zero element.

Fig. 1

Let *f* be given by the directed graph $G(f)$ in Fig. 2:

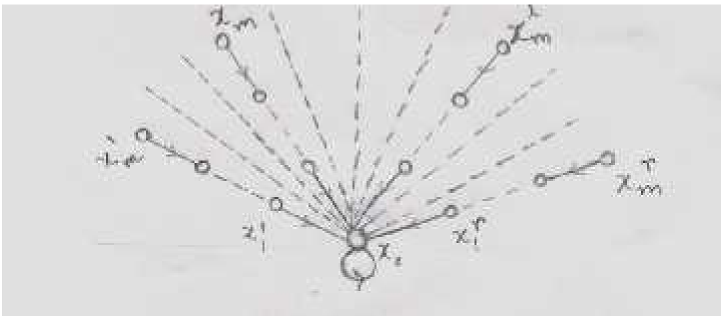


Fig. 2

consisting of *r* directed subgraphs each being a chain of length *m* and each with the loops at x_0. Since each $g \in End\ f$ must map x_0 onto itself and since each maximal chain ending at x_0 has the same length *m*, *End f* may be identified with the semigroup of all endomorphisms of an endomapping f' of X whose directed graph is $C_m \times \{1,\ 2, \cdots\cdots, r\}$, C_m being the chains shown in the directed graph. It therefore follows from the theorems of wreath product that $End\ f \cong E(m) \text{s}\ F_r$. Then $End\ f \cong E(m) \text{s}\ F_r$ ... (1).

Here, F_r is the full transformation semigroup on a set with *r* elements.

It is observed that if *f* is given by the directed graph *G*(*f*) in Fig. 3:

Fig. 3

with $m > n$, i.e., the directed graph consists of two chains of unequal lengths, then $End(X, f)$ will consist of maps:

$(a)\quad T_1 \to T_1,\quad T_2 \to T_2$

$(a)\quad T_1 \to T_1,\quad T_2 \to T_1$

$(a)\quad T_1 \to T_2,\quad T_2 \to T_1$

$(a)\quad T_1 \to T_2,\quad T_2 \to T_2$, T_1 and T_2 being subgraphs. Therefore, one may have

$$(2)\quad \begin{cases} End\ f = (End\ f_1 \times End\ f_2) \cup (End\ f_1 \times Hom(T_2, T_1)) \cup \\ \qquad (Hom(T_1,T_2) \times Hom(T_1 \times T_2)) \cup (Hom(T_1,T_2), End\ f_2)\ , \end{cases}$$

where, f_1 and f_2 are *f* restricted to $\{x_0,\ x_1, \cdots,\ x_m\}$ and $\{x_0,\ x_1', \cdots,\ x_n'\}$, respectively and $Hom(T_i,T_j)\ \ (i, j = 1, 2\ ;\ i \neq j)$ denotes the set of maps the directed graph T_i into the directed graph T_j of f_i and f_j , respectively.

with

$$(3)\quad \begin{cases} (End\ T_i)\ Hom(T_j, T_i) \subseteq Hom(T_j, T_i) \\ Hom(T_i, T_j)\ Hom(T_j, T_i) \subseteq End\ T_j\ . \end{cases}$$

Also, if $\phi = \begin{pmatrix} x_0 & x_1 & \cdots\cdots x_{m-n} & x_{m-n+1} \cdots\cdots x_n \\ x_0 & x_0 & \cdots\cdots x_0 & x_1' \ \cdots\cdots\ x_n' \end{pmatrix}$ and $\varphi = \begin{pmatrix} x_0 & x_1' & \cdots\cdots & x_n' \\ x_0 & x_1 & \cdots\cdots & x_n \end{pmatrix}$, then it is easy to see that

$$(4)\quad \begin{cases} Hom(T_1, T_2) = (End\, T_2)¤ \\ Hom(T_2, T_1) = (End\ T_1)¨\ \ . \end{cases}$$

It follows from the above facts that

$$(5)\quad \begin{cases} (End\ f_1) \times Hom(T_2, T_1) = (End\ f_1)^{2\,¨}\ , \\ (Hom(T_1,T_2)(End\ f_1) = (End\ f_2)¤(End\ f_1), \\ (End\ f_2) \times Hom(T_1, T_2) = (End\ f_2)^{2\,¨}\ , \\ (Hom(T_2,T_1)(End\ f_2) = (End\ f_1)¨\ (End\ f_2), \\ (Hom(T_1,T_2))(Hom(T_2,T_1)) = (End\ f_2)¤(End\ f_1)¨\ , \\ (Hom(T_2,T_1))(Hom(T_1,T_2)) = (End\ f_1)¨\ (End\ f_2)¤\ . \end{cases}$$

It, therefore, may be stated that:

Theorem 4: The semigroup-structure of *End f* of *f* given by Fig. 3 is completely given by the expressions from (1) to (5).

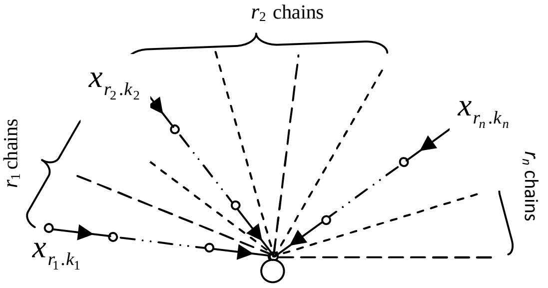


Fig. 4

Theorem 5: Let *f* be given by the directed graph $G(f)$ as in Fig.4 with $f^k(X)$ a singleton, where $k = \max\{k_1, \cdots, k_m\}$. Then

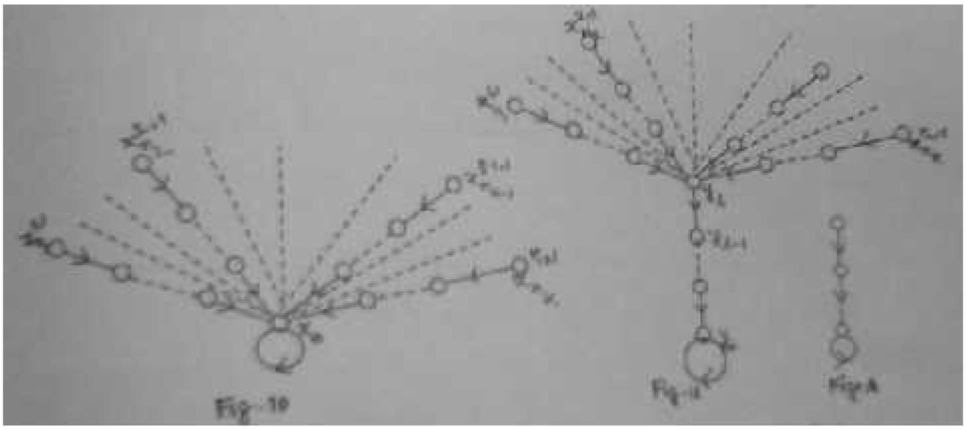
(6) $End\ f = \bigcup [End\ T^{i_1} \times \cdots\cdots \times End\ T^{i_u} \times (\times_{u<v,\ v'<k}^{v \neq v'} Hom(T^{i_v}, T^{i_{v'}}))]$,

the union being taken over all permutations $\begin{pmatrix} 1 & 2 & 3 & \cdots\cdots k \\ i_1 & i_2 & i_3 & \cdots\cdots i_k \end{pmatrix}$.

Here,

$$(7)\quad \begin{cases} End\,T^i = End\,T^{i,\alpha} \varsigma F_{i_r} \qquad (\alpha \in \{1,\ \ 2,\ \ 3, \cdots\cdots,\ \ i_r\}), \\ Hom\,(T^{i_v}, T^{i_{v'}}) = \times_{1 \le \beta \le r_{i_{v'}}}^{1 \le \alpha \le r_{i_v}} Hom(T_\alpha^{i_v}, T_\beta^{i_{v'}})\,. \end{cases}$$

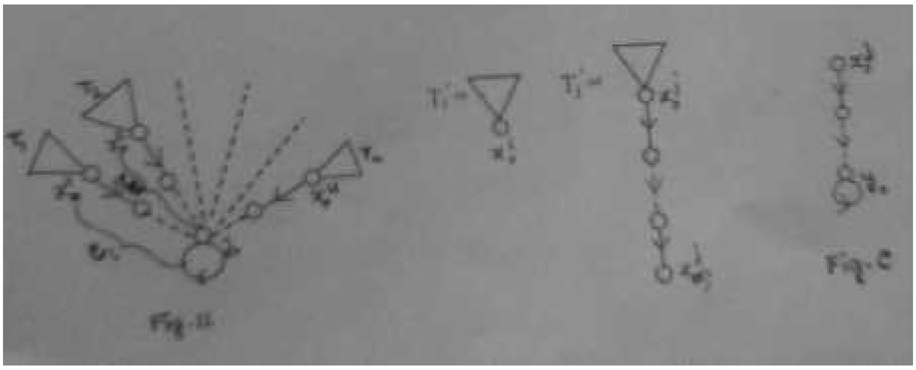
The products of $End\ T^{i,\alpha}$ with themselves and with $Hom(T_\alpha^i, T_\beta^j)$ as well as the products of $Hom(T_\alpha^i, T_\beta^j)$ among themselves are given by (5). Also, the $End\ T^{i,\alpha}$'s are isomorphic to one another, since $T^{i,\alpha}$'s are chains of the same length.



One may consider the situation that the directed graph representing the endomapping consists of a finite number of disjoint directed rooted trees shown as above:

Now let (*X, f*) be given by the directed graph in Fig. 11. Then a map $g : X \to X$ is in *End*(*X, f*) if and only if *g* induces an endomorphism of chain C [fig.A] and two such endomorphisms combine to yield an element of *End*(*X, f*). It is noted that the endomorphism semigroup of the graph of (*X, f*) in this case and that of the directed rooted tree with root at y_0 obtained by removing the loop at y_0 are isomorphic to each other. Thus, if one denotes *End*(*X,f*) in fig.10 by *E*, then *End*(*X,f*) in Fig.11 is given by $End(X, f) \cong E \times E(l)$ (direct product), $E(l)$ being the endomorphism semigroup of chain *C*.

Let (*X, f*) be now given by a more general form Fig. 12



where $T_1, \ldots, T_u$ are graphs of the form in the earlier case, which are isomorphic to directed trees each with root at y_0. An endomorphism of (*X,f*) maps every tree T_i into a tree T_j (*i, j* not necessarily distinct) in such a way that

(a) y_0 is mapped onto y_0,

(b) x_i^0 is mapped onto a point in C, where C is shown in the Fig. C.

the subtree $T_i^{'}$ of T_i into the subtree $T_j^{'}$, where $x_{\alpha_j}^j = f(x_0^i)$ is such that the directed edges are mapped onto the corresponding directed edges.

Thus,

$$(6) \quad \begin{aligned} & End(X, f) \\ & = \bigcup [End\, T^{i_1} \times \cdots\cdots \times End\, T^{i_u} \times (\times_{u<v,\, v'<k}^{v \neq v'} Hom(T^{i_v}, T^{i_{v'}}))], \end{aligned}$$

the union being taken over all permutations $\begin{pmatrix} 1 & 2 & 3 & \cdots\cdots k \\ i_1 & i_2 & i_3 & \cdots\cdots i_k \end{pmatrix}$.

Here, it is easily verified that

(i) if the length l_i of C_1 is greater than or equal to the length l_j of C_2 (C_1 and C_2 are indicated in the Fig. 12), then $Hom(T_i, T_j) = \{(\sigma_{j,2}, \sigma_{j,1})(f_{ij}, \sigma_{i,1})\}$, where $f_{ij} = \begin{pmatrix} x_0^i \cdots x_{l_1,2}^i \cdots x_{l_i,s}^i & y_0 \\ x_0^j \quad x_{l_1,2}^j \cdots x_{l_j}^j & y_0 \end{pmatrix}$,

$\sigma_{i,1} \in End(T_i^{'})$, $\sigma_{j,2} \in End(C)$, $\sigma_{j,2} \in End(T_j^{'})$, (as in the Fig. C).

(ii) if $l_i \le l_j$, then $Hom(T_i, T_j) = \{(\sigma_{j,2}, \sigma_{j,1})(f_{ij}^{'}, \sigma_{i,j}^{'})\}$

In the above, $f_{ij}^{'} : T_i \to T_j$ maps every subgraph of T_j which is a chain or tree of length $\le l_j$ with root x_0^i onto the subgraph C (as in the Fig. C) of T_j isomorphically.

Also, $\sigma_{i,1}^{'} \in End\, T_i^{''}$, where $T_i^{''}$ is the directed rooted tree obtained from $T_i^{'}$ by (a) deleting the maximal chain-subtrees in T_i with root x_0^i and length $< l_j$ and (b) collapsing the chain-subtrees of T_i with root x_0^i and length $\ge l_j$ by identifying all of $x_0^i, x_i^i, \cdots, x_{l_j}^i$ so that they represent the same point while the directed edges $x_{\alpha+1}x_\alpha$ $(0 \le \alpha \le l_{j_1})$ vanish.

Multiplication of the elements of $Hom(T_i,T_j)$ with those of $Hom(T_j,T_k)$ is the composition of the maps representing the elements; and multiplication of the elements of $End\ T_1 \times \cdots \times End\ T_n$ with the elements of $Hom(T_i,T_j)$ is defined similarly.

CONCLUDING REMARK

Structures for endomorphism semigroup of endomappings given by directed graphs of some particular rooted trees are almost general case that Majumdar (2011) studied. Such a study for general case, though it is complicated, will be taken up in future.

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