J. Bangladesh Acad. Sci., Vol. 43, No. 2, 205-209, 2019
DOI: https://doi.org/10.3329/jbas.v43i2.45742

THE TOWER OF HANOI PROBLEM WITH EVILDOER DISCS

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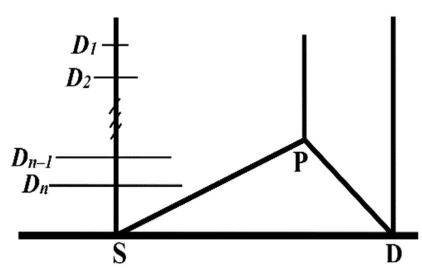
ABSTRACT

This paper deals with a variant of the classical Tower of Hanoi problem with $n\ (\geq 1)$ discs, of which r discs are evildoers, each of which can be placed directly on top of a smaller disc any number of times. Denoting by $E(n, r)$ the minimum number of moves required to solve the new variant, is given a scheme find the optimality equation satisfied by $E(n, r)$. An explicit form of $E(n, r)$ is then obtained.

Keywords: Tower of Hanoi, divine rule, evildoer

INTRODUCTION

The Tower of Hanoi problem, in general form, is as follows : Given are $n\ (\geq 1)$ discs $D_1, D_2, \ldots, D_n$ of different sizes, and three pegs, S, P and D. At the start of the game, the discs rest on the *source peg*, S, in a *tower* in increasing order, from top to bottom. The objective is to shift the tower to the *destination peg*, D, in minimum number of moves, where each move can shift only the topmost disc from one peg to another, under the "divine rule" that no disc is ever placed on a smaller one. It is well-known that the total number of moves required to solve the Tower of Hanoi (ToH) problem with n discs is $2^n - 1$. The Tower of Hanoi appears as a problem of mathematical recreation in Ball (1892) and Gardner (1956). In fact, the Tower of Hanoi puzzle with 8 discs was invented by Lucas back in 1883. For a historical account of the problem, the reader is referred to Hinz (2013, Section 0.3). The initial state of the classical Tower of Hanoi problem is shown in the figure below.



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Over the past decades, the Tower of Hanoi problem has seen many generalizations, some of which have been reviewed by Majumdar (2012, 2013) and Hinz *et al.* (2013).

Chen *et al.* (2007) have introduced a new variant of the Tower of Hanoi problem with $n \geq 1$ discs, which allows $r\ (\geq 1)$ violations of the "divine rule". In the new variant, the problem is to shift the tower of n discs from the peg S to the peg D in minimum number of moves, where for (at most) r moves, some disc may be placed directly on top of a smaller one.

Let the minimum number of moves required to solve the above problem be $S(n, r)$. Then, the following theorem, is found due to Chen *et al.* (2007) (in a slightly modified form).

Theorem 1.1 **:** For any $n \geq 1$, $r \geq 1$,

$$S(n,r) = \begin{cases} 2n-1, & \text{if } 1 \leq n \leq r+2 \\ 4n-2r-5, & \text{if } r+2 \leq n \leq 2r+3 \\ 2^{n-2r}+6r-1, & \text{if } n \geq 2r+3 \end{cases}$$

The following open problem has been proposed by Chen *et al.* (2007).

To H with r Evildoers : In the classical Tower of Hanoi problem, any r (of the $n\ (\geq 1)$) discs are evildoers, where each *evildoer* can be placed (directly) on top of a smaller disc any number of times.

Let $E(n, r)$ be the minimum number of moves required to solve the classical Tower of Hanoi (ToH) problem with $n \geq 1$ discs and r $(1 \leq r < n)$ number of evildoer(s). Clearly, in the (n, r)-problem (with n discs, r of which are evildoers),

$$E(n, r) < S(n, r).$$

This paper finds an explicit form of $E(n, r)$. This is done in Proposition 2.3 in Section 2. Some remarks are made in the final section, Section 3.

THE PROBLEM AND THE SOLUTION

The problem that is considered is as follows : There are three pegs, S, P and D, and n (≥ 1) discs of varying sizes. Initially, the discs rest on the source peg in a tower, in increasing order from top to bottom. Of the n discs, r (≥ 1) discs are evildoers, each of which can be placed directly on top of a smaller one any number of times. The problem is to shift the tower from the peg S to the destination peg D, in minimum number of moves, such that each move shifts only the topmost disc from one peg to another.

Let the minimum number of moves required to solve the above problem be $E(n, r)$.

To find an expression for E(n, r), it is first noted that n and r must satisfy the condition that $n \geq 2r + 4$. Also, it is recalled that, when $n \geq 2r+4$, under the optimal scheme (given in Theorem 1.1), the topmost $n - 2r - 1$ discs D_1, D_2, ..., D_{n-2r-1} require $2(2^{n-2r-1} - 1)$ number of moves, and the next 2r largest discs require 6r moves. Thus, any of these 2r discs is a possible candidate for an evildoer (though disc D_{n-1} can safely be deleted from the list); moreover, in order to have an effect of an evildoer, (at most) $n - 2r - 2$ (smallest) discs can be placed in a tower (in (at most) $2^{n-2r-2} - 1$ number of moves).

Of particular interest is E(n, 1). Chen *et al.* (2007), based on computer search, report that,

$$E(n, 1) = S(n, 1) \text{ for } 1 \leq n \leq 7,$$

but $E(8, 1) = 57$ when the disc D_6 is chosen as the evildoer. In a recent paper, Majumdar *et al.* (2019) have given explicitly the scheme for $E(8,1)$ when the disc D_6 is an evildoer.

The following result can now be stated, whose proof is simple and is omitted here.

***Lemma 2.1* :** For any k (≥ 2) fixed, let $k = 2m + j$ for some integers $m \geq 1$ and $0 \leq j \leq 1$. Let

$$f(x; k) = 2^x + 2^{k-x},$$

$0 \leq x \leq k$ is an integer.

Then, $f(x; k)$ attains its minimum at $x = m$. The following results are obtained.

***Lemma 2.2* :** For the $(n, 1)$-problem, if the disc D_i is an evildoer, then the disc D_{i+1} is an evildoer for the $(n + 1, 1)$-problem.

***Proof* :** Lemma 1 in Majumdar *et al.* (2019) can be considered.

***Proposition 2.1* :** For the $(n, 1)$-problem with $n \geq 8$, the disc D_{n-2} is the (unique) evildoer.

***Proof* :** Corollary 1 in Majumdar *et al.* (2019) can be considered.

Proposition 2.2 generalizes Lemma 2.2 to the (n, r)-problem. For its proof, result below is needed.

***Lemma 2.3* :** For any k $(> r + 1 \geq 2)$ fixed, let

$$f(x_1, x_2, \ldots, x_r; k) = 2^{x_1} + 2^{x_2} + \ldots + 2^{x_r} + 2^{k-(x_1+x_2+\ldots+x_r)},$$

where $0 < x_i < k$ for all $1 \leq i \leq r$. Then, $f(x_1, x_2, \ldots, x_r; k)$ attains its minimum when

$$x_i = \frac{k}{r+1} \quad \text{for all } 1 \leq i \leq r,$$

and the minimum value is $(r + 1)\, 2^{\frac{k}{r+1}}$.

***Proof* :** is simple and is left as an exercise.

Lemma 2.3 has the following consequence.

***Corollary 2.1* :** Let $k = (r + 1)m + j$ for some integers $m \geq 1$ and $0 \leq j \leq r$. Then,

$$F(x_1, x_2, \ldots, x_r; k) = 2^{x_1} + 2^{x_2} + \ldots + 2^{x_r} + 2^{k-(x_1+x_2+\ldots+x_r)},$$

(where $0 < x_i < k$ are integers for all $1 \leq i \leq r$) attains its minimum at

$$x_i = m + 1,\ 1 \leq i \leq j;\ x_{j+1} = x_{j+2} = x_r = m,$$

with the minimum value

$$F(k) = (r + j + 1)2^m.$$

It may be mentioned here that, if $j = 0$ in Corollary 2.1, then $F(x_1, x_2, \ldots, x_r; k)$ attains its minimum at a unique point, otherwise, it has multiple solutions.

Proposition 2.2 **:** For the (n, r)-problem, if the r discs $D_{l_1}, D_{l_2}, \ldots, D_{l_r}$ (counted from the top of the tower) are the evildoers, then the discs $D_{l_1+1}, D_{l_2+1}, \ldots, D_{l_r+1}$ (counted from the top of the tower) are the evildoers for the $(n + 1, r)$-problem.

Proof **:** To find the total number of moves required to transfer the n discs from the peg S to the peg D in the (n, r)-problem, it is sufficient to consider the number of moves involved in the dismantling of the tower of the topmost $n–1$ discs on the peg S. The movements of the topmost k discs, can be first considered which requires $2^k–1$ moves, where $k < n – (r + 1)$.

At some stage of the dismantling process, the tower of k smallest discs is divided into $r+1$ subtowers, say, $T_1, T_2, \ldots, T_{r+1}$, of sizes $m_1, m_2, \ldots, m_{r+1}$ respectively, where

$$m_1 + m_2 + \ldots + m_{r+1} = k.$$

At the last stage of the dismantling process, the $r+1$ subtowers are moved to the peg D, using the r evildoers, where the subtower T_k requires $2^{m_k} – 1$ $(1 \le k \le r+1)$ number of moves. Let the next $n – k –1$ largest discs require N moves. Thus, the total number of moves required to dismantle the tower of the topmost $n – 1$ discs is

$$2^k + 2^{m_1} + 2^{m_2} + \ldots + 2^{m_r} + 2^{m_{r+1}} – (r+2) + N.$$

Now, by assumption,

$$2\left[2^k + 2^{m_1} + 2^{m_2} + \ldots + 2^{m_r} + 2^{m_{r+1}} – (r+2) + N\right] + 1$$
$$< 2^{n – 2r} + 6r – 1$$
$$= S(n, r)$$

that is,

$$2^{k+1} + 2^{m_1+1} + 2^{m_2+1} + \ldots + 2^{m_r+1} + 2^{m_{r+1}+1} + 2N$$
$$< 2^{n – 2r} + 8r + 2. \qquad (1)$$

Now, $n – k – 1 \ge r + 1$, and each of the r evildoers requires (at least) 2 moves, once to free them, and then again during the process of the movements of the $r + 1$ subtowers $T_1, T_2, \ldots, T_{r+1}$. Thus, $N \ge 2r + 1$. Also, $2^{m_k} \ge 2$ for all $1 \le k \le r + 1$. Therefore,

$$2^{k+1} + 2^{m_1+1} + 2^{m_2+1} + \ldots + 2^{m_r+1} + 2^{m_{r+1}+1} + 2N$$
$$\ge 2^{k+1} + 2^{m_1} + 2^{m_2} + \ldots + 2^{m_r} + 2^{m_{r+1}+1} + N + 4r + 1,$$

which, by virtue of (1), gives

$$2^{k+1} + 2^{m_1} + 2^{m_2} + \ldots + 2^{m_r} + 2^{m_{r+1}+1} + N$$
$$< 2^{n – 2r} + 4r + 1. \qquad (2)$$

Attention is now confined to the $(n+1, r)$-problem. At the first stage of the dismantling of the topmost n discs, the tower of the topmost $k+1$ discs is transferred, which is subsequently divided into $r+1$ subtowers. Without loss of generality, the sizes of these $r + 1$ subtowers may be taken as $m_1, m_2, \ldots, m_r, m_{r+1}+1$ respectively. At the last stage, these subtowers are moved, using the evildoers $D_{l_1+1}, D_{l_2+1}, \ldots, D_{l_r+1}$. The total number of moves required to solve the $(n + 1, r)$-problem is

$$2\left[2^{k+1} + 2^{m_1} + 2^{m_2} + \ldots + 2^{m_r} + 2^{m_{r+1}+1} – (r+2) + N\right] + 1$$
$$< 2^{n + 1 – 2r} + 6r – 1 \text{ (by the inequality (2))}$$
$$= S(n + 1, r).$$

All these complete the proof of the proposition.

Now the main result of this paper is proved.

Proposition 2.3 **:** Let

$$n = (r + 1)m + j$$

for some integers $m \ge 1$, $1 \le j \le r$.

Then, there exists an integer $\boldsymbol{N}$ such that for $n \ge \boldsymbol{N}$,

$$E(n, r) = 2^{n – 2r – 1} + (r + j + 1)2^{m – 1} + 10r – 1.$$

Proof **:** To find $E(n, r)$, the scheme below is followed.

Step 1 : Move the tower of the topmost $n –2r–2$ discs, $D_1, D_2, \ldots, D_{n – 2r – 2}$, from the peg S to the peg D, in a tower, in $2^{n – 2r – 2} – 1$ moves.

Step 2 : With the next $2r$ discs on S, form r pairs of discs (D_i, D_{i+1}). For each pair (D_i, D_{i+1}), the disc D_i is first moved to the peg P, next D_{i+1} is shifted to the peg D (violating the "divine rule"), and then the disc D_i is moved again (from the peg P) to the peg D.

This step requires $3r$ moves, and the "divine rule" is violated r times. It is noted that the r discs $D_{n-2r}, D_{n-2r+2}, \ldots, D_{n-2}$ each violates the "divine rule". Also it is noted that, the r pairs of discs $(D_{n-3}, D_{n-2}), (D_{n-5}, D_{n-4}), \ldots, (D_{n-2r-1}, D_{n-2r})$ rest on the peg D, in this order, on the tower of $n-2r-2$ smallest discs.

Step 3 : Move the disc D_{n-1} (from the peg S) to the peg P.

Step 4 : Transfer the topmost $2r$ discs, now resting on the peg D, one-by-one, the r discs $D_{n-3}, D_{n-5}, \ldots, D_{n-2r-1}$ (in this order) to the peg P, and the r evildoers $D_{n-2}, D_{n-4}, \ldots, D_{n-2r}$ (in this order) to the peg S.

This step involves $2r$ moves.

After Step 4, the tower of discs $D_1, D_2, \ldots, D_{n-2r-2}$ on the peg D are obtained, which is now divided into $r+1$ subtowers, $T_1, T_2, \ldots, T_{r+1}$, of sizes $m_1, m_2, \ldots, m_{r+1}$ respectively (according to the criterion of Corollary 2.1).

Step 5 : Move the $r+1$ subtowers $T_1, T_2, \ldots, T_{r+1}$, using the r evildoers (on the peg S), one-by-one, to the peg P. This is done as follows : Move the tower T_k, followed by the transfer of the evildoer $D_{n-2r+2(k-1)}$, for $1 \le k \le r$, and then transfer the largest tower T_{r+1} on top of the evildoer D_{n-2}. Note that, the tower T_k needs $2^{m_k}-1$ number of moves.

Thus, the number of moves involved in this step is $F(n-2r-2)-1$.

The total number of moves required in Step 1 – Step 5 above is:

$$(2^{n-2r-5}-1)+3r+1+2r+[F(n-2r-2)-1]$$
$$=2^{n-2r-1}+F(n-2r-2)+5r-1.$$

After Step 5, the peg S contains the largest disc D_n only, and the peg D is empty.

Step 6 : Move the disc D_n to the peg D. After transferring the largest disc D_n to the destination peg, Steps 1–5 are repeated in reverse order (with appropriate choices of the pegs) to complete the tower on the destination peg D. This is affected in the steps below.

Step 7 : Move the subtowers $T_{r+1}, T_r, \ldots, T_1$

(in this order, on the peg P) to the peg S and the evildoers to the peg D.

After Step 7, the tower of the topmost smallest $n-2r-2$ discs on the peg S, and the r evildoer discs $D_{n-2r}, D_{n-2r+2}, \ldots, D_{n-2}$ (in this order) on the peg D on top of the largest disc D_n are obtained.

Step 8 : Place the evildoer D_{n-2r} (from the peg D) to the peg S on top of the tower of the $n-2r-2$ discs, followed by the transfer of the disc D_{n-2r-1} (from the peg P) on top of D_{n-2r}. This process is continued to free the disc D_{n-1}.

After Step 8, the pair of discs (D_i, D_{i+1}), $i=n-3, n-5, \ldots, n-2r-1$ (in this order on the peg S, and the free disc D_{n-1} on the peg P are obtained. This step requires $2r$ number of moves.

Step 9 : Move the disc D_{n-1} to the peg D.

Step 10 : With the pair of discs (D_i, D_{i+1}) on the peg S, the disc D_i is first moved to the peg P, then the disc D_{i+1} is transferred to the peg D, and finally, the disc D_i is moved to the peg D. This step requires $3r$ number of moves.

Step 11 : Move the tower of $n-2r-2$ discs on the peg S to the peg D.

The number of moves required under this scheme is

$$2[2^{n-2r-2}+F(n-2r-2)+5r-1]+1$$
$$=2^{n-2r-1}+2F(n-2r-2)+10r-1.$$

Therefore,

$$E(n,r)=2^{n-2r-1}+2F(n-2r-2)+10r-1. \quad (2.1)$$

Now, let

$$n=(r+1)m+j$$

for some integers $m \ge 1$, $1 \le j \le r$.

Then, $n-2r-2=(r+1)(m-2)+j$, and hence, by Corollary 2.1,

$$F(n-2r-2)=(r+j+1)2^{m-2}.$$

Plugging in the above expression of $F(n-2r-2)$ in (3.1) and simplifying, the desired expression of $E(n,r)$ is obtaind.

It now remains to be shown that, for any $r\ (\geq 1)$ fixed, there is an integer $\boldsymbol{N}$ such that, for $n \geq N$,

$$E(n,r) < S(n,r),$$

that is,

$$2^{n-2r-1}+(r+j+1)2^{m-1}+10r-1$$
$$< 2^{n-2r}+6r-1$$

that is

$$(r+j+1)2^{m-1}+4r$$
$$< 2^{n-2r-1} = 2^{(m-2)r+m+j-1}$$

that is,

$$2^{m-2}\,[2^{(m-2)r+j}-(r+j+1)] > 2r. \qquad (2.2)$$

Clearly, in order that the inequality (2.2) holds, $m \geq 2$. Then, for any $r \geq 1$ fixed, there are integers m and j (with $0 \leq j \leq r$) such that (2.2) holds. With the minimum such m and j,

$$\boldsymbol{N}=(r+1)m+j. \qquad (2.3)$$

Thus, the proposition is established.

In course of proving Proposition 2.3, the following is also proved.

Corollary 2.2 **:** For the (n, r)-problem, the r discs, $D_{n-2}, D_{n-4}, \ldots, D_{n-2r}$, are the evildoers.

CONCLUDING REMARKS

This paper gives a scheme to solve the Tower of Hanoi problem with $r\ (\geq 1)$ number of evildoers. Denoting by $E(n, r)$ the minimum number of moves required to solve the problem, an explicit form of $E(n,r)$ is given in Proposition 2.3. It is interesting to find that $E(n, r)$ has a closed-form simple expression.

From Theorem 1.1, it is seen that, if r relaxations are allowed, then the number of moves is reduced approximately by the multiplicative factor of 2^{2r}. And from Proposition 2.3, it is seen that the r evildoers reduce the number of moves further.

It may be recalled that, in deriving $S(n, r)$ in Theorem 1.1, the r discs $D_{n-1}, D_{n-3}, \ldots, D_{n-2r+1}$ each violates the "divine rule". This result may be compared with that given in Corollary 2.2.

Proposition 2.3 proves that, for any $r\ (\geq 1)$ fixed, there is a number $\boldsymbol{N}$ such that (2.2) is satisfied. Since no explicit form is available, $\boldsymbol{N}$ is to be found by trial-and-error.

REFERENCES

Ball, W.W.R. 1892. Mathematical Recreations and Essays. MacMillan Book Co. Ltd., London (11th Revised Edition by H.S.M. Coxeter, 1939).

Chen, X., Tian, B. and Wang, L. 2007. Santa Claus' Towers of Hanoi. *Graphs and Combinatorics*, **23**(Supplement): 153 – 167.

Claus, N. (= Lucas, E.) 1883. La Tour d Hanoi. *Jeu de Calcul. Sci. Nat.*, **1**(8): 127 – 128.

Gardner, M. 1956. Mathematical Puzzles and Diversions. Penguin Book, U.K.

Hinz, A.M., S. Klavzar, U. Milutinovic and C. Petre. 2013. The Tower of Hanoi – Myths and Maths. Springer, NY.

Majumdar, A.A.K. 2012. The Classical Tower of Hanoi Problem and Its Generalizations Vol. **1** : Multi-Peg Generalization. Lambert Academic Publishing, U.S.A.

Majumdar, A.A.K. 2013. The Classical Tower of Hanoi Problem and Its Generalizations, Vol. 2 : Other Generalizations. Lambert Academic Publishing, U.S.A.

Majumdar, A.A.K. and Islam, S.M.S. 2019. The Tower of Brahma with One Evildoer : The Life-Time of the World. *Journal of Science and Technology*, **17**: 111 – 118.

(Received revised manuscript on 30 November 2019)