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PROPERTIES OF SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

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ABSTRACT

In this paper, three notions of separation axioms in bitopological space are discussed. Some relations of topology and bitopology in such notions have been found. Further, that these notions are hereditary and topological property are proved.

Keywords: Topology, Bitopology, Separation of axiom, Hereditary

INTRODUCTION

The concept of bitopological spaces first introduced by Kelly (1962) provided a natural foundation for building new branches. After the introduction of the definition of a bitopological space by Kelly, a large number of topologists Reilly (1972), Lal (1978), Aarts (1990), Datta (1972), Patty (1967), Hyung (1979), Suganya (2015), Selvanayaki (2011) turned their attention to the generalization of different concepts of a single topological space. Shanin (1943) first define T_0 space in topological space. Kandil (1991, 1995) introduced the concept of fuzzy bitopological space. After then Hossain (2017) introduced the concept of pairwise T_0bitopological space. Some relationship and their property of given such notions in bitopological space are established.

Notations

Through this paper X, Y will be a non empty set $\mathcal{S}, \mathcal{T}, \mathcal{W}, \mathcal{Z}$ the topology on $X. (X, \mathcal{S}, \mathcal{T})$ and $(Y, \mathcal{W}, \mathcal{Z})$ be bitopological spaces. U, V are open sets and its elements are x, y, x_1, x_2, y_1, y_2.

Preliminaries

Definition 2.1. Let x be a non empty set. A class $\mathcal{T}$ of subsets of X is a topology on X iff $\mathcal{T}$ satisfies the following axioms

(a) X and ϕ belong to $\mathcal{T}$.

(b) The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

(c) The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The member of $\mathcal{T}$ are then called $\mathcal{T}$ open sets or simply open sets and X together with $\mathcal{T}$ .Hence the pair $(X, \mathcal{T})$ is called a topological space (Lipschutz 1965).

Definition 2.2. A space X on which are defined two topologies $\mathcal{S}$ and $\mathcal{T}$ is called a bitopological space and denoted by $(X, \mathcal{S}, \mathcal{T})$ (Kelly 1962).

Definition 2.3. Let A be a non empty subset of a topological space $(X, \mathcal{T})$. The class $\mathcal{T}_A$ all intersections of A with $\mathcal{T}$ open subsets of X is a topology on A, it is called the relative topology on A or the relativization of $\mathcal{T}$ to A, and the topological space $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$ (Lipschutz 1965).

Definition 2.4. A mapping $f: (X, \mathcal{S}, \mathcal{T}) \longrightarrow (Y, \mathcal{W}, \mathcal{Z})$ is called P-continuous (respectively P-open, P-closed) if the induced mappings $f: (X, \mathcal{S}) \longrightarrow (Y, \mathcal{W})$ and $f: (X, \mathcal{T}) \longrightarrow (Y, \mathcal{Z})$ are continuous (respectively open, closed) (Kelly 1962).

Definition 2.5. A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is called T_0 space if $\forall x, y \in X$ with $x \neq y$ then $\exists U \in \mathcal{S} \cup \mathcal{T}$such that $x \in U, y \notin U$ or $x \notin U, y \in U$ (Murdeshwar and Naimpally 1966).

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Definition 2.6. A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is called T_1 space if $\forall x, y \in X$ with $x \neq y$ then $\exists U \in \mathcal{S}$ and $V \in \mathcal{T}$ such that $x \in U, y \notin U$ and $x \notin U, y \in U$ (Reilly 1972).

Definition 2.7. A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is called T_2 space if $\forall x, y \in X$ with $x \neq y$ then $\exists U \in \mathcal{S}, V \in \mathcal{T}$ such that $x \in U, y \in V$ and $U \cap V = \phi$ (Kelly 1962).

PROPERTIES OF T_0 , T_1 AND T_2 BITOPOLOGICAL SPACES

In this section some of the properties of separation axioms in bitopological space and some of its features are discussed.

Theorem 3.1. To show that T_0 is a hereditary property.

Proof: Suppose $(X, \mathcal{S}, \mathcal{T})$ is a T_0 space and $A \subseteq X$, has to be proved that $(A, \mathcal{S}_A, \mathcal{T}_A)$ is also T_0 space. let, $x, y \in A$ with $x \neq y$, then $x, y \in X$ with $x \neq y$. Since $(X, \mathcal{S}, \mathcal{T})$ is T_0 space then $\exists U \in \mathcal{S} \cup \mathcal{T}$ such that $x \in U, y \notin U$ or $x \notin U, y \in U$.

Then, $U \in \mathcal{S} \cup \mathcal{T}$

$\Rightarrow U \in \mathcal{S}$ or $U \in \mathcal{T}$

$\Rightarrow U \cap A \in \mathcal{S}_A$ or $U \cap A \in \mathcal{T}_A$

$\Rightarrow U \cap A \in \mathcal{S}_A \cup \mathcal{T}_A$

Again since $x, y \in A$ then $x \in U \cap A, y \notin U \cap A$ or $x \notin U \cap A, y \in U \cap A$.

Hence $(A, \mathcal{S}_A, \mathcal{T}_A)$ is also T_0 space.

Theorem 3.2. To show that T_1 is a hereditary property.

Proof: Suppose $(X, \mathcal{S}, \mathcal{T})$is a T_1 space and $A \subseteq X$, that $(A, \mathcal{S}_A, \mathcal{T}_A)$ is also T_1 space has to be proved. Let, $x, y \in A$ with $x \neq y$ then $x, y \in X$ with $x \neq y$. Since $(X, \mathcal{S}, \mathcal{T})$ is T_1 space then $\exists U \in \mathcal{S}$ and $V \in \mathcal{T}$ such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Then, $U \in \mathcal{S}$ and $V \in \mathcal{T} \Rightarrow U \cap A \in \mathcal{S}_A$ and $V \cap A \in \mathcal{T}_A$

Again since $x, y \in A$ then $x \in U \cap A, y \notin U \cap A$ and $x \notin V \cap A, y \in V \cap A$.

Hence $(A, \mathcal{S}_A, \mathcal{T}_A)$ is also T_1 space.

Theorem 3.3. To show that T_2 is a hereditary property.

Proof: Suppose $(X, \mathcal{S}, \mathcal{T})$ is a T_2 space and $A \subseteq X$, $(A, \mathcal{S}_A, \mathcal{T}_A)$ is also T_2 space has to be proved. Let, $x, y \in A$ with $x \neq y$ then $x, y \in X$ with $\neq y$. Since $(X, \mathcal{S}, \mathcal{T})$ is T_2 space then $\exists U \in \mathcal{S}$ and $V \in \mathcal{T}$ such that $x \in U, y \in V$ and $U \cap V = \phi$.

Then, $U \in \mathcal{S}$ and $V \in \mathcal{T}$

$\Rightarrow U \cap A \in \mathcal{S}_A$ and $V \cap A \in \mathcal{T}_A$

Again since $x, y \in A$ then $x \in U \cap A, y \in V \cap A$ and $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \phi \cap A = \phi$.

Hence $(A, \mathcal{S}_A, \mathcal{T}_A)$ is also T_2 space.

Theorem 3.4. To show that T_0 space is a topological property.

Proof: Let $f: (X, \mathcal{S}, \mathcal{T}) \longrightarrow (Y, \mathcal{W}, \mathcal{Z})$ be a homeomorphism and $(X, \mathcal{S}, \mathcal{T})$ is T_0 space. We shall prove that $(Y, \mathcal{W}, \mathcal{Z})$ is also T_0 space.

Let, $y_1, y_2 \in Y$ with $y_1 \neq y_2$, since f is onto then $\exists x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. Again since f is one, one with $y_1 \neq y_2 \Rightarrow f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$. Further since $(X, \mathcal{S}, \mathcal{T})$ is T_0 space and $x_1, x_2 \in X$, with $x_1 \neq x_2$ then $\exists U \in \mathcal{S} \cup \mathcal{T}$ such that $x_1 \in U, x_2 \notin U$ or $x_1 \notin U, x_2 \in U$.

Let, $x_1 \in U, x_2 \notin U$. Then, $U \in \mathcal{S} \cup \mathcal{T} \Rightarrow f(U) \in f(\mathcal{S} \cup \mathcal{T})$ as f is open and continuous then $f(U) \in f(\mathcal{S}) \cup f(\mathcal{T}) \in \mathcal{W} \cup \mathcal{Z}$

Also $x_1 \in U \Rightarrow f(x_1) \in f(U) \Rightarrow y_1 \in f(U)$ and $x_2 \notin U \Rightarrow f(x_2) \notin f(U) \Rightarrow y_2 \notin f(U)$. i.e for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $f(U) \in \mathcal{W} \cup \mathcal{Z}$ is obtained such that $y_1 \in f(U), y_2 \notin f(U)$.

$\therefore (Y, \mathcal{W}, \mathcal{Z})$ is a T_0 space. i.e every homeomorphic image of T_0 space is also T_0 space.

Hence, T_0 is a topological property.

Theorem 3.5. To show that T_1 is a topological property.

Proof: Let $f:(X,\mathcal{S},\mathcal{T}) \longrightarrow (Y,\mathcal{W},\mathcal{Z})$ be a homeomorphism and $(X,\mathcal{S},\mathcal{T})$ is T_1 space. That $(Y,\mathcal{W},\mathcal{Z})$ is also T_1 space. Let, $y_1, y_2 \in Y$ with $y_1 \neq y_2$, since f is onto then $\exists x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. Again since f is one, one with $y_1 \neq y_2 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow x_1 \neq x_2$. Then $x_1, x_2 \in X$with $x_1 \neq x_2$. Again since $(X,\mathcal{S},\mathcal{T})$ is T_1 space then $\exists U \in \mathcal{S}$ and $V \in \mathcal{T}$ such that $x_1 \in U, x_2 \notin U$ and $x_1 \notin V, x_2 \in V$. Further since f is open then $f(U) \in \mathcal{W}$ and $f(V) \in \mathcal{Z}$. Also $x_1 \in U \Longrightarrow y_1 = f(x_1) \in f(U)$, $x_2 \notin U \Longrightarrow y_2 = f(x_2) \notin f(U)$ and $x_1 \notin V \Longrightarrow y_1 = f(x_1) \notin V, x_2 \in V \Longrightarrow y_2 = f(x_2) \in f(V)$. *i.e*, for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $f(U) \in \mathcal{W}$ and $f(V) \in \mathcal{Z}$ are obtained such that $y_1 \in f(U), y_2 \notin f(U)$ and $y_1 \notin f(V), y_2 \in f(V)$.

$\therefore (Y,\mathcal{W},\mathcal{Z})$ is a T_1 space. *i.e*, every homeomorphic image of a T_1 space is also a T_1 space. Hence, T_1 is a topological property.

Theorem 3.6. To show that T_2 is a topological property.

Proof: Let $f:(X,\mathcal{S},\mathcal{T}) \longrightarrow (Y,\mathcal{W},\mathcal{Z})$ be a homeomorphism and $(X,\mathcal{S},\mathcal{T})$ is T_2 space. That $(Y,\mathcal{W},\mathcal{Z})$ is also T_2 space has to be proved. Let, $y_1, y_2 \in Y$ with $y_1 \neq y_2$, since f is onto then $\exists x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. Again since f is one, one with $y_1 \neq y_2 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow x_1 \neq x_2$. Now $x_1, x_2 \in X$ with $x_1 \neq x_2$. Again since $(X,\mathcal{S},\mathcal{T})$ is T_2 space then $\exists U \in \mathcal{S}$ and $V \in \mathcal{T}$ such that $x_1 \in U, x_2 \in V$ and $U \cap V = \phi$. Further since f is open then $f(U) \in \mathcal{W}$ and $f(V) \in \mathcal{Z}$.

Let $f(U) \cap f(V) \neq \phi$ then $\exists\, z \in X$ such that $z \in f(U) \cap f(V)$

$\Longrightarrow z \in f(U)$ and $z \in f(V)$

$\Longrightarrow \exists\, p_1 \in U$ and $p_2 \in V$

such that $z = f(p_1)$ and $z = f(p_2)$ with $f(p_1) = f(p_2)$

$\Longrightarrow p_1 = p_2$ as f is one one.

$\Longrightarrow p_1 \in U$ and $p_1 \in V$

$\Longrightarrow p_1 \in U \cap V \Longrightarrow U \cap V \neq \phi$

which is a contradiction, the fact is that $U \cap V = \phi \Longrightarrow f(U) \cap f(V) = \phi$.

$\therefore$ For any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $f(U) \in \mathcal{W}$ and $f(V) \in \mathcal{Z}$ is obtained such that $y_1 \in f(U), y_2 \in f(V)$ and $f(U) \cap f(V) = \phi$.

i.e, $(Y,\mathcal{W},\mathcal{Z})$ is a T_2 space .

$\therefore$ Every homeomorphic image of a T_2 space is also a T_2 space. Hence, T_2 is a topological property.

Theorem 3.7. Let $(X,\mathcal{T})$ be a T_0 space and $(X,\mathcal{S})$ be any topological space then $(X,\mathcal{S},\mathcal{T})$ is a T_0 space.

Proof: Let $(X,\mathcal{T})$ be a T_0 space then for any $x, y \in X$ with $x \neq y$ then $\exists U \in \mathcal{T}$

such that $x \in U, y \notin U$ or $x \notin U, y \in U$.

Since $U \in \mathcal{T} \Longrightarrow U \in \mathcal{T} \cup \mathcal{S}$.

From above it is clear that $(X,\mathcal{S},\mathcal{T})$ is a T_0 space.

Theorem 3.8. Let $(X,\mathcal{S},\mathcal{T})$ be a T_0 space then need not be $(X,\mathcal{T})$ and $(X,\mathcal{S})$ are both T_0 space.

Proof: Let $X = \mathbb{N}$. Let $\mathcal{T}$ and $\mathcal{S}$ are two topology on X. Where $\mathcal{S} = \{X, \phi\}$ and $\mathcal{T}$ is generated by A_i which contain ϕ and X.

When $A_i = \{1,2,3,4,5,6,7, \dots \dots i\}$, $i \in \mathbb{N}$

then it is clear that $(X,\mathcal{S},\mathcal{T})$ is a bitopological space and also it is T_0 space. Also $(X,\mathcal{T})$ is a T_0 space but $(X,\mathcal{S})$ is not T_0 space.

Theorem 3.9. If $(X,\mathcal{T})$ is T_1 space and $(X,\mathcal{S})$ is any topological space then $(X,\mathcal{S},\mathcal{T})$ need not be T_1 space.

Proof: Let, $X = \{a, b, c\}$ and

$$\mathcal{T} = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$$

$$\mathcal{S} = \{X, \phi\}$$

Then $(X, \mathcal{T}), (X, \mathcal{S})$ be two topological space and $(X, \mathcal{S}, \mathcal{T})$ be a bitopological space. It is clear that $(X, \mathcal{T})$ is a T_1 space but $(X, \mathcal{S}, \mathcal{T})$ is not a T_1 space since for any $a, b \in X$ with $a \neq b$. No open set $U \in \mathcal{S}$ such that $a \in U, b \notin U$ and $a \notin U, b \in U$ can be found.

Theorem 3.10. If $(X, \mathcal{S}, \mathcal{T})$ is a T_1 space then need not be $(X, \mathcal{S})$ and $(X, \mathcal{T})$ be T_1 space.

Proof: Let, $X = \{a, b, c\}$

$$\mathcal{S} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$$

$$\mathcal{T} = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$$

The bitopological space $(X, \mathcal{S}, \mathcal{T})$ is a T_1 space but $(X, \mathcal{S})$ is not a T_1 space because $b, c \in X$ with $b \neq c$ but there does not exist any open set $U \in \mathcal{S}$ such that $c \in U$ with $b \notin U$. Again $(X, \mathcal{T})$ is not a T_1 space because $a, b \in X$ with $a \neq b$ but there does not exist any open set $U \in \mathcal{T}$ such that $a \in U$ with $b \notin U$.

Theorem 3.11. If (X, d) is T_2 space and $(X, \mathcal{I})$ is any topological space then $(X, d, \mathcal{I})$ need not be T_2 space.

Proof: $X = \mathbb{R}$ and d be a maetrix in X then the matrix space (X, d) is always a T_2 space. Again let $\mathcal{I}$ be the indiscrete topology on X then $(X, \mathcal{I})$ be a topological space.

$\therefore (X, d, \mathcal{I})$ is a bitopological space.

For if any $x, y \in X$ with $x \neq y$ then $\nexists U \in d$ and $V \in \mathcal{I}$ such that $x \in U, y \in V$and $U \cap V = \phi$. Since $\mathcal{I}$ has only non empty open set in X are X and ϕ.

Hence, $(X, d, \mathcal{I})$ is not a T_2 space.

Theorem 3.12. If $(X, \mathcal{S}, \mathcal{T})$ is T_2 space then need not be $(X, \mathcal{S})$ and $(X, \mathcal{T})$ are T_2 space.

Proof: Let, $X = \{a, b\}$, $\mathcal{S} = \{X, \phi, \{a\}\}$ and $\mathcal{T} = \{X, \phi, \{b\}\}$. Hence, $(X, \mathcal{S}, \mathcal{T})$ is T_2 space but $(X, \mathcal{S})$ is not T_2 space because $a, b \in X$ with $a \neq b$ but $\nexists U, V \in \mathcal{S}$ such that, $a \in U, b \in V$ and $U \cap V = \phi$.

$(X, \mathcal{T})$ is also not T_2 space because $a, b \in X$ with $a \neq b$ but $\nexists U, V \in \mathcal{T}$ such that, $a \in U, b \in V$ and $U \cap V = \phi$.

CONCLUSION

The main result of this paper is introducing some concepts of separation axioms in bitopological spaces which satisfy topological and hereditary properties. Also that topological space does not imply bitopological space and vice versa in given such notions is also obseved.

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