THE COMMUTATIVITY OF PRIME Γ_N-RINGS WITH LEFT AND RIGHT *k*-DERIVATIONS

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ABSTRACT

By introducing the notions of left *k*-derivation and right *k*-derivation of a gamma ring, we determine some significantly important results on the commutativity of prime Γ_N-rings of characteristic not equal to 2 and 3 with left *k*-derivation and right *k*-derivation, and also with the composition of such two *k*-derivations.

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1 INTRODUCTION

The purpose of introducing the concept of a Γ-ring is to generalize that of a classical ring. In the last few decades, a number of modern algebraists have determined a lot of fundamental properties of Γ-rings and extended numerous significant results in classical ring theory to gamma ring theory. Note that the notion of a Γ-ring was first introduced by N. Nobusawa[1] and then generalized by W. E. Barnes[2]. They obtained many important fundamental properties of Γ-rings, and also S. Kyuno[3], J. Luh[4], G. L. Booth[5] and some other prominent mathematicians characterized much more significant results in the theory of gamma rings. Here, we start with the following definition.

Let M and Γ be two additive abelian groups. If there exists a mapping $(a,\alpha,b)\to a\alpha b$ of $M\times\Gamma\times M\to M$ which satisfies the conditions

(a) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ and

(b) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$,

then M is called a **Γ-ring** in the sense of Barnes[2], or simply, a Γ-ring.

For example, suppose that R is a ring with identity 1 and $M_{m,n}(R)$ is the set of all $m\times n$ matrices over R. Then M is a Γ-ring with respect to the usual addition and multiplication of matrices if we choose $M = M_{m,n}(R)$ and $\Gamma = M_{n,m}(R)$. In particular, if we let $M = M_{1,2}(R)$ and $\Gamma = \left\{\begin{pmatrix} n.1 \\ 0\end{pmatrix} : n \text{ is an integer}\right\}$, then M is a Γ-ring.

In addition to the definition of a Γ-ring given above, if there exists another mapping $(\alpha,a,\beta)\to\alpha a\beta$ of $\Gamma\times M\times\Gamma\to\Gamma$ satisfying the conditions (a*) $(\alpha+\beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a+b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta+\gamma) = \alpha a\beta + \alpha a\gamma$,

(b*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$, and

(c*) $a\alpha b = 0$ implies $\alpha = 0$ for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$,

then *M* is called a Γ-ring in the sense of Nobusawa(1), or simply, a Γ_N **-ring**. Note that G. L. Booth(5) has also used this notation to express Nobusawa Γ-rings.

For example, let $D_{m,n}$ be the set of all rectangular $m \times n$ matrices over some division ring *D*. Considering $M = D_{m,n}$ and $\Gamma = D_{n,m}$, we see that *M* is a Γ_N-ring under the usual addition and multiplication of matrices.

It follows easily from the definitions of Γ-ring and Γ_N-ring that *M* is a Γ-ring does not imply Γ is an *M*-ring in general, but *M* is a Γ_N-ring always implies Γ is an *M*-ring.

If *M* is a Γ-ring, then *M* is called *prime* if $a\Gamma M\Gamma b = 0$ (with $a, b \in M$) implies either $a = 0$ or $b = 0$. Note that this concept of prime Γ-ring were introduced by J. Luh(4), and some analogous results corresponding to the prime rings were obtained by him as well as by S. Kyuno(3).

An additive subgroup *U* of *M* is said to be a *left* (or, *right*) *ideal* of *M* if $M\Gamma U \subset U$ (or, $U\Gamma M \subset U$), whereas *U* is called a *two-sided ideal*, or simply, an *ideal* of *M* if *U* is a left as well as a right ideal of *M* (i.e., if $m\gamma u \in U$ and $u\gamma m \in U$ for all $m \in M$, $\gamma \in \Gamma$ and $u \in U$). Similarly, an additive subgroup Ω of an *M*-ring Γ (if *M* is considered as a Γ_N-ring) is said to be a left (or, right) ideal of Γ if $\Gamma M\Omega \subset \Omega$ (or, $\Omega M\Gamma \subset \Omega$, and Ω is called a two-sided ideal, or simply, an ideal of Γ if Ω is both a left and a right ideal of Γ (i.e., if $\gamma m\omega \in \Omega$ and $\omega m\gamma \in \Omega$ for all $\gamma \in \Gamma$, $m \in M$ and $\omega \in \Omega$).

Recall that a Γ-ring *M* is said to be of *characteristic not equal to n* (where *n* is a positive integer greater than 1), written as $charM \neq n$, if $nx = 0$ implies $x = 0$ for all $x \in M$. Moreover, the set $Z(M)$ = $\{a \in M : a\alpha m = m\alpha a \text{ for all } \alpha \in \Gamma \text{ and } m \in M\}$ is called the *center* of the Γ-ring *M*.

Consider again that *M* is a Γ-ring. Then *M* is called a *commutative* Γ-ring if $x\gamma y = y\gamma x$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. If $a, b \in M$ and $\alpha \in \Gamma$, then $[a,b]_\alpha$ = $a\alpha b - b\alpha a$ is called the *commutator* of *a* and *b* with respect to α .

In a Γ-ring *M*, an element $m \in M$ is called a *left nonzero divisor* if $m\beta x = 0$ implies $x = 0$ for all $\beta \in \Gamma$. Similarly, an element $m \in M$ is called a *right nonzero divisor* if

$x\beta m = 0$ implies $x = 0$ for all $\beta \in \Gamma$. When this said element is both a left nonzero divisor and a right nonzero divisor, then it is called a *two-sided nonzero divisor*, or simply a *nonzero divisor*. In other words, a Γ-ring *M* is said to have no zero divisor if $a\gamma b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in M$ and $\gamma \in \Gamma$.

The notions of derivation and Jordan derivation of a Γ-ring have been introduced by M. Sapanci and A. Nakajima(6). Later, in view of the concept of Jordan left derivation of a usual ring developed by K. W. Jun and B. D. Kim(7), some important results due to left derivation and Jordan left derivation of a Γ-ring has been determined by Y. Ceven(8). But, H. Kandamar(9) has been introduced the notion of *k*-derivation of a Γ-ring and he obtained a number of important results on this concept. Here, we introduce the notions of left *k*-derivation and right *k*-derivation of a Γ-ring and we construct some characterizations of these concepts on certain Γ_N-rings to extend some significant results of certain Γ-rings with left derivation and right derivation shown by M. Asci and S. Ceran(10).

Let *M* be a Γ-ring and let $d: M \to M$ and $k: \Gamma \to \Gamma$ be two additive mappings. Then *d* is called a *left derivation* of *M* if $d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$, and *d* is called a *right derivation* of *M* if $d(a\alpha b) = d(a)\alpha b + d(b)\alpha a$ is satisfied for all $a, b \in M$ and $\alpha \in \Gamma$. But, if $d(a\alpha b) = a\alpha d(b) + ak(\alpha)b + b\alpha d(a)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$, then *d* is called a *left k-derivation* of *M*, and if $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + d(b)\alpha a$ is satisfied for all $a, b \in M$ and $\alpha \in \Gamma$, then *d* is called a *right k-derivation* of *M*.

Finally, for a Γ-ring *M*, if $d: M \to M$ and $k: \Gamma \to \Gamma$ are two additive mappings such that $a, b \in M$ and $\alpha \in \Gamma$, then *d* is called a *derivation* of *M* if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ and *d* is called a *k-derivation* of *M* if $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$.

To determine a number of significantly important results on the commutativity of prime Γ_N-rings of characteristic not equal to 2 and 3 with left *k*-derivation and right *k*-derivation, and also with the composition of such two *k*-derivations, we proceed as follows.

2. MAIN RESULTS

For the sake of completeness of the study of this paper we recall some necessary important results already proved earlier which are needed to reach our goal. To start the discussion we state first the following well-known lemma proved by M. Soyturk(11) [Lemma 1].

Lemma 2.1 Let *M* be a Γ-ring and *Z* the center of *M*. Then the following are true for all $a, b, c \in M$ and $\beta, \gamma \in \Gamma$:

(i) $[a\gamma b, c]_\beta = a\gamma[b, c]_\beta + [a, c]_\beta \gamma b + a\gamma(c\beta b) - a\beta(c\gamma b)$;

(ii) If $a \in Z$, then $[a\gamma b, c]_\beta = a\gamma[b, c]_\beta$;

(iii) If $a \in Z$, then $a\gamma[b,c]_\beta = a\beta[b,c]_\gamma$.

Especially, if *M* is a prime Γ-ring, then for all $a,b,c \in M$ and $\beta,\gamma \in \Gamma$:

(iv) If $a \in Z$ and $a\Gamma b = 0$, then either $a = 0$ or $b = 0$;

(v) If $a \in Z$ and $a\Gamma b \subset Z$, then either $a = 0$ or $b \in Z$;

(vi) If $a \in Z$ and $a\gamma[b,c]_\gamma = 0$, then either $a = 0$ or $[b,c]_\gamma = 0$.

Except otherwise mentioned, *throughout the article hereafter*, *M* represents a prime Γ_N-ring (implying from the very definition that Γ is then an *M*-ring), *Z* is the center of *M*, *U* is a nonzero ideal of *M*, and Ω is a nonzero ideal of the associated *M*-ring Γ.

Now, we state some useful results that have already been discussed and proved by H. Kandamar[9] and by M. Soyturk[11] as follows.

Lemma 2.2 For all $a,b \in M$ and $\alpha,\beta \in \Gamma$,

(i) $a\Omega b = 0 \Rightarrow$ either $a = 0$ or $b = 0$; (ii) $\alpha U\beta = 0 \Rightarrow$ either $\alpha = 0$ or $\beta = 0$;

(iii) $a\Gamma U\Gamma b = 0 \Rightarrow$ either $a = 0$ or $b = 0$; (iv) $\alpha M\Omega M\beta = 0 \Rightarrow$ either $\alpha = 0$ or $\beta = 0$;

(v) If $u\alpha v = 0$ for all $u,v \in U$, then $\alpha = 0$; (vi) If $\gamma a\delta = 0$ for all $\gamma,\delta \in \Omega$, then $a = 0$.

As the immediate consequences from (iii) and (iv) of this lemma, we get

Corollary 2.1 For all $a,b \in M$ and $\alpha,\beta \in \Gamma$,

(i) $a\Omega U\Omega b = 0 \Rightarrow$ either $a = 0$ or $b = 0$; (ii) $\alpha U\Omega U\beta = 0 \Rightarrow$ either $\alpha = 0$ or $\beta = 0$.

Also, we need the following important results proved in H. Kandamar[9] and M. Soyturk[11]:

Lemma 2.3 For all $a \in M$ and $\alpha \in \Gamma$,

(i) $a\Gamma U$ (or, $U\Gamma a$) = 0 $\Rightarrow a = 0$; (ii) $\alpha U\Gamma$ (or, $\Gamma U\alpha$) = 0 $\Rightarrow \alpha = 0$;

(iii) $a\Omega M$ (or, $M\Omega a$) = 0 $\Rightarrow a = 0$; (iv) $\alpha M\Omega$ (or, $\Omega M\alpha$) = 0 $\Rightarrow \alpha = 0$.

Consequently, it follows from this lemma that

Corollary 2.2 For all $a \in M$ and $\alpha \in \Gamma$,

(i) $a\Omega U$ (or, $U\Omega a$) = 0 $\Rightarrow a = 0$; (ii) $\alpha U\Omega$ (or, $\Omega U\alpha$) = 0 $\Rightarrow \alpha = 0$.

The following result plays a pivotal role in this article.

Lemma 2.4 If $U \subset Z$, then M is commutative.

Proof. Please refer to the proof given by M. Soyturk[11] [Lemma 2(i)].

Then we go forward with our main results step by step as follows:

Lemma 2.5 With our notations as above, the following are true:

(i) If *d* is a left *k*-derivation of *M* such that $d(U) = 0$ along with $k(\Omega) = 0$, then $d = 0$;

(ii) If *d* is a right *k*-derivation of *M* and $d(M)$ is a right nonzero divisor such that $a\Omega d(U) = 0$ for all $a \in M$ along with $k(\Omega) = 0$, then $a = 0$;

(iii) If *d* is a left *k*-derivation of *M* and $d(M)$ is a left nonzero divisor such that $d(U)\Omega a = 0$ for all $a \in M$ along with $k(\Omega) = 0$, then $a = 0$;

(iv) If $charM \neq 2$ and *d* is a right *k*-derivation of *M* such that $d^2(U) = 0$ along with $k(\Omega) = 0$, then $d = 0$;

(v) If d_1 is a left k_1-derivation of *M* and d_2 is a right k_2-derivation of *M* such that $charM \neq 2$, $d^2(U) \subset U$ and $d_1 d_2(U) = 0$ along with $k_1(\Omega) = 0$ and $k_2(\Omega) = 0$, then either $d_1 = 0$ or $d_2 = 0$.

Proof. (i) Let $u \in U$, $\alpha \in \Omega$ and $m \in M$. Then we have 0 = $d(u\alpha m)$ = $u\alpha d(m)$ + $uk(\alpha)m$ + $m\alpha d(u)$ = $u\alpha d(m)$. This implies, $U\Omega d(M) = 0$. Hence, by Corollary 2.2(i), we obtain $d(M) = 0$, i.e., $d = 0$.

(ii) Let $u \in U$, $\alpha, \beta \in \Omega$ and $a, m \in M$. Then we get 0 = $a\alpha d(u\beta m)$ = $a\alpha d(u)\beta m$ + $a\alpha uk(\beta)m$ + $a\alpha d(m)\beta u$ = $a\alpha d(m)\beta u$. Hence, $(a\alpha d(M))\Omega U = 0$. So, by Corollary 2.2(i), this yields $a\alpha d(M) = 0$. But, since $d(M)$ is a right nonzero divisor, therefore $a = 0$.

(iii) Let $u \in U$, $\alpha, \beta \in \Omega$ and $a, m \in M$. Then we obtain 0 = $d(m\beta u)\alpha a$ = $m\beta d(u)\alpha a$ + $mk(\beta)u\alpha a$ + $u\beta d(m)\alpha a$ = $u\beta d(m)\alpha a$. So, $U\Omega(d(M)\alpha a) = 0$. Thus, by Corollary 2.2(i), we get $d(M)\alpha a = 0$. But, since $d(M)$ is a left nonzero divisor, we get $a = 0$.

(iv) Let $u \in U$ and $\alpha \in \Omega$. Then we have 0 = $d^2(u\alpha u)$ = $d(d(u)\alpha u$ + $uk(\alpha)u$ + $d(u)\alpha u)$ = $2d(u)\alpha d(u)$. Since $charM \neq 2$, we get $d(u)\alpha d(u) = 0$. Thus, we obtain $d(U)\Omega d(U) = 0$, and consequently, $d(U) = 0$. Hence, by (i), we conclude that $d = 0$.

(v) Let $u, v \in U$ and $\alpha \in \Omega$. Then we get 0 = $d_1 d_2(u\alpha v)$ = $d_1(d_2(u)\alpha v$ + $uk_2(\alpha)v$ + $d_2(v)\alpha u$ = $d_2(u)\alpha d_1(v)$ + $d_2(v)\alpha d_1(u)$. Putting $d_2(u)$ for *u*, we have $d_2^2(u)\alpha d_1(v) = 0$. That is, $d_2^2(U)\Omega d_1(U) = 0$. Hence, by Lemma 2.2(i), either $d_2^2(U) = 0$ or $d_1(U) = 0$, and therefore, we obtain either $d_2 = 0$ or $d_1 = 0$ [by (iv) and (i), respectively]. This completes the proof of the lemma.

Theorem 2.1 Let *d* be a nonzero right *k*-derivation of *M* such that $k(\Omega) = 0$ and $charM \neq 2$. If $d(U) \subset Z$, then *M* is commutative.

Proof. Let $u \in U$, $\gamma \in \Omega$, $z \in Z$ and $y \in M$. Then we have

$$0 = [d(u\gamma z), y]_\gamma = [d(u)\gamma z + uk(\gamma)z + d(z)\gamma u, y]_\gamma = d(z)\gamma[u, y]_\gamma.$$

That means, $d(Z)\Omega[U, M]_\gamma = 0$. Hence, by Lemma 2.2(i), either $d(Z) = 0$ or $[U, M]_\gamma = 0$. If $d(Z) = 0$, then $d^2(U) \subset d(Z) = 0$, implying $d^2(U) = 0$, and so, $d = 0$ [by Lemma 2.5(i)], which is a contradiction to our assumption. Therefore, $[U, M]_\gamma = 0$ for all $\gamma \in \Omega$, and consequently, $U \subset Z$, and hence, by Lemma 2.4, *M* is commutative.

Theorem 2.2 Let *d* be a nonzero right *k*-derivation of *M* such that $k(\Omega) = 0$ and $charM \neq 2, 3$. If $d^2(U) \subset Z$ and $d(U) \subset U$, then *M* is commutative.

Proof. Let $u \in U$, $\gamma \in \Omega$, $\beta \in \Gamma$ and $y \in M$. Then we get

$$0 = [d^2(d(u)\gamma d(u)), y]_\beta = [d(d^2(u)\gamma d(u) + d(u)k(\gamma)d(u) + d^2(u)\gamma d(u)), y]_\beta$$

$$= 2[d(d^2(u)\gamma d(u)), y]_\beta = 2[d^3(u)\gamma d(u) + d^2(u)k(\gamma)d(u) + d^2(u)\gamma d^2(u), y]_\beta$$

$$= 2d^3(u)\gamma[d(u), y]_\beta.$$

Hence, the hypothesis $charM \neq 2$ implies that $d^3(u)\gamma[d(u), y]_\beta = 0$, and so, $d^3(u)\Omega[d(u), y]_\beta = 0$. By Lemma 2.2(i), either $d^3(u) = 0$ or $[d(u), y]_\beta = 0$; i.e., either $d^3(u) = 0$ or $d(u) \in Z$ for all $u \in U$. If $d^3(u) = 0$ for all $u \in U$, then we obtain 0 = $[d^2(u\beta d(u)), y]_\gamma$ = $[d(d(u)\beta d(u) + uk(\beta)d(u) + d^2(u)\beta u), y]_\gamma$ = $3d^2(u)\beta[d(u), y]_\gamma$. Again, since $charM \neq 3$, we get $d^2(u)\beta[d(u), y]_\gamma = 0$. Hence, by Lemma 2.1(vi), either $d^2(u) = 0$ or $[d(u), y]_\gamma = 0$; i.e., either $d^2(u) = 0$ or $d(u) \in Z$ for all $u \in U$.

Let $H=\{u\in U: d^2(u)=0\}$ and $K=\{u\in U: d(u)\in Z\}$. Then H and K are additive subgroups of U, and also $U=H\cup K$. But, if $U=H$, then by Lemma 2.5(iv), $d=0$, which is a contradiction to our hypothesis. Hence, by Brauer's trick [**meaning that** a group cannot be a set-theoretic union of its two proper subgroups so that $U=H\cup K$ implies either $U=H$ or $U=K$ in this case], we have $U=K$, and so, by Theorem 2.1, M is commutative.

Theorem 2.3 Let d_1 be a nonzero left k_1-derivation of M and d_2 a nonzero right k_2-derivation of M such that $d_1d_2(U)\subset Z$, $d^2(U)\subset U$ and $charM\neq 2, 3$ along with $k_1(\Omega)=0$ and $k_2(\Omega)=0$. If $d_1d_2^2(U)=0$, then M is commutative.

Proof. Let $u\in U$, $\gamma\in\Omega$, $\beta\in\Gamma$ and $y\in M$. Then we obtain

$$0=[d_1d_2(d_2(u)\gamma d_2(u)),y]_\beta=[d_1(d_2^2(u)\gamma d_2(u)+d_2(u)k_2(\gamma)d_2(u)+d_2^2(u)\gamma d_2(u)),y]_\beta$$

$$=2[d_1(d_2^2(u)\gamma d_2(u)),y]_\beta=2[d_2^2(u)\gamma d_1d_2(u)+d_2^2(u)k_1(\gamma)d_2(u)+d_2(u)\gamma d_1d_2^2(u),y]_\beta$$

$$=2[d_2^2(u)\gamma d_1d_2(u),y]_\beta=2[d_1d_2(u)\gamma d_2^2(u),y]_\beta=2d_1d_2(u)\gamma[d_2^2(u),y]_\beta.$$

So, $d_1d_2(u)\gamma[d_2^2(u),y]_\beta=0$ (since $charM\neq 2$). Thus, $d_1d_2(u)\Omega[d_2^2(u),y]_\beta=0$. By Lemma 2.2(i), either $d_1d_2(u)=0$ or $[d_2^2(u),y]_\beta=0$; i.e., either $d_1d_2(u)=0$ or $d_2^2(u)\in Z$. Let $H=\{u\in U: d_1d_2(u)=0\}$ and $K=\{u\in U: d_2^2(u)\in Z\}$. Then H and K are subgroups of U and $U=H\cup K$. If $U=H$, either $d_1=0$ or $d_2=0$ [by Lemma 2.5(v)], a contradiction. Hence, $U=K$, by Brauer's trick (as stated earlier), and therefore, by Theorem 2.2, M is commutative.

Lemma 2.6 Let $a\in M$ and $Z\neq 0$. If $[U,a]_\gamma=0$ for all $\gamma\in\Gamma$, then $a\in Z$.

Proof. Please see the proof given by M. Soyturk[(11)] [Lemma 5].

Lemma 2.7 Let $a\in M$, d_1 a nonzero left k_1-derivation of M and d_2 a nonzero right k_2-derivation of M such that $d_1d_2(U)\subset Z$ and $d^2(U)\subset U$ with $k_1(\Omega)=0$ and $charM\neq 2$. If $[d_1(U),a]_\beta=0$, then $[d_2(U),a]_\beta=0$ for all $\beta\in\Gamma$.

Proof. Let $a\in M$, $u\in U$, $\gamma\in\Omega$ and $\beta\in\Gamma$. Then we have

$$0=[d_1(d_2(u)\gamma d_2(u)),a]_\beta=[d_2(u)\gamma d_1d_2(u)+d_2(u)k_1(\gamma)d_2(u)+d_2(u)\gamma d_1d_2(u),a]_\beta$$

$$=2[d_2(u)\gamma d_1d_2(u),a]_\beta=2[d_1d_2(u)\gamma d_2(u),a]_\beta=2d_1d_2(u)\gamma[d_2(u),a]_\beta.$$

Thus, $d_1d_2(u)\gamma[d_2(u),a]_\beta=0$ (as $charM\neq 2$). So, $d_1d_2(u)\Omega[d_2(u),a]_\beta=0$.

Applying Lemma 2.2(i), either $d_1d_2(u)=0$ or $[d_2(u),a]_\beta=0$ for all $u\in U$ and $\beta\in\Gamma$. Let $H=\{u\in U: d_1d_2(u)=0\}$ and $K=\{u\in U:[d_2(u),a]_\beta=0\}$. Then *H* and *K* are subgroups of *U* and $U=H\cup K$. But, since $d_1\neq 0$ and $d_2\neq 0$, *U* cannot be equal to *H* [by using Lemma 2.5(v)]. Therefore, we obtain $U=K$, by Brauer's trick. As a result, $[d_2(U),a]_\beta=0$ for all $\beta\in\Gamma$.

Theorem 2.4 Let $a\in M$, d_1 a nonzero left k_1-derivation of *M* and d_2 a nonzero right k_2-derivation of *M* such that $d_1d_2(U)\subset Z$ and $d^2(U)\subset U$ with $k_1(\Omega)=0$, $k_2(\Omega)=0$ and $charM\neq 2$. If $[d_1(U),a]_\beta=0$ for all $\beta\in\Gamma$, then $a\in Z$.

Proof. Given $a\in M$, from Lemma 2.7, we get $[d_2(U),a]_\beta=0$ for all $\beta\in\Gamma$. Now, let $u\in U$, $\gamma\in\Omega$ and $\beta\in\Gamma$. Then we have

$$0=[d_1d_2(u\gamma u),a]_\beta=[d_1(d_2(u)\gamma u+uk_2(\gamma)u+d_2(u)\gamma u),a]_\beta$$

$$=2[d_1(d_2(u)\gamma u),a]_\beta=2[d_2(u)\gamma d_1(u)+d_2(u)k_1(\gamma)u+u\gamma d_1d_2(u),a]_\beta$$

$$=2[u\gamma d_1d_2(u),a]_\beta=2[d_1d_2(u)\gamma u,a]_\beta=2d_1d_2(u)\gamma[u,a]_\beta.$$

Since $charM\neq 2$, we get $d_1d_2(u)\gamma[u,a]_\beta=0$. Thus, $d_1d_2(u)\Omega[u,a]_\beta=0$. Using Lemma 2.2(i), either $d_1d_2(u)=0$ or $[u,a]_\beta=0$ for all $u\in U$ and $\beta\in\Gamma$. If we let $H=\{u\in U: d_1d_2(u)=0\}$ and $K=\{u\in U:[u,a]_\beta=0\}$, then *H* and *K* are subgroups of *U* and $U=H\cup K$. But, since $d_1\neq 0$ and $d_2\neq 0$, therefore, *U* cannot be equal to *H* [by Lemma 2.5(v)]. Hence, by Brauer's trick, $U=K$. This means, $[u,a]_\beta=0$ for all $\beta\in\Gamma$, and consequently, $a\in Z$.

Theorem 2.5 Let $a\in M$, d_1 a nonzero left k_1-derivation of *M* and d_2 a nonzero right k_2-derivation of *M* such that $d_1d_2(U)\subset Z$ and $d^2(U)\subset U$ with $k_1(\Omega)=0$, $k_2(\Omega)=0$ and $charM\neq 2,3$. If $[d_1(U),a]_\beta\subset Z$ for all $\beta\in\Gamma$, then $a\in Z$.

Proof. If we consider $Z=0$, then $d_1d_2(U)=0$, implying $d_1=0$ or $d_2=0$ [by Lemma 2.5(v)], a contradiction. Therefore, we can assume that $Z\neq 0$. Now, let $a\in M$, $u\in U$, $z\in Z$, $\gamma\in\Omega$ and $\beta\in\Gamma$. Then we get

$$Z\ni[d_1(u\gamma z),a]_\beta=[u\gamma d_1(z)+uk_1(\gamma)z+z\gamma d_1(u),a]_\beta=[u\gamma d_1(z)+z\gamma d_1(u),a]_\beta$$

$= [u\gamma d_1(z), a]_\beta + [z\gamma d_1(u), a]_\beta = [u, a]_\beta \gamma d_1(z) + z\gamma[d_1(u), a]_\beta$

This implies, $[U, a]_\beta \gamma d_1(Z) \subset Z$ for all $\beta \in \Gamma$ and $\gamma \in \Omega$. Hence, by Lemma 2.1(v), either $[U, a]_\beta \subset Z$ for all $\beta \in \Gamma$ or $d_1(Z) = 0$.

Now, if $d_1(Z) \neq 0$, then $[U, a]_\beta \subset Z$ for all $\beta \in \Gamma$. Then, for any $u \in U$, $m \in M$ and $\beta \in \Gamma$, we get $[[u, a]_\beta, m]_\beta = 0$. Hence, by replacing *u* by $u\beta u$, we have

$$0 = [[u\beta u, a]_\beta, m]_\beta = [u\beta[u, a]_\beta + [u, a]_\beta \beta u, m]_\beta = 2[u\beta[u, a]_\beta, m]_\beta.$$

Since $charM \neq 2$, we obtain $0 = [u\beta[u, a]_\beta, m]_\beta = [u, m]_\beta \beta[u, a]_\beta$ for all $u \in U$, $m \in M$ and $\beta \in \Gamma$. Hence, by Lemma 2.1(vi), for all $u \in U$, $m \in M$ and $\beta \in \Gamma$, either $[u, m]_\beta = 0$ or $[u, a]_\beta = 0$. This yields, $[u, a]_\beta = 0$ for all $u \in U$ and $\beta \in \Gamma$. Therefore, by applying Lemma 2.6, we conclude that $a \in Z$.

But, if $d_1(Z) = 0$, then $d_1(d_1 d_2(U)) = 0$. For any $u \in U$ and $\gamma \in \Omega$, we get

$$0 = d_1(d_1 d_2(d_2(u)\gamma d_2(u))) = d_1(d_1(d_2^2(u)\gamma d_2(u) + d_2(u)k_2(\gamma)d_2(u) + d_2^2(u)\gamma d_2(u)))$$

$$= 2d_1(d_1(d_2^2(u)\gamma d_2(u))) = 2d_1(d_2^2(u)\gamma d_1 d_2(u) + d_2^2(u)k_1(\gamma)d_2(u) + d_2(u)\gamma d_1 d_2^2(u))$$

$$= 2[d_1(d_2^2(u)\gamma d_1 d_2(u)) + d_1(d_2(u)\gamma d_1 d_2^2(u))]$$

$$= 2[d_2^2(u)\gamma d_1 d_1 d_2(u) + d_2^2(u)k_1(\gamma)d_1 d_2(u) + d_1 d_2(u)\gamma d_1 d_2^2(u)$$

$$+ d_2(u)\gamma d_1 d_1 d_2^2(u) + d_2(u)k_1(\gamma)d_1 d_2^2(u) + d_1 d_2^2(u)\gamma d_1 d_2(u)]$$

$$= 2[d_1 d_2(u)\gamma d_1 d_2^2(u) + d_1 d_2^2(u)\gamma d_1 d_2(u)] =$$

$$2[d_1 d_2(u)\gamma d_1 d_2^2(u) + d_1 d_2(u)\gamma d_1 d_2^2(u)]$$

$$= 4d_1 d_2(u)\gamma d_1 d_2^2(u)$$

Again, since $charM \neq 2$, $d_1 d_2(u)\gamma d_1 d_2^2(u) = 0$. Thus, $d_1 d_2(u)\Omega d_1 d_2^2(u) = 0$. By Lemma 2.2(i), either $d_1 d_2(u) = 0$ or $d_1 d_2^2(u) = 0$ for all $u \in U$. Saying $H = \{u \in U : d_1 d_2(u) = 0\}$ and $K = \{u \in U : d_1 d_2^2(u) = 0\}$, we see that *H* and *K* are subgroups of *U* and $U = H \cup K$. But, since $d_1 \neq 0$ and $d_2 \neq 0$, *U* cannot be equal to *H* [by Lemma 2.5(v)]. Thus, by Brauer's trick, $U = K$, which implies that $d_1 d_2^2(U) = 0$. Therefore, by Theorem 2.3, *M* is commutative, and hence, we obtain $a \in Z$. This completes the proof.

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