- Short Communication

J. Bangladesh Acad. Sci., Vol. 43, No. 2, 211-212, 2019
DOI: https://doi.org/10.3329/jbas.v43i2.45743

THE REVE'S PUZZLE WITH SINGLE RELAXATION

A.A.K. MAJUMDAR
Ritsumeikan Asia-Pacific University, Beppu-shi 874–8577, Japan

ABSTRACT

This paper considers a variant of the Reve's puzzle with n (≥ 1) discs which allows at most one violation of the "divine rule". Denoting by $S_4(n)$ the minimum number of moves required to solve the new variant, an explicit form of $S_4(n)$ is given.

Key words: Tower of Hanoi puzzle, divine rule, sinner's tower, Reve's puzzle

The Tower of Hanoi puzzle, due to Lucas (1883), is as follows : Given are n (≥ 1) discs d_1, d_2, …, d_n of different sizes, and three pegs, S, P and D. At the start of the game, the discs rest on the *source peg*, S, in a tower in increasing order, from top to bottom. The objective is to shift this tower to the *destination peg*, D, in minimum number of moves, where each move can shift only the topmost disc from one peg to another, under the "divine rule", which demands that, during the transfer process of the discs, no disc could ever be placed on top of a smaller one. One immediate generalization of the Tower of Hanoi puzzle is the Reve's puzzle with four pegs, due to Dudeney (1958). Let $M_4(n)$ be the minimum number of moves required to solve the Reve's puzzle. Then, $M_4(n)$ satisfies the following recurrence relation :

$$M_4(n) = \min_{1 \leq k \leq n-1} \left\{ 2M_4(k) + 2^{n-k} - 1 \right\}, \; n \geq 4, \qquad (1)$$

with

$$M_4(0) = 0; \; M_4(n) = 2n - 1 \text{ for all } 1 \leq n \leq 3. \qquad (2)$$

For details, the reader is referred to Majumdar (1994). The following result is needed.

Lemma 1 **:** For any $n \geq 1$,

$$M_4(n+1) - M_4(n) > 4 \text{ for all } n \geq 6.$$

Chen, Tian and Wang (2007) have introduced a new variant of the Tower of Hanoi problem which allows r (≥ 1) violations of the "divine rule". Let $S_3(n)$ denote the minimum number of moves required to solve the Tower of Hanoi problem with n discs and single relaxation of the "divine rule". Then, the following lemma is obtained.

*Corresponding author: <aakmajumdar@gmail.com>.

Lemma 2 **:** For any $n \geq 1$,

$$S_3(n) = \begin{cases} 2n - 1, & \text{if } 1 \leq n \leq 3 \\ 2^{n-2} + 5, & \text{if } n \geq 4 \end{cases}$$

The problem is considered follows : Given are four pegs, S, P_1, P_2 and D, and a tower of n (≥ 1) discs (of varying sizes) on the source peg S, in small-on-large ordering. The objective is to move this tower to the peg D, using the auxiliary pegs P_1 and P_2, in minimum number of moves, where each move shifts the topmost disc from one peg to another, and for only one move, some disc may be placed directly on top of a smaller one. Let $S_4(n)$ be the minimum number of moves required to solve the above problem. An explicit form of $S_4(n)$ is given below.

Theorem 1 **:** For $n \geq 1$,

$$S_4(n) = \begin{cases} 2n - 1, & \text{if } 1 \leq n \leq 4 \\ M_4(n-1) + 2, & \text{if } 4 \leq n \leq 7 \\ M_4(n-2) + 6, & \text{if } n \geq 7 \end{cases}$$

Proof **:** The proof is trivial if $1 \leq n \leq 4$.
So, let $n \geq 5$. In this case, the following two possible schemes are considered.

Case 1 :

1. move the topmost $k\ (\geq 1)$ discs, $d_1, d_2, d_3 \ldots, d_k$, from the peg S to the peg P_1, say, using the four pegs available, in (minimum) $M_4(k)$ moves.

2. shift the remaining $n-k$ discs on S to the peg D, using the three pegs available, in (minimum) $S_3(n-k)$ moves.

3. finally, transfer the tower of k discs from P_1 to D, again in (minimum) $M_4(k)$ moves, to complete the tower on the destination peg D.

The total number of moves involved is $2M_4(k) + S_3(n-k) = 2M_4(k) + 2^{n-k-2}+5$, and k is to be determined such that the total number of moves is minimum. Thus, in this scheme, the minimum number of moves required is:

$$\min_{1 \leq k \leq n-1} \left\{2M_4(k) + 2^{n-2-k}\right\} + 5 \qquad (3)$$
$$= M_4(n-2) + 6,$$

where the last expression follows from the equation (1). It may be recalled here that, $M_4(n-2)$ is attained at a point k satisfying $k \leq n-3 < n-1$. Thus, extending the range of k (to $n-1$) in (3), the value of $M_4(n-2)$ is not affected.

Case 2 :

1. move the topmost $k\ (\geq 1)$ discs from S to the peg P_1, say, using the four pegs available, in (minimum) $M_4(k)$ moves,

2. shift the disc d_{k+1} from the peg S to the peg P_1, violating the "divine rule",

3. transfer the tower of $n-k-1$ discs from S to D, (using the three available pegs) in (minimum) $2^{n-k-1}-1$ moves,
4. move the disc d_{k+1} from P_1 to D,
5. finally, shift the tower (of k discs) on P_1 to D.

The minimum number of moves required under this scheme is

$$\min_{1 \leq k \leq n-1} 2\left\{M_4(k)+1\right\} + 2^{n-1-k} - 1$$
$$= M_4(n-1)+2.$$
(4)

Since, by Lemma 1,

$M_4(n-2) + 6 < M_4(n-1) + 2$ for all $n \geq 8$,

the result follows. It is noted that, in (4) above, $M_4(n-1)$ is attained at a point k satisfying $k \leq n-2 < n-1$.

Hence, the theorem is proved.

REFERENCES

Chen, X., Tian, B. and Wang, L. 2007. Santa Claus' Towers of Hanoi. *Graphs and Combinatorics*, **23** (Supplement): 153 – 167.

Claus, N. (= Lucas, E.) 1883. La Tour d Hanoi. *Jeu de Calcul. Sci. Nat.*, **1**(8): 127 – 128.

Dudeney, H.E. 1958. *The Canterbury Puzzles*. Dover, 4th Edition.

Majumdar, A.A.K. 1994. The Generalized Four-Peg Tower of Hanoi Problem. *Optimization*, **29**: 349 – 360.

(Received revised manuscript on 24 December 2019)