J. Bangladesh Acad. Sci. 44(2);139-143: December 2020 DOI: 10.3329/jbas.v44i2.51458

Journal of Bangladesh Academy of Sciences

Journal homepage: http://www.bas.org.bd/publications/jbas.html

Research Article

Fuzzy pairwise regular bitopological spaces in quasi-coincidence sense

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ARTICLE INFO

Article History

Received: 06 September 2020

Revised: 11 October 2020

Accepted: 21 October 2020

Keywords: Fuzzy bitopological space; Quasi-Coincidence; Fuzzy pairwise regular space; Hereditary; Mappings.

ABSTRACT

This paper introduces three notions of fuzzy pairwise regular between bitopological spaces in quasi-coincidence sense. Then, we investigate some relations between ours and other counterparts. We observe that all these concepts are preserved under one-one, onto, fuzzy closed, fuzzy open, and fuzzy continuous mappings. Also the hereditary property is satisfied by these concepts.

Introduction

As an important research field in fuzzy set theory,fuzzy topology was explored by Chang (1968) based on Zadeh (1965) concept of fuzzy sets. Since then, much attention has been paid to generalize the basic concepts of general topology in a fuzzy setting. In a fuzzy topology, the concept of regular and normal separations defined first by Hutton and Reilly (1980), and many fuzzy mathematicians have contributed various forms of separation axioms and other aspects; in particular, fuzzy normality (Hutton, 1975; Miah et al.,2018); fuzzy regularity (Ali, 1990); fuzzy hausdorffness (Hossain and Ali,2005; Amin et al., 2014; Miah and Amin, 2017b), separations on fuzzy topological spaces (Ali et al.,1990; Miah et al., 2017a; Miah et al., 2017b), fuzzy bitopological spaces (Kandil and El-Shafee, 1991; Amin et al., 2014a); connectedness (Rana et al.,2017), to the theory of fuzzy topological and bitopological spaces. Also, many concepts in classical topology have been extended to fuzzy bitopological spaces. The concepts of fuzzy pairwise regular bitopological spaces were first introduced by Kandil and El-Shafee (1991). Later, Safiya et al. (1994), Nouh (1996), Mukherjee (2002), Lee (2003), Ramadan et al. (2006), Kumar (1994), Tapi and Navalakhe (2012), also introduced other types of regular fuzzy bitopological spaces. The purpose of this paper is to define three notions of fuzzy pairwise regular (in short FPR) bitopological spaces in a quasi-coincidence sense. We establish some relations among ours and other such notions. We show that one of our definitions is stronger than such as Safiya et al. (1994). Also, hereditary is satisfied with these concepts. We observe that all with these concepts are preserved under one-one, onto and fuzzy continuous mapping.

Basic Notions and Preliminary Results

This section gives some elementary concepts and results in a fuzzy set, which will be used in

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the sequel. Through this paper, X will be a nonempty set, I = [0,1], I_0 = (0,1] and FP stands for fuzzy pairwise. The class of all fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by u, v, w, etc. A crisp subset of X will be denoted by capital letters U, V, W etc. In this paper, (X, t) and (X, s, t) will be denoted fuzzy topological space and fuzzy bitopological space, respectively. $x_r qu$ denotes x_r is quasi-coincident with u and $x_r \bar{q} u$ denotes that x_r is not quasi-coincident with u throughout this paper.

Definition: A fuzzy set μ in a set X is a function from X into the closed unit interval $I = [0, 1]$. For every $x \in X,\ \mu(x) \in I$ is called the grade of membership of x. Throughout this paper, I^X will denote the set of all fuzzy sets from X into the closed unit interval I. A member of I^X may also be a called fuzzy subset of X (Zadeh, 1965).

Definition: A fuzzy set μ in X is called a fuzzy singleton iff $\mu(x) = r, (0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. We call x_r is a fuzzy point if $0 < r < 1$. The class of all fuzzy singletons in X will be denoted by $S(X)$, (Wong, 1974).

Definition: A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q\mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q} \mu$. (Kandil et al., 1995).

Definition: A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X, t) is called a fuzzy topological space (fts, in short) and members of t are called t-open (or simply open) fuzzy sets. A fuzzy set μ is called a t- closed (or simply closed) fuzzy set if $1 - \mu \in t$ (Chang, 1968).

Definition: Let f be a real valued function on a topological space. If $\{x: f(x) > \alpha\}$ is open for every real $\alpha \in I_1$, then f is called lower semi-continuous function (Rudin, 1974).

Definition: Let X be a nonempty set and T be a topology on X. Let $t = \omega(T)$ be the set of all lower semi-continuous (lsc, in short) functions from (X, T) to I (with usual topology). Thus $\omega(T) = \{\mu \in I^X: \mu^{-1}(\alpha,\ 1] \in T\}$ for each $\alpha \in I_1$. It can be shown that $\omega(T)$ is a fuzzy topology on X. (Ali et al., 1990)

Let P be a property of topological spaces and FP be its fuzzy topology analog. Then FP is called a 'good extension' of P "iff the statement $(X,\ T)$ has P iff $(X,\ \omega(T))$ has FP" holds suitable for every topological space (X, T) (Ali, 1990).

Definition: A fuzzy bitopological space (fbts, in short) is a triple (X, s, t) where s and t are arbitrary fuzzy topologies on X (Kandil et al., 1995).

Definition: A bitopological space $(X, S,\ T)$ is called pairwise regular (in short, PR) if, for each point x in X and each S-closed set P with $x \notin P$, there exist $U \in S,\ V \in T$ such that $x \in V, P \subseteq V$ and $U \cap V = \emptyset$ (Kelly, 1963).

Definition: A bitopological space $(X, S,\ T)$ is called pairwise normal (in short, PN) if, given S-closed set A and a T-closed set B with $A \cap B = \emptyset$, there exist $U \in S,\ V \in T$ such that $A \subseteq V, B \subseteq U$ and $U \cap V = \emptyset$ (Kelly, 1963).

Fuzzy Pairwise Regular Bitopological Spaces

Definition: A fuzzy bitopological space (X, s, t) is called

(a) $FPR(i)$ iff $x_r \notin w$, w is an s-closed fuzzy set, there exist $u \in s$ and $v \in t$ such that $x_r \in u, w \subseteq v$ and $u\bar{q}v$.

(b) $FPR(ii)$ iff $x_r \notin w$, w is an s-closed fuzzy set, there exist $u \in s$ and $v \in t$ such that $x_r \in u, w \subseteq v$ and $u \cap v = 0$.

(c) $FPR(iii)$ iff $x_r\bar{q}w$, w is an s-closed fuzzy set, there exist $u \in N(x_r, s)$ and $v \in N(w, t)$ such that $u\bar{q}v$. (Safiya et al., 1994).

(d) $FPR(iv)$ iff $x_r\bar{q}w$, w is an s-closed fuzzy set, there exist $u \in N(x_r, s)$ and $v \in N(w, t)$ such that $u \cap v = 0$.

Theorem: *Let (X, s, t) be a fuzzy bitopological space. Then we have the following implications:* (b)⇒ (a), (d)⇒ (c).

Proof: (b)⇒ (a), (d)⇒ (c) are obvious since $u \cap v = 0$ implies that $u\bar{q}v$.

The following counter example shows that (a) ⇏(b) as well as (c) ⇏(d) in general.

Example: Let $X = \{x, y\}$ and s be a fuzzy topology on X generated by$\{u\} \cup \{constants\}$, where $u(x) = 1, u(y) = 0.3$. Let t be a fuzzy topology on X generated by $\{v\} \cup \{constants\}$, where $v(x) = 0, v(y) = 0.7$. Then we see that (X, s, t) is $FPR(i)$ but not $FPR(ii)$. It is also clear that (X, s, t) is $FPR(iii)$ but not $FPR(iv)$. Hence (a) ⇏(b), (c) ⇏(d).

Theorem: *Let (X, s, t) be a fuzzy bitopological space, $A \subseteq X$ and $s_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$, then*

(a) (X, s, t) is $FPR(i)$ ⇒ (A, s_A, t_A) is $FPR(i)$;

(b) (X, s, t) is $FPR(ii)$ ⇒ (A, s_A, t_A) is $FPR(ii)$;

(c) (X, s, t) is $FPR(iii)$ ⇒ (A, s_A, t_A) is $FPR(iii)$;

(d) (X, s, t) is $FPR(iv)$ ⇒ (A, s_A, t_A) is $FPR(iv)$.

Proof: Suppose (X, s, t) is $FPR(i)$. We have to show that (A, s_A, t_A) is $FPR(i)$. Let w be a s_A-closed fuzzy set and $x_r \in S(A)$ such that $w(x) < r$. This implies that $w^c \in s_A$ and $w^c(x) > 1 - r$. So there exists an $u \in s$ such that $u/a = w^c$ and clearly u^c is closed in s. Now

$u^c(x) = (u/A)^c(x) = w(x) < r$ since $x \in A$.

Since (X, s, t) is $FPR(i)$, then there exists $p \in s$, $v \in t$ such that $x_r \in p$, $u^c \subseteq v$ and $p\bar{q}v$. Since $p \in s$, $v \in t$, then $p/A \in s_A$, $v/A \in t_A$. Now we have

$x_r \in p$/A, $(u/A)^c \subseteq v/A$ and $(p/A)\bar{q}(v/A)$. So $x_r \in p$/A, $w \subseteq v/A$ and $(p/A)\bar{q}(v/A)$. Therefore (A, s_A, t_A) is $FPR(i)$.

Similarly, (b), (c) and (d) can be proved

Theorem: *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and let $f: X \longrightarrow Y$ be bijective, FP-continuous and FP-open.* Then

(X, s, t) is $FPR(j)$ ⇒ (Y, s_1, t_1) is $FPR(j)$, where $j = i, ii, iii, iv$.

Proof: Suppose the fuzzy bitopological space (X, s, t) is $FPR(i)$. We have to show that (Y, s_1, t_1) is $FPR(i)$. Let w be s_1-closed and $y_r \in S(Y)$ with $w(y) < r$. Then $f^{-1}(w) \in s^c$ as f is FP-continuous. Since f is bijective, then for $y \in Y$ there exits $x \in X$ such that $f(x) =$

y. Now $f^{-1}(w)(x) = w(f) = w(y) < r$. Since (X, s, t) is $FPR(i)$, there are $u \in s$, $v \in t$ such as $x_r \in u$, $f^{-1}(w) \subseteq v$ and $u\bar{q}v$.

Now, we see that $f(u)(y) = \{\sup u(x) : f(x) = y\} < r$. So $y_r \in f(u)$.

Also, u$\bar{q}v$ implies that u$(x) + v(x) \leq 1$ for all $x \in X$.

Now for all $f(x) \in Y$, we have f(u)f$(x) + f(v)(f(x)) =$ u$(x) + v(x) \leq 1$ as f is bijective. So $f(u)\bar{q}f(v)$.

Again since $f^{-1}(w) \subseteq v$, then $w \subseteq f(v)$. It is clear that $f(u) \in s_1$, $f(v) \in t_1$ as f is FP-open. So $f(u) \in s_1$, $f(v) \in t_1$ such that $y_r \in f(u)$, $w \subseteq f(v)$ and $f(u)\bar{q}f(v)$. Hence (Y, s_1, t_1) is $FPR(i)$.

The proofs are similar for $j = ii, iii, iv$.

Theorem: *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and $f: X \longrightarrow Y$ be bijective, FP-continuous and FP-closed.* Then

(Y, s_1, t_1) is $FPR(j) \Rightarrow (X, s, t)$ is $FPR(j)$, where $j = i, ii, iii, iv, v$.

Proof: Suppose the fuzzy bitopological space (Y, s_1, t_1) is $FPR(i)$. We have to show that (X, s, t) is $FPR(i)$. Let w be s-closed and $x_r \in S(X)$ with $w(x) < r$. Then $f(w) \in {s_1}^c$ as f is FP-close and let $f(x) = y$. Now, we have

$f(w)(y) = \{\sup w(x) : f(x) = y\} < r$, since f is one-one

Since (Y, s_1, t_1) is $FPR(i)$, then there exist $u \in s_1$, $v \in t_1$ such that

$y_r \in u$, $f(w) \subseteq v$ and $u\bar{q}v$.

Now, we see that $f^{-1}(u)(x) = u(f(x)) = u(y) \leq r$. So $x_r \in f^{-1}(u)$.

Also, it is clear that $f(w) \subseteq v$ implies that $w \subseteq f^{-1}(v)$ as f is bijective.

Again $u\bar{q}v$ implies that

$u(y) + v(y) \leq 1$ for all $y \in Y$.

Now for all $x \in X$, we have $f^{-1}(u)(x) + f^{-1}(v)(x) = u(f(x)) + v(f(x)) = u(y) + v(y) \leq 1$. So $f^{-1}(u)\bar{q}f^{-1}(v)$.

Since f is FP-continuous, then $f^{-1}(u) \in s$, $f^{-1}(v) \in t$. So $f^{-1}(u) \in s$, $f^{-1}(u) \in t$ such that $x_r \in f^{-1}(u)$, $w \subseteq f^{-1}(v)$ and $f^{-1}(u)\bar{q}f^{-1}(v)$. Hence (X, s, t) is $FPR(i)$.

Similarly, we can prove for $j = ii, iii, iv$.

Conclusion

One of the main results of this paper is introducing some new definitions of fuzzy regular bitopological spaces in the sense of quasi-coincidence. We present their hereditary and order-preserving properties. We compare the results with other existing notions and their counterparts' and show that one of our definitions is stronger than such one (Safiya et al., 1994).

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