*J. Bangladesh Acad. Sci., Vol. 40, No. 2, 187-197, 2016*

# GENERALIZATION OF SEMIPRIME SUBSEMIMODULES

F. FARZALIPOUR* AND P. GHIASVAND[1]

*Department of Mathematics, Payame Noor University P.O.BOX 19395-3697 Tehran, Iran*

## ABSTRACT

The authors introduce the concept of almost semiprime subsemimodules of semimodules over a commutative semiring *R*. They investigated some basic properties of almost semiprime and weakly semiprime subsemimodules and gave some characterizations of them, especially, for (fnitely generated faithful) multiplication semimodules. They also study the relations among the semiprime, weakly semiprime and almost semiprime subsemimodules of semimodules over semirings.

Key words: Almost semiprime, Multiplication semimodule, Weakly semiprime

## INTRODUCTION

The concept of semirings and semimodules has been studied by several authors (Chaudhury and Bonde 2010 a,b, Golan 1999, Gupta and Chowdhury 2008). Prime subsemimodules of semimodules over a commutative semirng was studied by Atani (2010). Semiprime subsemimodules of a semimodule over a commutative semiring have been studied by Yesilot *et al.* 2010. The authors studied the weakly semiprime and almost semiprime subsemi-modules of a semimodule over a commutative semiring with nonzero identity. They introduced some notation and terminology. By a commutative semiring they mean an algebraic system $R = (R, +, .)$ such that $R = (R, +)$ and $R = (R, .)$ are commutative semigroup, connected by $a(b + c) = ab + bc$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = 0$ and $r$ ě $0 = 0$ ě $r = 0$ for all $r \in R$. Throughout this paper let $R$ be a commutative semiring. A semiring $R$ is said to be semidomain whenever $a, b \in R$ with $ab = 0$, implies that $a = 0$ or $b = 0$. A subtractive ideal (= $k$-ideal) $I$ is an ideal such that if $x, x + y \in I$, then $y \in I$. A (left) semimodule $M$ over a semiring $R$ is a commutative additive semigroup which has a zero element, together with a mapping from $R \times M$ into $M$ such that $(r + s)m = rm + sm$, $r(m + n) = rm + rn$, $r(sm) = (rs)m$ and 2000 Mathematics Subject Classi_cation: 16D10, 16D80, 13A15 $0m = r0_M = 0_MT = 0_M$ for all $m; n \in M$ and $r, s \in R$. Let $M$ be a semimodule over a semiring $R$ and let $N$ be a subset of $M$, we say that $N$ is a subsemimodule of $M$ when $N$ is itself an $R$-semimodule with respect to the operations for $M$ (so $0M \in N$).

*Corresponding author: <farzalipour@pnu.ac.ir>.
[1]Department of Mathematics, Payame Noor University (PNU), P.O.BOX 19395-3697, Tehran, Iran.

It is easy to see that if $r \in R$, then $rM = \{rm : m \in M\}$ is a subsemimodule of $M$. A subtractive subsemimodule (=$k$-subsemimodule) $N$ is subsemimodule such that if $x$, $x + y \in N$, then $y \in N$. A proper subsemimodule $N$ of $R$-semimodule $M$ is called prime, if $rm \in N$ where $r \in R$ and $m \in M$, then $m \in N$ or $rM \subseteq N$. A semimodule $M$ is called prime if the zero subsemimodule of $M$ is prime subsemimodule. The semiring $R$ is a semimodule over itself. In this case, the subsemimodules of $R$ are called ideals of $R$. If $R$ is a semiring (not necessarily a semidomain) and $M$ an $R$-semimodule, then we define the subset $T(M)$ as $T(M) = \{m \in M : rm = 0$ for some $0 \neq r\ 2\ R\}$. It is clear that if $R$ a semidomain, then $T(M)$ is a subsemi-module of $M$ (see [4]). Let $R$ is a semidomain and $M$ an $R$-semimodule, then $M$ is called torsion if $T(M) = M$ and $M$ is called torsion free if $T(M) = 0$. For any two subsemimodules $N$ and $K$ of an $R$-semimodule $M$, the residual of $N$ by $K$ is de_ned as the set $(N : K) = \{r \in R : rK \subseteq N\}$ which is clearly an ideal of $R$ (Alani). In particular, the ideal $(0 : M)$ is called the annihilator of $M$. Let $N$ be a sub-semimodule of $M$ and $I$ be an ideal of $R$, the residual subsemimodule of $N$ by $I$ is defined as $(N :_M I) = \{m \in M : Im \subseteq N\}$ which is clearly a subsemimodule of $M$. These two residual ideal and subsemimodule were proved to be useful in studying many concepts of semimodules. A proper subsemimodule $N$ of an $R$-semimodule $M$ is called semiprime, if whenever $r \in R$, $m \in M$ and $k \in \mathbf{Z}+$ such that $r^k m \in N$, then $rm \in N$. An $R$-semimodule $M$ is called a second semimodule provided that for every element $r \in R$, the $R$-endomorphism of $M$ produced by multiplication by $r$ is either surjective or zero, this implies that $(0 : M) = p$ is a prime ideal of $R$, and $M$ is said to be $p$-second. An $R$-semimodule $M$ is called a multiplication semimodule provided that, for every subsemimodule $N$ of $M$, there exists an ideal $I$ of $R$ so that $N = IM$ (or equivalently, $N = (N : M)M$). An ideal $I$ of a semiring $R$ is called multiplication, if it is multiplication as $R$-semimodules. An $R$-semimodule $M$ is called a cancellation semimodule if for all ideals $I$ and $J$ of $R$, $IM = JM$ implies that $I = J$.

## RESULTS AND DISCUSSION

### Some results on almost semiprime subsemimodules

**Definition 2.1.** *(i)* Let $R$ be a commutative semiring. A proper ideal $I$ of $R$ is called almost semiprime if whenever $a^k b \in I - I^2$ for $a, b \in R$ and $k \in \mathbf{Z}+$, then $ab \in I$.

*(ii)* Let $R$ be a commutative semiring and $M$ be an $R$-semimodule. A proper subsemimodule $N$ of $M$ is called almost semiprime if whenever $r \in R$, $m \in M$ and $k \in \mathbf{Z}+$ such that $r^k m \in N - (N : M)N$, then $rm \in N$.

Let $M$ be an $R$-semimodule and $N$ a subsemimodule of $M$. $N$ is called idem-potent in $M$ if $N = (N : M)N$. Thus any proper idempotent subsemimodule of $M$ is almost semiprime. If $M$ is a multiplication $R$-semimodule and $N = IM$ and $K = JM$ are two subsemimodules of $M$, then the product $NK$ of $N$ and $K$ is defined as $NK = (IM)(JM) = (IJ)M$. In particular, we have $N^2 = NN = [(N : M)M]\ [(N : M)M] = (N : M)^2 M$. A subsemimodule $N$ of an $R$-semimodule $M$ is called a pure (RD-) subsemimodule if $IN = N \cap IM$ ($rN = N \cap rM$) for any ideal $I$ of $R$ (for any $r \in R$).

**Example 1.** It is clear that every semiprime subsemimodule is almost semiprime. But the convers is not true in general. For example, set $\mathbf{Z}^* = \mathbf{Z}+ \cup \{0\}$. Consider $\mathbf{Z}^*$-semimodule $M = \mathbf{Z}^*_{24}$ (the non negative integers modulo 24) and the subsemimodule $N = < 8 >$. Then $(N : M)N = N$, and so $N$ is an almost semiprime subsemimodule of $M$. But $N$ is not semiprime in $M$, because $2^2 \in 2 \in N$, but $2 \cdot 2 \notin N$.

Now the authors study almost semiprime subsemimodules in quotient semimodules.

Atani and Atani (2010) introduced the concept of quotient semimodules, a subsemimodule $N$ of an $R$-semimodule $M$ is called a partitioning subsemimodule (=$Q$-subsemimodule) if there exists a non-empty subset $Q$ of $M$ such that

(1) $RQ \subseteq Q$, where $RQ = \{rq : q \in Q\}$;

(2) $M = \cup \{q + N : q \in Q\}$

(3) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \varnothing \Leftrightarrow q_1 = q_2$.

Let $M$ be a semimodule over a semiring $R$, and let $N$ be a $Q$-subsemimodule of $M$. We put $M/N = \{q + N : q\ 2\ Q\}$. Then $M=N$ forms a commutative additive semigroup which has zero element under the binary operation $\oplus$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3$ is a unique element of $Q$ such that $q_1 + q_2 + N \subseteq q_3 + N$. By the definition of $Q$-subsemimodule, there exist a unique $q_0 \in Q$ such that $0_M + N \subseteq q_0 + N$. Then $q_0 + N$ is a zero element of $M/N$. But, for every $q \in Q$ from (1) one obtains $0_M = 0_{Rq} \in Q$; hence $q_0 = 0$.

Now let $r \in R$ and suppose that $q_1 + N$, $q_2 + N \in M/N$ are such that $q_1 + N = q_2 + N$ in $M/N$. Then $q_1 = q_2$, the authors must have $rq_1 + N = rq_2 + N$. The authors can unambiguously define a mapping from $R \times M/N$ into $M/N$ (sending $(r, q_1 + N)$ to $rq_1 + N$) and it is routine to check that this turns the commutative semigroup $M/N$ into an $R$-semimodule. The authors call this $R$-semimoule the residue class semimodule or factor semimodule of $M$ modulo $N$.

In the almost semiprime subsemimodules case, $N$ is an almost semiprime sub-semimodule of $M$, then $N/K$ is an almost semiprime subsemimodule of $M/K$ for any $Q$-subsemimodule $K \subseteq N$. But the covers part may not be true. For example, for any non almost semiprime subsemimodule $N$ of $M$, the authors have $N/N = 0$ is an almost semiprime subsemimodule of $M/N$. But the authors have the following theorem:

**Theorem 2.2.** *Let $N$ be a $k$-subsemimodule and $K$ a $Q$-subsemimodules of an $R$-semimodule $M$ with $K$ ċ $(N : M)N$. Then $N$ is an almost semiprime sub-semimodule of $M$ if and only if $N/K$ is an almost semiprime subsemimodule of the $R$-semimodule $M/K$.*

**Proof**. Let $N$ be an almost semiprime subsemimodule of $M$ and assume that $r \in R$, $q_1 + K \in M/K$ and $k \in \mathbf{Z}^+$ such that $r^k(q_1 + K) \in N/K - (N/K : M/K)N/K$. So $r^k q_1 + K = q_2 + K$ where $q^2 \in Q \cap N$. Therefore $r^k q_1 \in N$ since $K \subseteq N$ and $N$ is a $k$-subsemimodule of $M$. Since by Lemma 5 of [4], $(N : M) = (N/K : M/K)$, so $r^k q_1 \notin (N : M)N$, hence $rq_1 \in N$ since $N$ is almost semiprime subsemimodule. Thus $r(q_1 + K) \in N/K$, as needed. Conversely, let $N/K$ be an almost semiprime subsemimodule of $M/K$ and assume that $r^k m \in N - (N : M)N$ for some $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. So there exists a unique element $q \in Q$ such that $m \in q + K$. Hence $m = q + k$ for some $k \in K$, and $r^k m = r^k q + r^k k$. Since $K \subseteq N$ and $N$ is $k$-subsemimodule, hence $r^k q \in N \cap Q$. Thus $r^k q + K \in N/K$, and it is clear that $r^k q + K \notin (N/K : M/K)N/K$ $((N : M) = (N/K : M/K))$. Therefore, $r(q + K) \in N/K$ because $N/K$ is almost semiprime, so $rq \in N$. Thus $rm = rq + rk \in N$, as required.

Let $R$ be a semiring and let $S$ be a multiplicatively closed subset of $R$. De_ne a relation on $R \times S$ as follows:

for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $at = bs$. Then $\sim$ is a equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of s which contains $(a, s)$ by $a/s$, and denoted the set of all equivalence classes of $\sim$ by $S^{-1}R$. Then $S^{-1} R$ can be given the structure of a commutative semiring under operations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = ab/st$ for all $a, b \in R$ and $s, t \in S$. This new semiring $S^{-1}R$ is called the semiring of fractions of $R$ with respect to $S$, its zero element is 0/1, its multiplicative identity element is 1/1 and each element of $S^{-1}R$ has a multiplicative inverse in $S^{-1}R$ (see [5]). Let $M$ be a semimodule over a semiring $R$. We de_ne a relation on $M \times S$. Assume that $(a, s), (b, t) \in M \times S$, we write $(m, s) \sim (n, t)$ if and only if $mt = ns$. Then $\sim$ is a equivalence relation on $M \times S$. For $(m, s) \in M \times S$, denote the equivalence class of $\sim$ which contains $(m; s)$ by $m/s$, and denoted the set of all equivalence classes of s by $S^{-1}M$. Then $S^{-1}M$ can be given the structure of a semimodule over the semiring $S^{-1}R$ under operations for which $m/s + n/t = (tm + sn)/st$, $(r/l)(m/s) = am/st$ for $r/l \in S^{-1}R$ and $m/s, n/t \in S^{-1}M$.

**Theorem 2.3.** *Let $S$ be a multiplicative closed subset of $R$ and $N$ an almost semiprime subsemimodule of $R$-semimodule $M$ with $S$ ą $(N : M) = \varnothing$. Then $S^{-1}N$ is an almost semiprime subsemimodule of the $S^{-1}R$-semimodule $S^{-1}M$.*

**Proof**. Let $N$ be an almost semiprime subsemimodule of $M$. Since $(N : M) \cap S = \varnothing$, then $S^{-1}N \neq S^{-1}M$. Assume that $(r/s)^k m/t \in S^{-1}N - (S^{-1} :_{S-1R} S^{-1}M)S^{-1}N$ where $r/s \in S^{-1}R$, $m/t \in S^{-1}M$ and $k \in \mathbf{Z}^+$. Hence $r^k m/s^k t = n/s'$ for some $n \in N$ and $s' \in S$, and so there exists $t' \in S$ such that $r^k s't'm = s^k tt'n \in N$. If $r^k s't'm \in (N : M)N$, then $r^k m/s^k t = r^k s't'm = s^k ts't' \in S^{-1}((N : R\ M)N) = S^{-1}(N :_R M)S^{-1}N \subseteq (S^{-1}N :_{S-1R} S^{-1}M)S^{-1}N$, a contradiction. So $rks't'm \in N - (N : M)N$, and $rs't'm \in N$ since $N$ is almost semiprime. Therefore $rm/st = rs't'm/sts't' \in S^{-1}N$, hence $S^{-1}N$ is an almost semiprime subsemimodule of $S^{-1}M$.

**Proposition 2.4.** *Let $R = R^1 \times R^2$ where each $R_i$ is a commutative semiring with non zero identity. Let $M_i$ be an $R_i$-semimodule and let $M = M_1 \times M_2$ be the $R$-semimodule with action $(r_1, r_2)$ $(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then*

*(i) $N_1$ is an almost semiprime subsemimodule of $M_1$ if and only if $N_1 \times M_2$ is an almost semiprime subsemimodule of $M$.*

*(ii) $N_2$ is an almost semiprime subsemimodule of $M_2$ if and only if $M_1 \times N_2$ is an almost semiprime subsemimodule of $M$.*

**Proof**. (i) Let $N_1$ be an almost semiprime subsemimodule of $M_1$. Assume that $(r_1, r_2)$ $k(m_1, m_2) \in N_1 \times M_2 - (N_1 \times M_2 : M)N_1 \times M_2$ where $(r_1, r_2) \in R$, $(m_1, m_2) \in M$ and $k \in \mathbf{Z}^+$. If $r^k{}_1m_1 \in (N_1 : M_1)N_1$, then $(r_1, r_2)$ $k(m_1, m_2) \in (N_1 : M_1)N_1 \times (M_2 : M_2)M_2 = ((N_1 : M_1) \times (M_2 : M_2))N_1 \times M_2 = (N_1 \times M_2 : M_1 \times M_2)N_1 \times M_2$, a contradiction. Hence as $N_1$ is almost semiprime and $r^k{}_1m_1 \in N_1 - (N_1 : M_1)N_1$, then $r_1m_1 \in N_1$, and so $(r_1, r_2)(m_1, m_2) \in N_1 \times M_2$. Conversely, Assume that $N_1 \times M_2$ is an almost semiprime subsemimodule of $M$. Let $r^k{}_1m_1 \in N_1 - (N_1 : M_1)N_1$ for $r_1 \in R_1$, $m_1 \in M_1$ and $k \in \mathbf{Z}^+$. Then $(r_1, 1)$ $k(m_1, 0) \in N_1 \times M_2 -(N_1 \times M_2 : M)N_1 \times M_2$ by (i). Therefore $(r_1, 1)$ $(m_1, 0) \in N_1 \times M_2$, since $N_1 \times M_2$ is almost semiprime, so $r_1m_1 \in N_1$, as needed.

(ii) is similar to (i).

Let $R$ be a commutative semiring with identity and $M$ be an $R$-semimodule. Then $R(M) = R(+)M$ with multiplication $(a, m)$ $(b, n) = (ab, an + bm)$ and with addition $(a, m) + (b, n) = (a + b, m + n)$ is a commutative semiring with identity $(1, 0)$, and $0(+)M$ is a nilpotent ideal of index 2. The semiring $R(+)M$ is said to be the idealization of $M$ or trivial extension of $R$ by $M$. We view $R$ as a subsemiring of $R(+)M$ via $r \to (r, 0)$. An ideal $J$ is said to be homogeneous if $J = I(+)N$ for some ideal $I$ of $R$ and some subsemimodule $N$ of $M$ such that $IM \subseteq N$.

**Lemma 2.5.** *Let $I(+)N$ be an ideal of $R(M)$. Then $(I(+)N)^2 \subseteq I^2(+)IN$.*

**Proof**. The proof is straightforward.

**Theorem 2.6.** *Let $I(+)N$ be a homogeneous ideal of $R(M)$. Then, if $I(+)N$ is an almost semiprime ideal of $R(M)$, then $I$ is an almost semiprime ideal of $R$ and $N$ is an almost semiprime subsemimodule of $M$.*

**Proof**. Assume that $I(+)N$ is an almost semiprime ideal of $R(M)$. Let $a, b \in R$ and $k \in \mathbf{Z}^+$ such that $a^kb \in I - I^2$. Then $(a, 0)$ $k(b, 0) \in I(+)N - (I(+)N)^2$. Because if $(a, 0)$ $k(b, 0) \in (I(+)N)^2$, then by Lemma 2.5, $(a, 0)$ $k(b, 0) \in I^2(+)IN$, hence $a^kb \in I^2$, a contradiction. Therefore $(a, 0)$ $(b, 0) \in I(+)N$, and $ab \in I$, so $I$ is an almost semiprime ideal of $R$. Let $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$ such that $r^km \in N - (N : M)N$. Therefore $(r, 0)$ $k(0, m) \in I(+)N - (I(+)N)^2$. Because if $(r, 0)$ $k(0, m) = (0, r^km) \in (I(+)N)^2 \subseteq I^2(+)IN$, then

$r^k m \in IN$. So $r^k m \in IN \subseteq (N : M)N$ since $I(+)N$ is a homogeneous ideal, a contradiction. Hence $(r, 0)\,(0, m) \in I(+)N$, so $rm \in N$. Thus $N$ is an almost semiprime subsemimodule of $M$.

**Proposition 2.7.** *Let M be an R-semimodule and N be an almost semiprime subsemimodule of M. Then*

*(i) If M is a second R-semimodule, then N is a second semimodule.*

*(ii) If M is a second R-semimodule, then N is an RD-subsemimodule of M.*

**Proof**. Let $N$ be an almost semiprime subsemimodule of $M$ and let $r \in R$. If $rM = 0$, then $rN \subseteq rM = 0$. Let $rM = M$. Now It is enough to show that $N \subseteq rN$. First, we show that $(N : M)N = 0$. Since $N$ is a proper subsemimodule of $M$, so for any $r \in (N : M)$, we have $rM = 0$. Therefore $(N : M)N = 0$. Let $n \in N$. We may assume that $n \neq 0$. Since $rM = M$, so $n = rm$ for some $m \in M$, and $m = rm'$ for some $m' \in M$. Hence $n = r^2 m' \in N - (N : M)N$, as $N$ is almost semiprim so $m = rm' \in N$. Hence $n = rm \in rN$, so $N \subseteq rN$. Therefore $rN = N$, and $N$ is second.

(ii) Let $r \in R$. If $rM = 0$, then $rN = 0$, so $rN = 0 = N \cap rM$. Suppose that $rM = M$, so by (i), $rN = N$, therefore $rN = N \cap rM$.

In the following Theorems, we give other characterizations of almost semiprime subsemimodules.

**Theorem 2.8.** *Let M be an R-semimodule and N a proper subsemimodule of M. Then the following are equivalent:*

*(i) N is an almost semiprime subsemimodule of M.*

*(ii) For* $r \in R$, $k \in \mathbf{Z}^+$, $(N : M <r^k>) = (N : M <r>) \cup ((N : M)N : M <r^k>)$.

*(iii) For* $r \in R$ *and* $k \in \mathbf{Z}^+$, $(N : M <r^k>) = (N : M <r>)$ *or* $(N : M <r^k>) = ((N : M)N : M <r^k>)$.

**Proof**. *(i)* $\Rightarrow$ *(ii)* Let $m \in (N : M <r^k>)$, then $r^k m \in N$. If $r^k m \neq (N :{}_R M)N$, as $N$ is almost semiprime, $rm \in N$, so $m \in (N : M <r>)$. Let $r^k m \in (N :{}_R M)N$, then $m \in ((N :{}_R M)N : M <r> k)$, hence $(N : M <r^k>) \subseteq (N : M <r>) \cup ((N : M)N : M <r^k>)$. The other containment holds for any subsemimodule $N$.

*(ii)* $\Rightarrow$ *(iii)* It is well known that if a subsemimodule is the union of two subsemimodules, then it is equal to one of them.

*(iii)* $\Rightarrow$ *(i)* Let $r^k m \in N - (N :{}_R M)N$ for some $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. Hence $m \in (N : M <r^k>)$ and $m \neq ((N :{}_R M)N : M <r^k>)$, so by assumption, $m \in (N : M <r>)$ and $rm$

$\in N$. Therefore *N* is almost semiprime. The following Theorem give from Theorem 2.8

**Theorem 2.9.** *Let M be an R-semimodule and N be a proper subsemimodule of M. Then N is almost semiprime in M if and only if for any subsemimodule K of M, a* Č *R and k* Č $\mathbf{Z}^+$ *with* $<a>^k K$ ċ *N and* $<a>^k K \not\subseteq (N :_R M)N$*, we have* $<a> K$ ċ *N.*

It is clear that if *N* is a semiprime subsemimodule of *M*, then $(N :_R M)$ is a semiprime ideal of *R*. But it may not be true in the case of almost semiprime subsemimodules.

**Example 2.** Let *M* denoted the cyclic $\mathbf{Z}^*$-semimodule $\mathbf{Z}^*4$ (the non negative integers modulo 4). Take $N = \{0\}$. Certainly, *N* is almost semiprime, but $(N :_R M) = 4\mathbf{Z}^*$ is not an almost semiprime ideal of $\mathbf{Z}^*$. Because $2^2 \in (N : M) - (N : M)^2$, but $2 \neq (N : M)$.

Now in the following Theorem, we give a characterization of almost semiprime subsemimodules in (finitely generated faithful) multiplication semimodules. We first need the following Lemma.

**Lemma 2.10.** *Let N be a subsemimodule of a finitely generated faithful multiplication (so cancellation) R-semimodule. Then* $(IN : M) = I(N : M)$ *for every ideal I of R.*

**Proof**. As *M* is multiplication *R*-semimodule, then we have $(IN : M)M = IN = I(N : M)M$. So since *M* is cacellation semimodule, the proof is hold.

**Theorem 2.11.** *Let M be a finitely generated faithful multiplication R-semimodule and N a proper subsemimodule of M. Then the following are equivalent:*

*(i) N is almost semiprime in M.*

*(ii)* $(N :_R M)$ *is almost semiprime in R.*

*(iii) N = PM for some almost semiprime ideal P of R.*

**Proof**. *(i)* Ĝ *(ii)* Suppose that *N* is an almost semiprime subsemimodule of *M*. Let $a, b \in R$ and $k \in \mathbf{Z}^+$ such that $a^k b \in (N : M) - (N : M)^2$. Then $<a>^k (bM) \subseteq N$ and $<a>^k (bM) \not\subseteq (N : M)N$. Indeed, if $<a>k\,(bM) \subseteq (N : M)N$, then by Lemma 2.10, $a^k b \in ((N : M)N : M) = (N : M)^2$, a contradiction. Now, *N* almost semiprime implies that $<a> (bM) \subseteq N$ by Theorem 2.9, so $ab \in (N : M)$, hence $(N : M)$ is almost semiprime in *R*.

*(ii)* Ĝ *(i)* In this direction, we need *M* to be just a multiplication semimodule. Let $r^k m \in N - (N : M)M$ where $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. Then $<r>^k (<m> : M) \subseteq (<r^k m> : M) \subseteq (N : M)$. Moreover, $<r>^k (<m > : M) \not\subseteq (N : M)^2$ because otherwise, if $<r>^k (<m> : M) \subseteq (N : M)^2 \subseteq ((N : M)N : M)$, then $<r>^k <m> = <r>^k (<m> : M)M \subseteq (N : M)N$, a contradiction. As $(N : M)$ is an almost semiprime ideal of *R*, then $<r> (<m>: M) \subseteq (N : M)$. Therefore $<r> <m> = <r> (<m> : M)M \subseteq (N : M)M = N$, and so $rm \in N$, as required.

*(ii)* Ĝ *(iii)* We choose $P = (N : M)$.

**3 The relations among almost semiprime and weakly semiprime subsemimodules**

**Definition 3.1.** *(i)* Let $R$ be a commutative semiring. A proper ideal $I$ of $R$ is called weakly semiprime if whenever $0 \neq a^k b \in I$ for some $a, b \in R$ and $k \in \mathbf{Z}^+$, then $ab \in I$.

*(ii)* Let $M$ be an $R$-semimodule. A proper subsemimodule $N$ of $M$ is called weakly semiprime if whenever $0 \neq r^k m \in N$ for some $r \in M$, $m \in M$ and $k \in \mathbf{Z}^+$; then $rm \in N$.

**Remark 1.** Let $M$ a semimodule over a commutative semiring $R$. Then semiprime subsemimodules $\Rightarrow$ weakly semiprime subsemimodudules $\Rightarrow$ almost semiprime subsemimodules.

**Example 3.** Consider the $\mathbf{Z}^*$-semimodule $M = \mathbf{Z}^*24$ and the proper subsemimodule $N = \langle 8 \rangle = \{0, 8, 16\}$. Then $0 = 0\cdot 8$, $8 = 16\cdot 8$ and $16 = 16\cdot 16$, so $(N : M)N = N$. Therefore $N$ is almost semiprime. On the other hand, $0 \neq 2^2\cdot 2 \in N$, but $2\cdot 2 \notin N$, and so $N$ is not weakly semiprime.

**Theorem 3.2.** *Let M be an R-semimodule, (N : M)N a Q-subsemimodule of M and N a proper k-subsemimodule of M. Then N is an almost semiprime subsemi-module of M if and only if N/(N : M)N is a weakly semiprime subsemimodule of the R-semimodule M/(N : M)N.*

**Proof**. Assume that $N$ is an almost semiprime subsemimodule of $M$. Let $q_0 + (N : M)N \neq r^k(q_1 + (N : M)N) \in N/(N : M)N$ where $r \in R$, $q_1 \in Q$ and $k \in \mathbf{Z}^+$. Hence $r^k q_1 + (N : M)N = q_2 + (N : M)N$ such that $q_2 \in Q \cap N$. Since $(N : M)N \subseteq N$ and $N$ is a $k$-subsemimodule, hence $r^k q_1 \in N$. If $r^k q_1 \in (N : M)N$, then $r^k q_1 \in q_0 + (N : M)N$, so $r_k q_1 + (N : M)N = q_0 + (N : M)N$, a contradiction. Hence $r^k q_1 \in N - (N : M)N$, and so $rq_1 \in N$ since $N$ is weakly semiprime. Therefore $r(q_1 + (N : M)N)) \in N/(N : M)N$, as needed. Conversely, assume that $N/(N : M)N$ is weakly semiprime in $M/(N : M)N$. Let $r^k m \in N - (N : M)N$ where $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. Then $m \in q + (N : M)N$ where $q \in Q$ is a unique element of $Q$. Thus $r^k m \in r^k q + (N : M)N$ and since $N$ is $k$-subsemimodule, then $r^k q \in N \cap Q$. Therefore $q_0 + (N : M)N \neq r^k q + (N : M)N \in N/(N : M)N$, hence $r(q + (N : M)N) \in N/(N : M)N$ since $N/K$ is weakly smiprime subsemimodule. So $rq \in N$, and so $rm \in N$, as required.

**Proposition 3.3.** *Let R be a semidomain and M be a torsion free R-semimodule. Then every weakly semiprime subsemimodule of M is semiprime.*

**Proof**. Let $N$ be a weakly semiprime subsemimodule of $M$. Let $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$ such that $r^k m \in N$. If $0 \neq r^k m$, then $N$ weakly semiprime gives that $rm \in N$. Suppose that $r^k m = 0$. If $rk \neq 0$, then $m \in T(M) = 0$, so $rm \in N$. If $rk = 0$, then $r = 0$, and hence $rm$

$\in N$. Therefore $N$ is semiprime.

**Proposition 3.4.** *Let $M$ be a prime $R$-semimodule. Then every weakly semiprime subsemimodule of $M$ is semiprime.*

**Proof**. Let $N$ be a weakly semiprime subsemimodule of $M$. Let $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$ such that $r^k m \in N$. If $0 \neq r^k m$, then $N$ weakly semiprime gives that $rm \in N$. Suppose that $r^k m = 0$, then $rm = 0$ or $r^{k-1}M = 0$ since $M$ is a prime semimodule. By following this method, we get $rm = 0 \in N$, hence $N$ is a semiprime subsemimodule of $M$.

**Proposition 3.5.** *Let $M$ be a second $R$-semimodule and $N$ a proper subsemimodule of $M$. Then $N$ is almost semiprime if and only if $N$ is weakly semiprime.*

**Proof**. We know that every weakly semiprime is almost semiprime. Let $N$ be an almost semiprime subsemimodule of $M$ and $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. By Proposition 2.7, we have $(N : M)N = 0$, hence $rkm \in N - (N : M)N$, and so $rm \in N$. Therefore $N$ is weakly semiprime subsemimodule of $M$.

Now we get other characterizations of weakly semiprime subsemimodule.

**Theorem 3.6.** *Let $M$ be an $R$-semimodule and $N$ a proper subsemimodule of $M$. Then the following are equivalent:*

*(i) $N$ is a weakly semiprime subsemimodule of $M$.*

*(ii) For $r$ Č $R$ and $k$ Č $\mathbf{Z}^+$; $(N :_M \langle r^k \rangle) = (0 :_M \langle r^k \rangle)$ Ć $(N :_M \langle r \rangle)$.*

*(iii) For $r$ Č $R$ and $k$ Č $\mathbf{Z}^+$; $(N :_M \langle r^k \rangle) = (0 :_M \langle r^k \rangle)$ or $(N :_M \langle rk \rangle) = (N :_M \langle r \rangle)$.*

**Proof**. (i) $\Rightarrow$ (ii) Let $m \in (N :_M \langle r^k \rangle)$, then $r^k m \in N$. If $r^k m \neq 0$, as $N$ is weakly semiprime, $rm \in N$, so $m \in (N :_M \langle r \rangle)$. Let $r^k m = 0$, then $m \in (0 :_M \langle r^k \rangle)$, hence $(N : M \langle r^k \rangle) \subseteq (N :_M \langle r \rangle)[(0 :_M \langle r^k \rangle)$. Clearly, $(N :_M \langle r \rangle) \cup (0 :_M \langle r^k \rangle) \subseteq (N :_M \langle r^k \rangle)$, therefore $(N :_M \langle r^k \rangle) = (0 :_M \langle r^k \rangle) \cup (N :_M \langle r \rangle)$.

*(ii)* Ĝ *(iii)* It is straightforward.

*(iii)* Ĝ *(i)* Let $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. Hence $m \in (N :_M \langle r^k \rangle)$ and $m \neq (0 :_M \langle r^k \rangle)$, so by assumption, $m \in (N :_M \langle r \rangle)$. Therefore $N$ is weakly semiprime.

**Theorem 3.7.** Let $M$ be an $R$-semimodule and $N$ be a proper subsemimodule of $M$. Then $N$ is weakly semiprime in $M$ if and only if for any subsemimodule $K$ of $M$, $a \in R$ and $k \in \mathrm{Z}^+$ with $0 \neq \langle a \rangle^k K \subseteq N$, we have $\langle a \rangle K \subseteq N$.

**Theorem 3.8.** *Let $N$ be a weakly semiprime subsemimodule of an $R$-semimodule $M$ with $T(M) = 0$. Then for any non zero ideal $I$ of $R$; $(N :_M I)$ is a weakly semiprime* *subsemimodule of M.*

**Proof**. Let $r \in R$, $m \in M$ and $k \in$ Z+ such that $0 \neq rkm \in (N : M\, I)$. Hence $<r>^k (mI) \subseteq N$. If $0 \neq\, <r>^k (mI) \subseteq N$, then by Theorem 3.7, $N$ weakly semiprime gives that $<r> (mI) \in N$, so $rm \in (N :_M I)$, as needed. Suppose that $<r>^k (mI) = 0$, so $r^k ma = 0$ for some non zero $a \in I$. Hence $r^k m \in T(M) = 0$, which is a contradiction. Therefore $(N :_M I)$ is weakly semiprime.

In Theorem 3.8, the assumption $T(M) = 0$ is necessary. To see this, consider $\mathbf{Z}^*$-semimodule $\mathbf{Z}^*_{16}$. Let $N = \{0\}$ and $I = 2\mathbf{Z}^*$. Clearly, $N$ is weakly semiprime subsemimodule of $M$, but $(N :_M I) = \{0\}$, $8g$ is not weakly semiprime.

**Theorem 3.9.** *Let I be an ideal of R and N a subsemimodule of M such that I(+)N be a weakly semiprime ideal of R(M). Then I is a weakly semiprime ideal of R and N is a weakly semiprime subsemimodule of M.*

**Proof**. Assume that $I(+)N$ is a weakly semiprime ideal of $R(M)$. Let $a, b \in R$ and $k \in \mathbf{Z}^+$ such that $0 \neq a^k b \in I$. Then $(0; 0) \neq (a; 0)\, k(b, 0) \in I(+)N$. Therefore $(a, 0)\,(b, 0) \in I(+)N$, and $ab \in I$, so $I$ is a weakly semiprime ideal of $R$. Now, let $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$ such that $0 \neq r^k m \in N$. Therefore $(0, 0) \neq (r, 0)\, k(0, m) \in I(+)N$, hence $(r, 0)\,(0, m) \in I(+)N$, so $rm \in N$. Thus $N$ is a weakly semiprime subsemimodule of $M$.

**Proposition 3.10.** *Let M be a faithful R-semimodule and N a weakly semiprime subsemimodule of M. Then (N : M) is a weakly semiprime ideal of R.*

**Proof.** Suppose $N$ is weakly semiprime, $a, b \in R$ and $k \in \mathbf{Z}^+$ such that $0 \neq a^k b \in (N : M)$. Then $0 = \,<a>^k (bM) \subseteq N$. Indeed, if $<a>^k (bM) = 0$, then $a^k b \in (0 : M) = 0$, a contradiction. Now, by Theorem 3.7 implies that $<a> (bM) \subseteq N$, so $ab \in (N : M)$, and $(N : M)$ is weakly semiprime in $R$.

Now we give characterizations of weakly semiprime subsemimodules in (finitely generated faithful) multiplication semimodules.

**Theorem 3.11.** *Let M be a finitely generated faithful multiplication R-semimodule and N be a proper subsemimodule of M. Then the following are equivalent:*

*(i) N is weakly semiprime in M.*

*(ii)* $(N :_R M)$ *is weakly semiprime in R.*

*(iii) N = QM for some weakly semiprime ideal Q of R.*

**Proof**. *(i)* Ĝ *(ii) By Proposition 3.10.*

*(ii)* Ĝ *(i)* In this direction, we need $M$ to be just a multiplication semimodule. Let $0= r^k m \in N$ where $r \in R$, $m \in M$ and $k \in \mathbf{Z}^+$. Then $<r>^k (<m> : M) \subseteq (<r^k m> : M) \subseteq (N :$

$M$). Moreover, $<r>^k (<m> : M) \neq 0$ because otherwise, if $<r>^k (<m> : M) = 0$, then $<r>^k <m> = <r>^k (<m> : M)M = 0$, a contradiction. As $(N : M)$ is a weakly semiprime ideal of $R$, then $<r> (<m> : M) \subseteq (N : M)$. Therefore $<r> <m> = <r> (<m> : M)M \subseteq (N : M)M = N$, and so $rm \in N$, as required.

($ii$) $\Leftrightarrow$ ($iii$) We choose $Q = (N : M)$.

## REFERENCES

Chaudhari, J. N. and D. R. Bonde. 2010a. Weakly prime subsemimodules of semimod-ules over semirings. *International J. Algebra* **4**(5): 167-174.

Chaudhari, J. N. and D. R. Bonde. 2010b. On partitioning and subtractive subsemimod-ules of semimodules over semirings, *Kyungpook Math. J.* **50**: 329-336.

Atani, R. Ebrahimi. 2010a. On prime subsemimodules of semimodules, *Int. J. of Alge-bra,* **4**(26): 1299-1306.

Atani, R. Ebrahimi and S. Ebrahimi Atani. 2010b. On subsemimodules of semimodules, *Buletinul Academiei De Stiinte a Repablicii Moldova Mathematica*. **2**(63): 20-30.

Golan, J. S. 1999. *Semiring and their applications*. Kluwer Academic publisher Dor-drecht.

Gupta, Vishnu and J. N. Chaudhari, 2008. Characterization of weakly prime subtrac-tive ideals in semirings, *Bull. Inst. Math. Acad. Sinica (New Series)* **3**: 347-352.

Yesilot, G., K. H. Oral and U. Tekir. 2010. On prime subsemimodule of semimodules. *International J. Algebra* **4**(1): 53-60.

(*Received revised manuscript on 16 September, 2016*)