*J. Bangladesh Acad. Sci., Vol. 43, No. 1, 85-93, 2019*
*DOI: https://doi.org/10.3329/jbas.v43i1.42237*

# SECOND ORDER SCHEME FOR KORTEWEG-DE VRIES (KDV) EQUATION

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## ABSTRACT

The kinematics of the solitary waves is formed by Korteweg-de Vries (KdV) equation. In this paper, a third order general form of the KdV equation with convection and dispersion terms is considered. Explicit finite difference schemes for the numerical solution of the KdV equation is investigated and stability condition for a first-order scheme using convex combination method is determined. Von Neumann stability analysis is performed to determine the stability condition for a second order scheme. The well-known qualitative behavior of the KdV equation is verified and error estimation for comparisons is performed.

**Keywords:** KdV Equation, Soliton, Solitary Wave, Finite Difference Scheme

## INTRODUCTION

In nature we can see, waves on the surface of the ocean are playing beautiful and dramatic phenomena that impact every aspect of life on the planet. So in this situation, one needs to include this phenomenon as a mathematical model and analysis. At first, in 1895 Korteweg and de Vries (Korteweg and De Vries 1895) developed the Korteweg-de Vries (KdV) equation to model weakly nonlinear waves. This equation has been used in several different fields to describe various physical phenomena of interest such as water wave (Hammack and Segur 1974), plasma physics (Berezin and Karpman 1964), and bubble-liquid mixture (Wijngaarden 1968) and so on. It is also applicable to pulse wave propagation in blood vessels. The solution of the KdV equation is resemblance as soliton, and it is newly found that signals carry within neurons in the form of solitons (Heimburg and Jackson 2005 2007, Andersen *et al*. 2009). These solitons may take place in proteins and DNA (deoxyribonucleic acid), and solitons are related to the low-frequency collective motion in proteins and DNA (Sinkala 2006).

In 1965 Zabusky and Kruskal (Zabusky and Kruskal 1965) obtained the numerical solution of the KdV equation. Moreover, different methods have been discussed in several papers to solve the KdV equation numerically. Collocation and Radial Basis Function (RBF) method (Dehghan and Shokri 2007) solves the KdV equation numerically with better accuracy, but the complex calculation is needed to solve this method numerically which takes much time. So investigation of Finite Difference Method (FDM) with sufficient accuracy for the KdV equation was undertaken. Shahrill *et al.* (2015) Zabusky and Kruskal presented a second-order finite difference scheme (ZK scheme) for the KdV equation. The scheme is considered in non-conservative form and the convection velocity is considered the average of the three neighboring grid points. For further investigation of the finite difference scheme first, first-order scheme is studied in both non-conservative and conservative forms. Then the second order scheme in both conservative and non-conservative forms for the KdV equation is considered.

Exact solution of the KdV equation is discussed first. Then explicit finite difference schemes for

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the numerical solution of the KdV equation is investigated and the stability conditions for the first and second order schemes are determined. Zabusky and Kruskal scheme is also presented in the same section. After that, the verification of the effects of convection and dispersion terms is discussed. Numerical results and explanation of graphical representations for various cases are discussed sequentially. In the end, some references are given.

**Exact Solution of the KdV equation**

The third order general form of the KdV equation is given by

$$\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} + \nu \frac{\partial^3 u}{\partial x^3} = 0. \qquad (1)$$

Where $\mu$ and $\nu$ are nonlinear and dispersion coefficients respectively. The equation has two different terms; one is the nonlinear term (with non-linear coefficient $\mu$) describing convection term and third order term (with dispersion coefficient $\nu$) illustrating dispersion term.

The exact solution of the KdV equation is obtained from (Kitavi 2013, Brauer 2000), and the solution resembles a single solitary wave. The solution is of the form:

$$u(x,t) = \frac{3c}{\mu} sech^2 \left[\frac{1}{2}\sqrt{\frac{c}{\nu}}(x - ct - \xi_0)\right]. \quad (2)$$

Where $c$ is the speed propagation of the wave, which is proportional to the amplitude $\frac{3c}{\mu}$ (linearly related to the amplitude), $\xi_0$ is the propagating constant (wave propagates left to right while increasing $\xi_0$ and wave moves right to left for decreasing $\xi_0$). For getting real and positive solitary wave solution, the quantity $c$ must be positive, and the wave moves to the right for $c > 0$. From the exact solution, the behavior of the dispersion term and the nonlinear term are observed. If the dispersion coefficient ($\nu$) increases, the width of the wave increases and similarly it is seen, when $\nu$ decreases, the width of the wave decreases, which represents the effect of dispersion term. And $\mu$ is inversely related to amplitude. When $\mu$ increasing, the amplitude of the wave decreasing and for decreasing $\mu$, the amplitude of the wave increasing, which represents the effect of the non-linear term.

**Finite Difference Methods of the KdV Equation**

In this section, some finite difference schemes of the KdV equation are discussed. The numerical solution of the KdV equation is investigated by explicit finite difference schemes. In (Shahrill *et al*. 2015). Zabusky and Kruskal presented a second-order explicit finite difference scheme (ZK scheme) for the KdV equation. The scheme is considered in non-conservative form and the convection velocity is considered the average of the three neighboring grid points. For further investigation of the finite difference scheme first, the first-order scheme in both non-conservative and conservative forms are studied. Then the second order scheme in both non-conservative and conservative forms for the KdV equation are considered. The equal grid size is taken into consideration in these schemes. The stability condition for the first-order scheme is determined by the convex combination method and this stability is discussed for the non-conservative form of the first-order scheme. Von Neumann stability analysis is presented for the non-conservative form of the second-order scheme.

(a) *First Order Scheme*

The first order scheme is obtained by performing the forward discretization of the time derivative, a backward discretization of the first order space derivative and second-order central difference in third order space derivative (this is considered the FTBSCS technique). Then the discrete form of KdV equation (1) reads as:

$$u_j^{n+1} = u_j^n - \frac{\mu\Delta t}{\Delta x} u_j^n \left(u_j^n - u_{j-1}^n\right) - \frac{\nu\Delta t}{2(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right). \qquad (3)$$

This is an explicit finite difference scheme for the KdV equation in non-conservative form. Now the equation (1) in conservative form is considered.

$$\frac{\partial u}{\partial t} + \mu \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) + \nu \frac{\partial^3 u}{\partial x^3} = 0 \qquad (4)$$

and the explicit finite difference scheme in conservative form by the same FTBSCS technique is formed. The equation (4) reads as:

$$u_j^{n+1} = u_j^n - \frac{\mu\Delta t}{2(\Delta x)}\left(\left(u_j^n\right)^2 - \left(u_{j-1}^n\right)^2\right) - \frac{\nu\Delta t}{2(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right). \quad (5)$$

This is an explicit finite difference scheme for KdV equation in conservative form.

(b) *Stability Condition of the First Order Scheme*

Here the stability condition of the first order scheme (non-conservative form) using convex combination (Eusha-Bin-Hafiz and Andallah 2016) is determined.

For convex combination consider

$\lambda_1 = \frac{\mu\Delta t}{\Delta x} \max_{n,j}\{u_j^n\} = \frac{\mu\Delta t}{\Delta x} \max_j \{u_j^0\}$ and

$\lambda_2 = \frac{\nu\Delta t}{2(\Delta x)^3}$ .

From Scheme (3)

$$u_j^{n+1} = u_j^n - \lambda_1\left(u_j^n - u_{j-1}^n\right) - \lambda_2\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right)$$

$$\Rightarrow u_j^{n+1} = (1-\lambda_1)u_j^n + (\lambda_1 - 2\lambda_2)u_{j-1}^n + 2\lambda_2 u_{j+1}^n - \lambda_2 u_{j+2}^n + \lambda_2 u_{j-2}^n$$

As it is known in convex combination, all the coefficients are positive and sum them is one. And also, Neumann Boundary Conditions $u_{j-2}^n = u_{j-1}^n$ and $u_{j+2}^n = u_{j+1}^n$ are used.

Therefore

$$u_j^{n+1} = (1-\lambda_1)u_j^n + (\lambda_1 - \lambda_2)u_{j-1}^n + \lambda_2 u_{j+1}^n$$

Since sum of all coefficients is one, then by the rule of convex combination,

$$0 \le (1-\lambda_1) \le 1\ ;\ 0 \le (\lambda_1 - \lambda_2) \le 1 \text{ and } 0 \le \lambda_2 \le 1$$

And hence the stability conditions are $\lambda_1 \le 1$ and $\lambda_1 \ge \lambda_2$, where $\lambda_1, \lambda_2 \ge 0$.

From stability condition two relations are obtained

$$\Delta t \le \frac{\Delta x}{\mu * \max_j \{u_j^0\}} \text{ and } \nu \le 2\mu(\Delta x)^2 * \max_j\{u_j^0\}$$

(c) *Second Order Scheme*

For the second order scheme, second order central difference in both time and space derivatives (CTCS technique) are performed. Then the KdV equation (1) reads as:

$$u_j^{n+1} = u_j^{n-1} - \frac{\mu\Delta t}{\Delta x}u_j^n\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{\nu\Delta t}{(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right). \quad (6)$$

This is second-order explicit central difference scheme for KdV equation in non-conservative form. Now the discrete form of the conservative equation (4) by CTCS technique is following:

$$u_j^{n+1} = u_j^{n-1} - \frac{\mu\Delta t}{2(\Delta x)}\left(\left(u_{j+1}^n\right)^2 - \left(u_{j-1}^n\right)^2\right) - \frac{\nu\Delta t}{(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right). \quad (7)$$

This is a second-order explicit central difference scheme for KdV equation in conservative form.

d) *Stability Condition of the Second Order Scheme*

Here the stability condition of the second order scheme (non-conservative form) is determined using Von Neumann stability analysis following (Kitavi 2013).

For Von Neumann stability analysis, putting $u_j^n = \xi^n e^{ikj\Delta x}$ in Scheme (6) the result is:

$$\xi^{n+1}e^{ikj\Delta x} = \xi^{n-1}e^{ikj\Delta x} - \frac{\mu\Delta t}{\Delta x}u_{max}\left(\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}\right) - \frac{\nu\Delta t}{(\Delta x)^3}\left[\xi^n e^{ik(j+2)\Delta x} - 2\xi^n e^{ik(j+1)\Delta x} + 2\xi^n e^{ik(j-1)\Delta x} - \xi^n e^{ik(j-2)\Delta x}\right]$$

where $u_j^n = \max_{n,j}\{u_j^n\} = \max_j \{u_j^0\} = u_{max}$ .

Canceling $u_j^n = \xi^n e^{ikj\Delta x}$ from both sides, using Euler's formula, and letting $k\Delta x = \omega$, then

$$\xi = \xi^{-1} - \frac{\mu\Delta t}{\Delta x} u_{max}(2i * \sin(\omega)) - \frac{\nu\Delta t}{(\Delta x)^3}[2i * \sin(2\omega) - 4i * \sin(\omega)]$$

Let

$$A = \frac{2\mu\Delta t}{\Delta x} u_{max}(\sin(\omega)) + \frac{2\nu\Delta t}{(\Delta x)^3}[\sin(2\omega) - 2\sin(\omega)] \quad (8)$$

Therefore $\xi = \xi^{-1} - iA$ is obtained, upon which multiplication by $\xi$,

$$\xi^2 + iA\xi - 1 = 0.$$

Now using quadratic formula

$\xi = \frac{\sqrt{4-A^2}}{2} - \frac{A}{2} i$ for $4 - A^2 \geq 0$, that is, $|A| \leq 2$ is obtained.

Consequently $|\xi| = \sqrt{\frac{4-A^2}{4} + \frac{A^2}{4}}$, which implies that $|\xi| = 1$ .

From (8)

$$A = \frac{2\mu\Delta t}{\Delta x} u_{max}(\sin(\omega)) + \frac{2\nu\Delta t}{(\Delta x)^3}[\sin(2\omega) - 2\sin(\omega)] .$$

To obtain maximum value which $A$ attains, let $y = \sin(2\omega) - 2\sin(\omega)$ and solve for the value of $\omega$ for which $\frac{dy}{d\omega} = 0$, that is,

$$\frac{dy}{d\omega} = 4cos^2(\omega) - 2\cos(\omega) - 2$$

Therefore, $2cos^2(\omega) - \cos(\omega) - 1 = 0$ ,

$\omega = 0$ or $\omega = \frac{2\pi}{3}$, since $\omega \in [0, \pi]$ .

Now when $\omega = 0$, $y = 0$ and also $A = 0$.

For $\omega = \frac{2\pi}{3}$, we have $y = -\frac{3\sqrt{3}}{2}$ and

$$|A| = \left|\frac{\sqrt{3}\mu\Delta t}{\Delta x} u_{max} - \frac{3\sqrt{3}\nu\Delta t}{(\Delta x)^3}\right| .$$

For stability $|A| \leq 2$ and so the stability region satisfy the inequality

$$\frac{\Delta t}{\Delta x} \leq \frac{2}{\sqrt{3}} * \frac{1}{\left|u_{max} * \mu - \frac{3\nu}{(\Delta x)^2}\right|} .$$

e) *Zabusky and Kruskal Scheme and Stability Condition*

Zabusky and Kruskal (ZK) scheme is a second-order explicit finite difference scheme. This scheme is derived by central difference approximations for both space and time. Then the equation (1) is as follows

$$u_j^{n+1} = u_j^n - \frac{\mu\Delta t}{3(\Delta x)}\left(u_{j+1}^n + u_j^n + u_{j-1}^n\right)\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{\nu\Delta t}{(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right) . \quad (9)$$

In fact, this scheme is a modification of the scheme (6), where the convection velocity $u_j^n$ is taken by $\frac{\left(u_{j+1}^n + u_j^n + u_{j-1}^n\right)}{3}$. This scheme is a three-level scheme and second-order accurate in time. The truncation error is of order $(o(\Delta t)^2 + o(\Delta x)^2)$. The linear stability condition for this scheme is following to (subsection d). The stability condition is as follows:

$$\frac{\Delta t}{\Delta x} \leq \frac{2}{\sqrt{3}} * \frac{1}{\left|u_{max} * \mu - \frac{3\nu}{(\Delta x)^2}\right|}$$

Where $u_{max}$ is the maximum value of $u$ depending on the amplitude of solitons.

**Verification of the Effect of Convection and Dispersion Terms of the KdV Equation**

In this section, the numerical solution is presented to understand the effect of nonlinear and dispersion terms for the scheme (6). These effects are presented for $\Delta t = 0.001$ and $\Delta x = 0.2$ and for different values of $\mu$ and $\nu$, the effect of the non-linear term and dispersion terms respectively at time, $t = 0.6$ is presented. In this case, space $-10$ to $10$ and time $0$ to 1 are considered. For fixed $\nu\,(= 1)$, $\mu$ $(= 6, 5, 4$ taking respectively) is changed, which represent the effect of convection term. Again, for fixed $\mu\,(= 6)$ we change in $\nu\,(= 1, 2, 3$ taking respectively), which is the effect of the dispersion term.

From Fig. 1 it is seen that the height is decreasing for increasing nonlinear coefficient, where considering $\mu = 4, 5, 6$ respectively for fixed $\nu = 1$, which shows the effect of non-linear term. From Fig. 2 it is observed that for fixed $\mu = 6$ and considering $\nu = 1, 2, 3$ (increasing) respectively, the width of the wave is spreading at the time, $t = 0.6$ which shows the effect of the dispersion term.

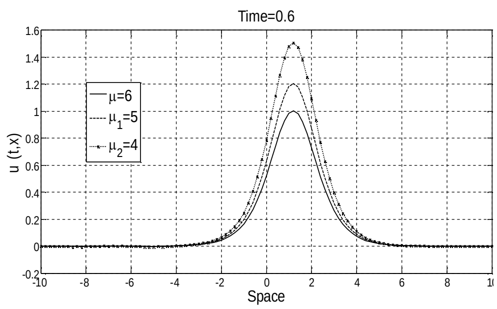


**Fig. 1.** Effect of non-linear term at time, $t = 0.6$.

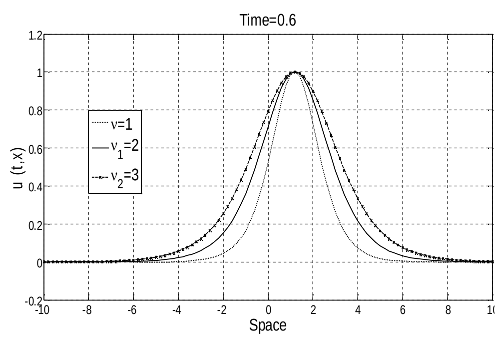


**Fig. 2.** Effect of dispersion term at time, $t = 0.6$.

**Numerical Presentations**

In this section, an error estimation of the explicit finite difference schemes is presented and the schemes are compared with each other. For error estimation $L_1$ norm is used, defined by

$$\|e\|_1 = \frac{\|u_e\, u_n\|_1}{\|u_e\|_1}$$

for all time where $u_e$ is the exact solution and $u_n$ is the numerical solution by the explicit finite difference scheme. Initial and boundary conditions are taken from the exact solution. As for zero boundary condition (Hirota 1971) of the exact solution at infinity, boundary value is approximately zero on the considered domain. Here errors are estimated for the different cases as - first-order scheme in non-conservative and conservative forms; second order scheme in non-conservative and conservative forms and Zabusky and Kruskal scheme. In exact solution two sets of date are considered: one is $\mu = 6, \nu = 1, c = 2, \xi_0 = 0$ and the other case is $\mu = 1, \nu = 1, c = 2, \xi_0 = 0$. For both two sets of date, the numerical solution of the KdV equation for $\mu = 6, \nu = 1$ and $\mu = 1, \nu = 1$are presented. These two cases are discussed unitedly. For numerical solution, taking $\Delta t = 0.0002$ and $\Delta x = 0.2000$ (for first order schemes) and $\Delta t = 0.0002$ and $\Delta x = 0.1000$ (for second order schemes and Zk scheme). And for error estimation, different sets of $\Delta t$ and $\Delta x$ are considered, and $\mu = 6$ and $\nu = 1$are considered. Graphical representations for various cases are given in (Figs. 3 - 16).

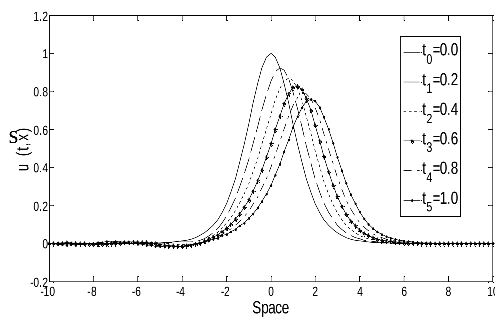


**Fig. 3.** Numerical solution of the first order non-conservative form for $\mu = 6, \nu = 1$.

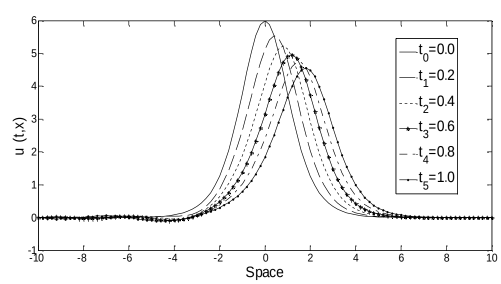


**Fig. 4.** Numerical solution of the first order non-conservative form for $\mu = 1, \nu = 1$.

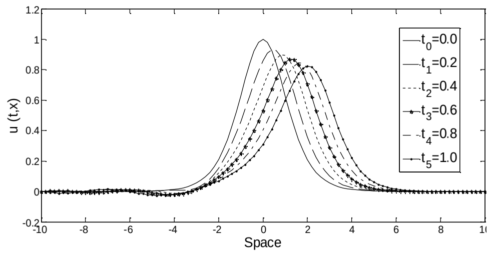


**Fig. 5.** Numerical solution of the first order conservative form for $\mu = 6, \nu = 1$.

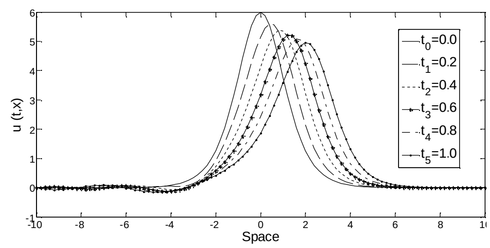


**Fig. 6.** Numerical solution of the first order conservative form for $\mu = 1, \nu = 1$.

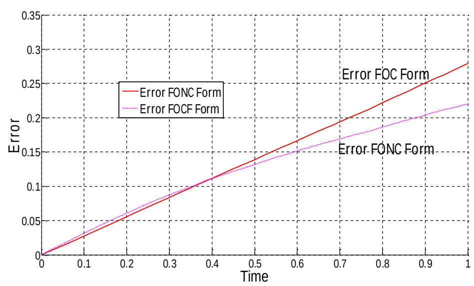


**Fig. 7.** Error comparing of the first order conservative form (FOC Form) and first order non-conservative form (FONC Form) for $\mu = 6, \nu = 1$.

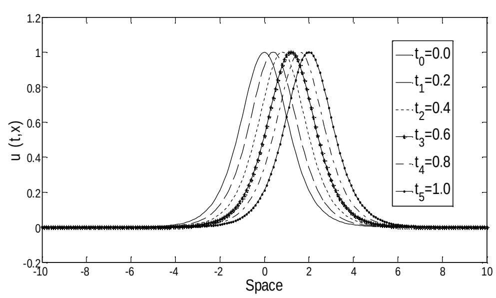


**Fig. 8.** Numerical solution of the second order non-conservative form for $\mu = 6, \nu = 1$.

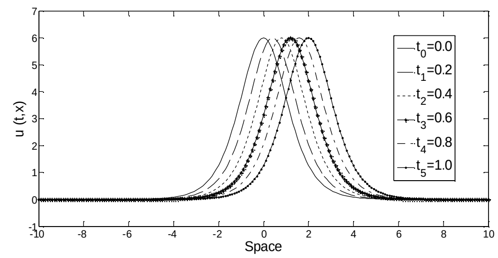


**Fig. 9.** Numerical solution of the second order non-conservative form for $\mu = 1, \nu = 1$.

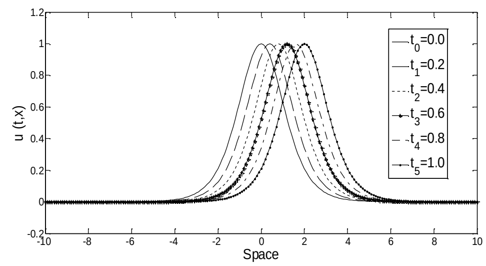


**Fig. 10.** Numerical solution of the second order conservative form for $\mu = 6, \nu = 1$.

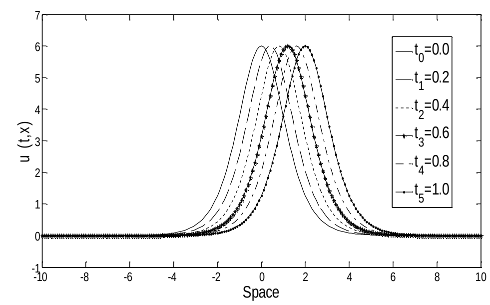


**Fig. 11.** Numerical solution of the second order conservative form for $\mu = 1, \nu = 1$.

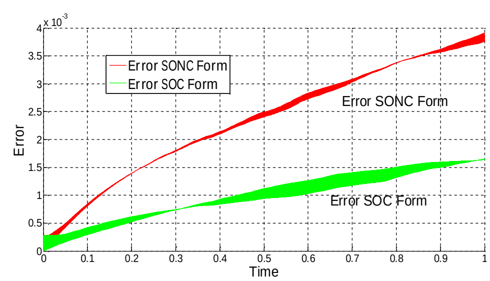


**Fig. 12.** Error comparing of the second order conservative form (SOC Form) and second order non-conservative form (SONC Form) for $\mu = 6, \nu = 1$.

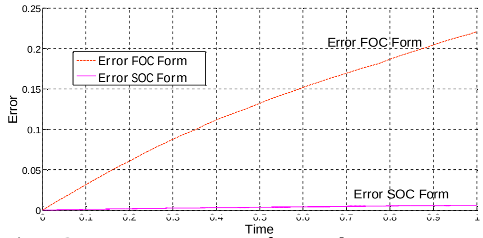


**Fig. 13.** Error comparing first order conservative form (FOC Form) and second order conservative form (SOC Form) for $\mu = 6, \nu = 1$.

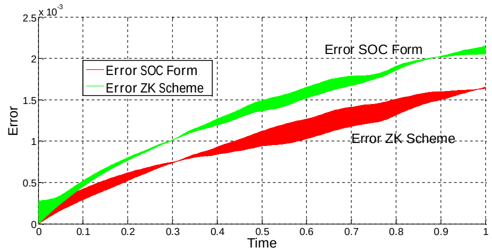


**Fig. 14.** Error comparing second order conservative form (SOC Form) and Zabusky and Kruskal (ZK Scheme) for $\mu = 6, \nu = 1$.

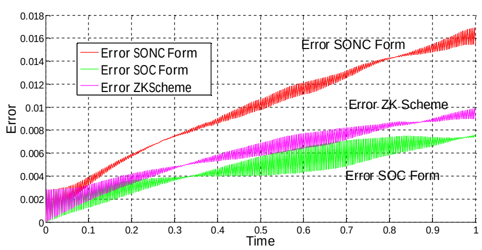


**Fig. 15.** Error comparing second order non-conservative (SONC Form), conservative forms (SOC Form) and Zabusky and Kruskal (ZK scheme) for $\mu = 6, \nu = 1$.

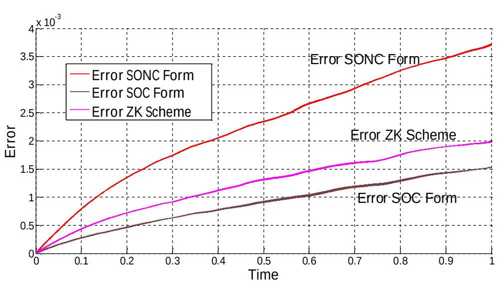


**Fig. 16.** Error comparing second order non-conservative (SONC Form), conservative forms (SOC Form) and Zabusky and Kruskal (ZK scheme) for $\mu = 6, \nu = 1$.

**Explanation of the Graphical Representations**

Figs. 3 and 5 represent the numerical solution of the first order non-conservative and conservative forms respectively, using $\mu = 6, \nu = 1$. Figs. 4 and 6 represent the numerical solution of the first order non-conservative and conservative forms respectively, using $\mu = 1, \nu = 1$. The figures show the amplitude of the waves decreasing for increasing time and width of the waves also spreading. Both are the effects of convection and dispersion terms respectively. Fig.7 represents error comparison of the first order conservative and non-conservative forms using $\Delta t = 0.0002$ and $\Delta x = 0.200$. Observing this figure, conservative form is better than non-conservative form. But the graphical representation is not so good, where the error is so high. Because the low order of the first order scheme which is killing the accuracy of the higher order $u_{xxx}$ approximation and the error is $o(\Delta x)$. In addition, first order scheme is unable to produce solitary waves.

Figs. 8 and 10 represent the numerical solution of the second order non-conservative and conservative forms respectively, using $\mu = 6, \nu = 1$. Figs. 9 and 11 represent numerical solution of the second order non-conservative and conservative forms respectively using $\mu = 1, \nu = 1$. Having glanced at four figures, waves propagate approximately same height and width, because the scheme is second order accuracy in both time and space. Moreover, comparing Figs. 3 and 8, solitary wave obtained by second order scheme.

Fig. 12 represents error comparison between second order conservative and non-conservative forms using $\Delta t = 0.0002$ and $\Delta x = 0.1000$. From Fig. 12, it is apparent that conservative form is much better than non-conservative form.

Fig. 13 represents error comparison of the first order conservative and second order conservative forms using $\Delta t = 0.0002$ and $\Delta x = 0.2000$ . Observing this figure, it can be said second conservative form is more suitable than the first conservative form.

Fig. 14 represents error comparison of the second order conservative and Zabusky and Kruskal scheme (ZK scheme) using $\Delta t = 0.0002$ and $\Delta x = 0.1000$. From Fig. 14, it is observed that second order conservative form is more accurate than ZK scheme.

Fig. 15 represents error comparison of the second order non-conservative, conservative forms and ZK Scheme using $\Delta t = 0.002$ and $\Delta x = 0.200$. Also, Fig. 16 represents error comparison of the second order non-conservative, conservative forms and ZK Scheme using $\Delta t = 0.00002$ and $\Delta x = 0.1000$. Observing these figures, it is apparent that second order conservative form is more selected scheme than the other schemes. From (Shahrill *et al.* 2015) it is observed that Zabusky and Kruskal scheme is more accurate scheme than the other schemes, where first-order scheme, Zabusky and Kruskal Scheme, Lax-Wendroff Scheme, Walkley Scheme are discussed. From above qualitative observation, it is seen that the error of the conservative form is less than ZK scheme, where both schemes are second-order schemes. In ZK scheme the average is taken for the convection velocity but no average is taken for the second-order conservative form. As a result, it can be said that the second order conservative scheme is more accurate than ZK scheme.

## CONCLUSION

In this paper, explicit finite difference schemes for the numerical solution of the KdV equation has been investigated. The stability condition for the first-order scheme using the convex combination method has been determined. Von Neumann stability analysis is performed to determine the stability condition for a second order scheme. The effect of convection and dispersion terms are being verified. The paper presents error estimations of the finite difference schemes and compared the schemes with each other. From all comparisons, it is observed that second-order conservative form is more accurate than other schemes because Zk scheme is non-conservative form, and in the ZK scheme the nonlinear term is taken into an average whereas no average is taken in conservative form. In the conservative form, flux on the nonlinear term has been used. The first-order scheme is not capable to produce the solitary wave. So from all error estimation, it can be said that the conservative scheme for KdV equation gives less error than the other scheme. Therefore, the second-order conservative form is more accurate than ZK non-conservative scheme.

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*(Received revised manuscript on 24 December 2018)*