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Research Article

Approximate solution of nonlinear differential system with time variation

Rezaul Karim, Pinakee Dey and Saikh Shahjahan Miah*

Department of Mathematics, Faculty of Science, Mawlana Bhashani Science and Technology University, Tangail, Bangladesh

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ABSTRACT

This paper develops a reliable algorithm based on the general Struble's technique and extended KBM method for solving nonlinear differential systems. Moreover, we find a solution based on the KBM and general Struble's technique of nonlinear autonomous systems with vary slowly with time, which is more powerful than the existing perturbation method. Finally, results are discussed, primarily to enrich the physical prospects, and shown graphically by utilizing MATHEMATICA and MATLAB software

Introduction

The most well-known standard methods for constructing the approximate analytical solutions to the nonlinear oscillators are the perturbation techniques. For the first time, Krylov and Bogoliubov (1974) introduced a new perturbation method in order to discuss the transient state solution of the equation presented by

$$\ddot{x}+\omega^2 x=\varepsilon F(x,\dot{x}) \tag{1.1}$$

where ε *is* a small parameter, ω is the angular frequency and x is the derivative of x with respect to t and so on. But in particular cases, it gives those periodic solutions obtained by Poincare's (Dey et al., 2008) method is a well-known perturbation method for determin-ingperiodic solutions of nonlinear ordinary differential equations with small nonlinearities.

When $\varepsilon=0$, equation (1.1) reduces to linear equation and solution of which is

$$x=a\cos(\omega t+\varphi) \tag{1.2}$$

where a and φ are arbitrary constant to be determined from the initial conditions.

Now in order to determine an approximate solution of the equation (1.1) for small but different from zero, Krylov and Bogoliubov (1974) assumed that the solution is still given by (1.2) with the derivative of the from

$$\dot{x}=-a\omega\sin(t+\varphi) \tag{1.3}$$

Therefore, the first approximation solution of Krylov and Bogliubov (1974) method is

$$\dot{a}=-\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi}\begin{bmatrix}F(\cos\psi,-a\sin\psi)\\ \sin\psi\end{bmatrix}d\psi \tag{1.4}$$

*Corresponding author:<skhshahjahan@gmail.com>

and

$$\dot{a} = -\frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi}\begin{bmatrix} F(\cos\psi, -a\sin\psi) \\ \sin\psi \end{bmatrix} d\psi$$

where a and φ are independent of time. The equation (1.4) are the differential equations of the first approximation in the form in which they are original; the method was developed by (Roy and Alam, 2004) for obtaining a periodic solution of a second-order nonlinear differential equation. The asymptotic method of Krylov-Bogoliubov-Mitroplshkii (KBM) [1.1-1.3] is a particularly convenient and extensively used method to study nonlinear differential systems with small nonlinearities. Through it is restricted to differential equations of the type (1.1), this method has been extened in plasma physics, theory of oscillations and control theory. Kruskal (1962) has extended this method to solve the equation of type

$$\ddot{x} = F(x, \dot{x}, \psi) \qquad (1.5)$$

The solutions of these fully nonlinear equations are based on the recurrent relations and are given in the forms of power series of the small parameter ϵ . Cap (1974) has investigated some nonlinear systems of the type

$$\ddot{x} + \omega^2 f(x) = \epsilon\, F(x, \dot{x}) \qquad (1.6)$$

by using elliptic functions in the sense of the Krylov and Bogoliubov (1974) method.

Later this technique has been amplified and justified mathematically by Bogoliubov and Mitropolkii (1961) and extended to a non-stationary vibration by Mitropokii (1964). They assumed the solution of the nonlinear differential equation (1.1) in the form

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \cdots + \in^n u_n(a,\psi) + 0(\in^{n+1}) \qquad (1.7)$$

where u_k (k=1,2, ..., n) are periodic functions of ψ with period 2^{π} and a, b and ψ are a function of time t, defined by

$$\dot{a} = -\varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1})$$

$$\dot{\psi} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \cdots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}) \qquad (1.8)$$

After replacing a and ψ by the function defined in equation (1.8), is a solution of (1.1). The function A_k and B_k generate the arbitrariness in the definitions of the functions u_k . To remove this arbitrariness, the following additional conditions are imposed.

$$\int_0^{2\pi}[u_1 k(a,\psi)]\cos\psi d\psi = 0$$

$$\int_0^{2\pi}[u_1 k(a,\psi)]\sin\psi d\psi = 0 \qquad (1.9)$$

These conditions guarantee the absence of secular terms in all successive approximations. Dey et al. (2008) extended the technique for damped forced nonlinear systems with varying coefficients. Later, Alam and Satter (1997) have presented a unified KBM method for solving third-order nonlinear systems. Alam (2002) has also presented a unified Krylov-Bogoliubov-Mitropolskii method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. Struble (1961) has developed a technique for treating weakly nonlinear oscillatory systems such as those governed by $\ddot{x} + \omega^2 x = \varepsilon F(x, \dot{x}, t)$ (1.10)

He expressed the asymptotic solution of this equation for small ε in the form

$$x = a\cos(\omega t - \theta) + \sum_{n=1}^{N} \varepsilon^n x_n(t) + O(\varepsilon^{y+1})$$

where a and θ are slowing varying functions of time.

Krylov and Bogoliubov (1947) originally developed a perturbation method to obtain an approximate solution of a second-order nonlinear differential system. The method was amplified and justified by Bogoliubov and Mitropolskii (1961), Mitropolskii (1964) has extended the method to nonlinear differential system with slowly varying coefficients. Following the extended KBM method, Bojadziev and Edwards (1981) and Arya and Bodadziev (1980) studied some damped oscillatory and purely non-oscillatory systems with slowly varying coefficients. Murty (1971) presented a unified KBM method for both under-damped and over-damped system with constant coefficients. Alam (2002) presented a unified formula to obtain a general solution of an *n*-th order ordinary differential equation with constant and slowly varying coefficients. Roy and Alam (2004) found an asymptotic solution of a differential system in which the coefficient changes in an exponential order of slowly varying time. Dev et al. (2008) has presented an extended KBM method for under-damped, damped and over-damped vibrating systems in which the coefficients change slowly and periodically with time. Recently, Alam et al. (2006) have developed the general Struble's techniques for several damping effects. This paper aims to find a solution based on the KBM and general Struble's nonlinear autonomous systems technique, which varies slowly with time, which is more powerful than the existing perturbation method and measures, better results, for strong nonlinearities.

Method of First Approximation of Krylov and Bogoliubov

We consider a method of finding an approximate solution of a nonlinear differential equation having the form

$$\ddot{y} + y = \varepsilon F(y, \dot{y}) \qquad (2.1)$$

where ε is a very small parameter, if $\varepsilon = 0$. Then equation (2.1) becomes

$$\ddot{y} + y = 0 \qquad (2.2)$$

The solution of equation (2.2) may be written as,

$$y = a(t)\cos(t + \varphi) \qquad (2.3)$$

where a and φ are arbitrary constant to be determined from the initial conditions.

Now equation (2.3) with respect to *t*, we get,

$$\dot{y} = -a(t)\sin(t + \varphi) \qquad (2.4)$$

where a and φ are functions of rather than being constants. If $\varepsilon \neq 0$ but it is sufficiently small. We can assume (2.1) has a solution of the form of equation (2.3) with derivative at the form of equation (2.4) provided, that is equation (2.3) becomes,

$$y = a(t)\cos(t + \varphi(t)) \qquad (2.5)$$

and

$$\dot{y} = -a(t)\sin(t + \varphi(t)) \qquad (2.6)$$

where a and φ are functions of *t*.

Differentiating (2.5) we get,

$$\ddot{y} = \ddot{a}\cos(t + \varphi(t)) - a\sin(t + \varphi(t))(1 + \varphi) \quad (2.7)$$

In order to make the form $\dot{y}$ given by equation (2.6) we must require

$$\dot{a}\cos(t + \omega(t) - \sin(t + \varphi(t) = 0 \qquad (2.8)$$

Again, differentiating (2.6) we get,

$$\ddot{y} = \ddot{a}\sin(t+\varphi(t) - \dot{a})\cos(t+\varphi(t))$$
$$= \ddot{a}\sin(t+\varphi(t))\cos(t+\varphi(t)) \quad (2.9)$$
$$- a\cos(t+\varphi(t))$$

Now using (2.9), (2.5) & (2.6) and let $t+\varphi=\psi$, we get from (2.1)

$$\dot{a}\sin\psi - a\cos\psi - a\ddot{\varphi}\cos\psi + a\cos\psi = \varepsilon \quad (2.10)$$

Now, equation (2.8) & (2.10) we get,

$$\dot{a}\cos\psi(t) - \dot{a}(t)\sin\psi(t) = 0 \quad (2.11)$$

and

$$\dot{a}\sin\psi(t) + a(t)\cos\psi(t)$$
$$= \varepsilon F[a(t)\cos\psi, -a(t)\sin\psi] \quad (2.12)$$

Multiplying (2.11) by $\cos\psi(t)$ and (2.12) by $\sin\psi(t)$ then adding we get

$$\dot{a} = -\frac{\varepsilon}{a} F[a(t)\cos\psi, -a(t)\sin\psi]\sin\psi(t)$$

(2.13)

Multiplying (2.11) by $\sin\psi(t)$ and (2.12) by $\cos\psi(t)$ then subtracting we get,

$$\dot{\varphi} = -\frac{\varepsilon}{a(t)}[a(t)\cos\psi, -a(t)\sin\psi\cos\psi(t)] \quad (2.14)$$

These are the exact equations for the function a and φ when the solution is of the form given by equation (2.5) and (2.6).

Now we apply the first approximation of Krylov and Bogliubov Equation

$$F[a(t)\cos\psi, -a(t)\sin\psi]\sin\psi(t)$$

and

$$F[a(t)\cos\psi, -a(t)\sin\psi]\cos(t)$$

in Fourier Series, we obtain

$$F\sin\psi = K_0(a) + \sum_{n=1}^{\infty}[K_n(a)\cos n\psi + L_n\sin n\psi]$$

and

$$F\cos\psi = P_0(a) + \sum_{n=1}^{\infty}[P_n(a)\cos n\psi + Q_n\sin n\psi]$$

where,

$$K_n(a) = \frac{1}{\pi}\int_0^{2\pi} F[a(t)\cos\psi, -a(t)\sin\psi]\sin\psi(t)\cos n\psi d\psi$$

$$P_0(a) = \frac{1}{\pi}\int_0^{2\pi} F[a(t)\cos\psi, -a(t)\sin\psi]\cos\psi(t) d\psi$$

$$P_n(a) = \frac{1}{\pi}\int_0^{2\pi} F[a(t)\cos\psi, -a(t)\sin\psi]\cos n\psi d\psi$$

$$L_n(a) = \frac{1}{\pi}\int_0^{2\pi} F[a(t)\cos\psi, -a(t)\sin\psi]\sin\psi(t)\sin n\psi d\psi$$

$$Q_n(a) = \frac{1}{\pi}\int_0^{2\pi} F[a(t)\cos\psi, -a(t)\sin\psi]\cos\psi(t)\sin n\psi d\psi$$

Equation (2.13) and (2.14) can be written as,

$$a = -\varepsilon K_0(a) - \sum_{n=0}^{\infty}[K_n(a)\cos n\psi + L_n\sin(n\psi)]$$

$$\varphi = -\frac{\varepsilon}{a(t)}\left(P_0(a) + \sum_{1}^{\infty}[P_n(a)\cos n\psi + Q_n\sin(n\psi)]\right)$$

The first approximation of Krylov –Bogliubov Consists of neglecting all terms on the R.H.S except first that is

$$\dot{a} = -\varepsilon K_0(a)$$
$$= -\frac{1}{2\pi}\int_0^{2\pi}(F[a(t)\cos\psi, -a(t)\sin\psi])\sin\psi d\psi \quad (2.15)$$

and

$$\dot{\varphi} = -\frac{\varepsilon}{a(t)} P_0(a)$$
$$= -\frac{\varepsilon}{2\pi}\int_0^{2\pi}(F[a(t)\cos\psi, -a(t)\sin\psi])\cos\psi d\psi \quad (2.16)$$

A justification for this procedure is as follows. We note that the right-hand sides are periodic with respect to the variable ψ with a period to 2π. Also $\dot{a} = O(\varepsilon)$ and $\dot{\varphi} = O(\varepsilon)$.

Example

Consider the damped linear oscillator

$$\ddot{y} + y + \varepsilon\dot{y} = 0 \tag{2.17}$$

For 1st approximation, we know

$F = -\dot{y}$

We have,

$$\psi = -\frac{\varepsilon}{2\pi a}\int_0^{2\pi} F(-a\sin\psi)\cos\psi d\psi$$

$$= -\frac{\varepsilon}{2\pi a}\int_0^{2\pi} a\sin\psi\cos\psi d\psi \tag{2.18}$$

Now, let $\sin\psi = z \Rightarrow \cos\psi d\psi = dz$

when $\psi = 0$ then z=0 and $\psi = 2\pi$ then z = 0

From equation (2.18), we have

$$\dot{\psi} = -\frac{\varepsilon}{2\pi}\int_0^{\infty} zdz = 0$$

and

$$B_1 = -\frac{\varepsilon}{2\pi a}\int_0^{\infty} F(-a\sin\psi)\cos\psi d\psi = 0$$

$$= -\frac{1}{2\pi a}\int_0^{2\pi} a\sin\psi\cos\psi d\psi \tag{2.19}$$

Now, let

$\sin\psi = z$ or $\cos\psi d\psi = dx$

when $\psi = 0$ then $z = 0$ and

when $\psi = 2\pi$ then $z = 0$

From equation (2.19), we have

$$\dot{\psi} = -\frac{1}{2\pi}\int_0^{2\pi} zdz = 0$$

$$\dot{a} = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} F(-a\sin\psi)\sin\psi d\psi$$

$$= -\frac{\varepsilon a}{2\pi}\int_0^{2\pi}(1-\cos 2\pi)d\psi$$

$$= \frac{\varepsilon a}{4\pi}\left[\pi - \frac{1}{2}\sin 2\pi\right]_0^{2\pi} = -\frac{1}{2}\varepsilon a$$

Therefore, $A_1 = \frac{1}{\varepsilon}\dot{a} = -\frac{a}{2}$.

In this case, the function,

$F(a\cos\psi, -a\sin\psi)\sin\psi(t)$

and

$F(a\cos\psi, -a\sin\psi)a\cos\psi$

and the corresponding Fourier Coefficients are

$$g_1 = \frac{1}{\pi}\int_0^{2\pi}(a\sin\psi)\cos\psi d\psi$$

$$= \frac{a}{\pi}\int_0^{0} zdz = 0$$

$$g_2 = \frac{1}{\pi}\int_0^{2\pi}(a\sin\psi)\cos 2\psi d\psi$$

$$= \frac{a}{\pi}\int_0^{2\pi}\left(\sin\psi - 2\sin^2\psi\right)d\psi$$

$$= \frac{a}{\pi}\int_0^{2\pi}\left(-\frac{1}{2}\sin\psi + \frac{1}{2}\sin 3\psi\right)d\psi = 0$$

Similarly, $g_i(a) = 0,\ i = 3,4,5,\cdots$

and

$$h_1 = \frac{1}{\pi}\int_0^{2\pi}(a\sin\psi)\sin\psi d\psi$$

$$= \frac{a}{2\pi}\int_0^{2\pi}\left(1-\cos 2\psi\right)d\psi = a$$

$$h_2 = \frac{1}{\pi}\int_0^{2\pi}(a\sin\psi)\sin 2\psi d\psi \tag{2.20}$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(2a\sin\psi\cos\psi\right)\cos 2\psi d\psi$$

Now, let $\sin\psi = z \Rightarrow \cos\psi d\psi = dz$

when $\psi = 0$ then $z = 0$ and

when $\psi = 2\pi$ then $z = 0$

From equation (2.20), we have

$$h_2 = \frac{2a}{\pi}\int_0^{2\pi} \sin^2\psi \cos\psi d\psi$$

$$= \frac{a}{\pi}\int_0^0 z^2 dz = 0$$

Similarly, $h_3 = h_4 = h_5 = 0$,U

$h_i(a) = 0,\ i = 2,3,4,\cdots$

Putting this in the Equation

$$U_1(a,\psi) = g_0(a) + \sum_{n=2}^{\infty}\left[g_n(a)\cos n\psi + h_n(a)\sin n\psi\right]/\left(1-n^2\right)$$

We get, $U_1(a,\psi) = 0$

Again,

$$A_2(a) = -\frac{1}{2}\left(2A_1B_1 + A_1\frac{dB_1}{da}a\right)$$
$$-\frac{1}{2\pi}\int_0^{2\pi}\left[U_1(a,\psi)F_1 y(a\cos\psi,-a\sin\psi)\right.$$
$$\left.+A_1\cos\psi - aB_1\sin\psi + \frac{\partial U_1}{\partial\psi}F_1 y(a\cos\psi,-a\sin\psi)\right]\sin\psi d\psi$$

$$= \frac{a}{4\pi}\int_0^{2\pi} a\sin\psi\cos\psi\sin\psi d\psi = \frac{a}{4\pi}\int_0^{2\pi}\sin^2\psi\cos\psi d\psi = 0$$

$$B_2(a) = \frac{1}{8} + \frac{a}{4\pi}\int_0^{2\pi}\sin\psi\cos^2\psi d\psi = \frac{1}{8} \quad (2.21)$$

This is the second approximation from which we have $Y = \cos\psi$ and Integrating both sides we get, $a = Ae^{-\varepsilon t/2}$ and $\psi = 1-\varepsilon^2/8$. Integrating (2.21) we get,

$$\psi(t) = (1-\varepsilon^2/8)t + \psi_0$$

where A and ψ_0 arbitrary constant. Hence the solution in the second approximation is given by

$$y = Ae^{-\varepsilon t/2}\cos(1-\varepsilon^2/8)t + \psi_0$$

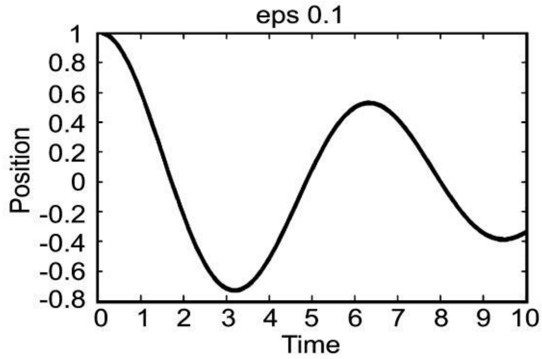

Fig. 1. Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) = 1, y'(0) = 0, \epsilon = 0.1$].

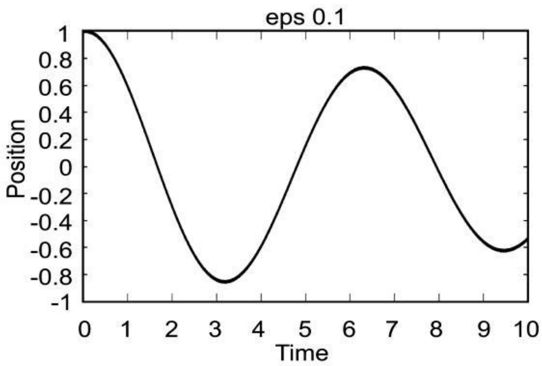

Fig. 2. Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) =$ 1, $y'(0) = 0, \epsilon = 0.2$].

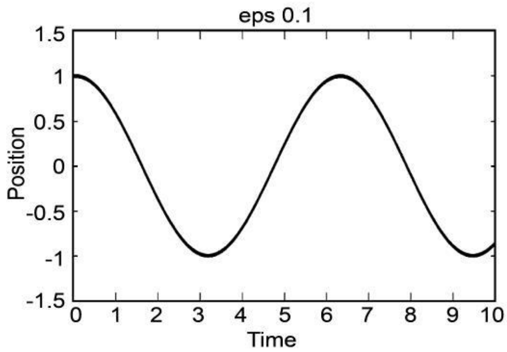

Fig. 3. Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) = 1,\ y'(0) = 0, \epsilon = 0.3$].

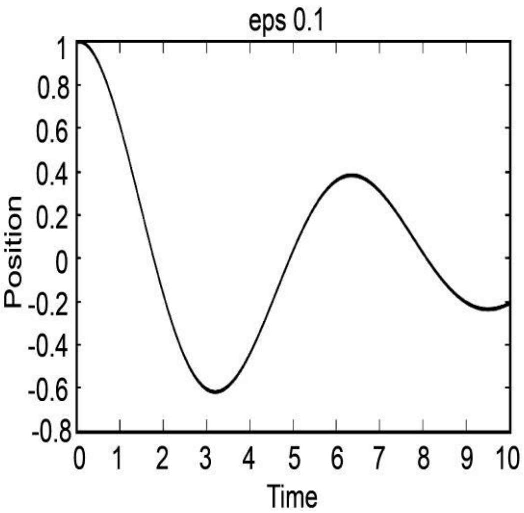

Fig. 4. Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) = 1, y'(0) = 0, \epsilon = 0.4$].

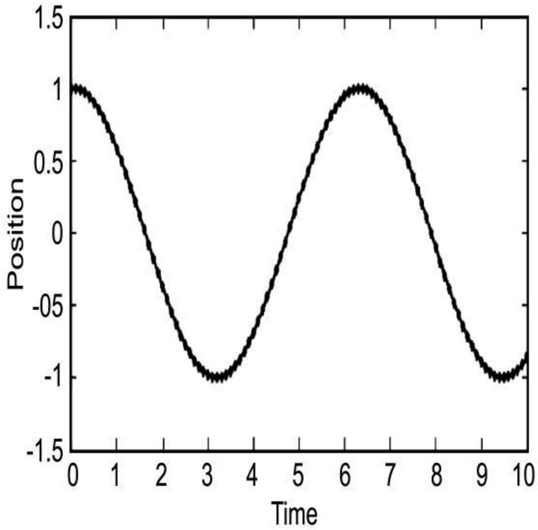

Fig. 5. Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) = 1, y'(0) = 0, \epsilon = 0.5$.

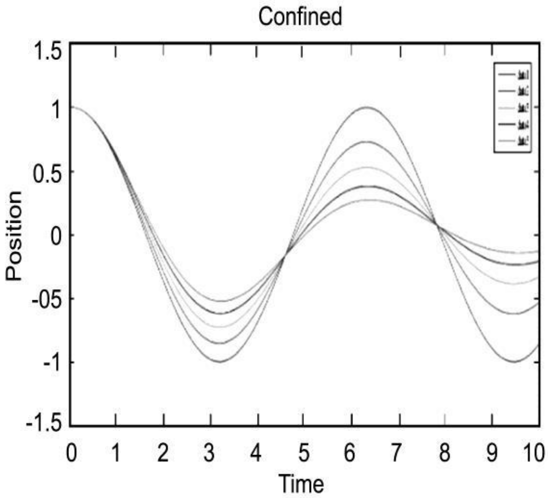

Fig. 6. Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) = 1, y'(0) = 0,\ \epsilon = 0.1, \epsilon = 0.2,\ \epsilon = 0.3,$ $\epsilon = 0.4, \epsilon = 0.5$].

Solution of Nonlinear Differential Systems

Let us consider a nonlinear non-autonomous differential system governed by

$$\ddot{x} + (\zeta_1^2 + \varsigma_2 \cos\tau)x = -\varepsilon f(x, \dot{x}, \tau), \quad \tau = \varepsilon t \quad 3.1)$$

where the over-dots denote differentiation with respect to *t*, ε is a small parameter, ζ_1 and ζ_2 are constants, and $\varsigma_1 = O(\varepsilon) = \varsigma_2$ is the slowly varying time, *f* is a given nonlinear function. We set $\omega^2(\tau) = \varsigma_1{}^2 + \varsigma_2 \cos\tau$, where $\omega(\tau)$ is known as internal frequency.

Putting $\varepsilon = 0$ and $\tau = \tau_0$ (constant), in Eq. (3.1), we obtain the unperturbed solution of (3.1) in the form

$$x(t,0) = x_{1,0} \exp(\lambda_1(\tau_0)t + x_{-1,0} \exp(\lambda_{21}(\tau_0)t \quad (3.2)$$

Let Eq.(3.1) has two eigen values, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ are constants, but when $\varepsilon \neq 0$, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ vary slowly with time. When $\varepsilon \neq 0$ we seek a solution of (3.1) in the form

$$x(t,\varepsilon) = x_1(t,\tau) + x_{-1}(t,\tau) + \varepsilon u_1(x_1, x_{-1}, t, \tau) + \varepsilon^2 u_2(x_1, x_{-1}, t, \tau) + \cdots \tag{3.3}$$

where x_1 and x_{-1} satisfy the equations

$$\begin{aligned} x_1' &= \lambda(\tau)x_1 + \varepsilon X_1(x_1, x_{-1}, \tau) + \varepsilon^2 X_1(x_1, x_{-1}, \tau) + \cdots \\ x_{-1}' &= \lambda_2(\tau)x_{-1} + \varepsilon X_{1-}(x_1, x_{-1}, \tau) \\ &\quad + \varepsilon^2 X_{-1}(x_1, x_{-1}, \tau) + \cdots \end{aligned} \tag{3.4}$$

Differentiating $x(t,\varepsilon)$ two times with respect to t, substituting for the derivatives $\ddot{x}$ and x in the original equation (3.1) and equating the coefficient of ε, we obtain

$$\begin{aligned} &(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_1 + (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_{-1} \\ &\quad + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} + \\ &\quad (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_1)(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_2) u_1 \\ &= -f^{(0)}(x_1, x_{-1}, \tau) \end{aligned} \tag{3.5}$$

where,

$$\lambda_1' = \frac{d\lambda_1}{d\tau},\ \lambda_2' = \frac{d\lambda_2}{d\tau},\ Dx_1 = \frac{\partial}{\partial x_1},\ Dx_{-1} = \frac{\partial}{\partial x_{-1}},\ f^{(0)} = f(x_0, \dot{x}_0, \tau)$$

Here it is assumed that $f^{(0)}$ can be expanded in Taylor's series as given below:

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) x_1^{r_1} x_{-1}^{r_2} \tag{3.6}$$

To obtain the solution of (3.1), it has been proposed in (Cap, 1974) that, u_1 excluded the terms $x_1^{r_1} x_{-1}^{r_2} e^{(r_1\lambda_1 + r_2\lambda_2)t}$ of $f^{(0)}$ where $r_1 - r_2 = \pm 1$. This restriction guarantees that the solution representations always exclude *secular*-type (*e.g.*, $t\cos t$ and $t\sin t$), otherwise a sizeable error would occur (Cap, 1974). By transforming the varia-bles $x_1 = \alpha e^{i\varphi}/2$ and $x_{-1} = \alpha e^{-i\varphi}/2$ and $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$, the existing form of the solution is determined (see also [3.1-3.3]). Here, α and φ are respectively amplitude and phase variables.

Example

Let us consider a nonlinear deferential autonomous system with slowly varying coefficients

$$\ddot{x} + (\zeta_1^2 + \zeta_2 \cos\tau)x = -\varepsilon x^3 \tag{3.7}$$

Here over dots denote differentiation with respect

to t. $x_0 = x_1 + x_{-1}$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(x_1^3 + 3x_1^2 x_{-1} + 3x_1 x_{-1}^2 + x_{-1}^3)$$
(3.8)

Following the assumption (discussed in section 2) u_1 excludes the terms $3x_1^2 x_{-1}$, $3x_1 x_{-1}^2$ and. We substitute in (3.5) and separate it into two parts

$$\begin{aligned} &(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_1 + (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_{-1} \\ &\quad + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} = -(3x_1^2 x_{-1} + 3x_1 x_{-1}^2) \end{aligned} \tag{3.9}$$

and

$$(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_1)(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_2) u_1 = -(x_1^3 + x_{-1}^3) \tag{3.10}$$

The particular solution of (3.10) is

$$u_1 = -1.5\{x_1^3(1 - \lambda_2/3\lambda_1)^{-1} - x_{-1}^3(1 - \lambda_1/3\lambda_2)^{-1}\} \tag{3.11}$$

Now we have to solve (3.9) for two functions X_1 and X_{-1} (discussed in section 2)

The particular solutions are

$$X_1 = -\lambda_1' x_1 (\lambda_1 - \lambda_2)^{-1} - 1.5 x_1^2 x_{-1} \lambda_1^{-1} \quad (3.12)$$

and

$$X_{-1} = \lambda_2' x_{-1} (\lambda_1 - \lambda_2)^{-1} - 1.5 x_1 x_{-1}^2 \lambda_2^{-1} \quad (3.13)$$

Substituting the functional values of X_1, X_{-1} from (3.12) and (3.13) into (3.4) and rearranging, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon\left(-\lambda_1' x_1 (\lambda_1 - \lambda_2)^{-1} - 1.5 x_1^2 x_{-1} \lambda_1^{-1}\right) \quad (3.14)$$

and

$$\dot{x}_{-1} = \lambda_2 x_{-1} + \varepsilon\left(\lambda_2' x_{-1} (\lambda_1 - \lambda_2)^{-1} - 1.5 x_1 x_{-1}^2 \lambda_2^{-1}\right) \quad (3.15)$$

The variational equations of α and φ, in the real form, transform (3.14) and (3.15) to

$$\dot{\alpha} = -\varepsilon x \omega' / 2\omega \quad (3.16)$$

and

$$\dot{\varphi} = \omega + 3\varepsilon x^2 / 8\omega^2 \quad (3.17)$$

where $\omega = \sqrt{\zeta_1^2 + \zeta_2 \cos\tau}$.

Therefore, the first-order solution of the equation (3.7) is

$$x(t, \varepsilon) = \alpha \cos\varphi + \varepsilon u_1$$

(3.18) where α and φ are the solutions of the equations (3.16) and (3.17) respectively, u_1 is given by (3.11). Substituting the values of X_1 and X_{-1} from (3.12) and (3.13) into

$$x(t,\ \varepsilon) = a(t)e^{\lambda_1 t} + b(t)e^{\lambda_2 t} + \varepsilon u_1(a, b, t) + \ldots$$

and solving them, we obtain the exiting solution of (3.4) similar to (3.16) and (3.17).

Conclusion

An approximate solution of second-order nonlinear differential systems with varying time has been found based on the KBM method and general Struble's technique. The method is simpler than the classical method and measures better result for strong nonlinearities. The method can be preceded to higher-order nonlinear systems (i.e., third order, fourth order, etc.).

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