J. Bangladesh Acad. Sci. 44(2); 197-200: December 2020 DOI: 10.3329/jbas.v44i2.51463

Journal of Bangladesh Academy of Sciences

Journal homepage: http://www.bas.org.bd/publications/jbas.html

Short Communication

New variants of the bottleneck tower of Hanoi problems

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ARTICLE INFO

Article History

Received: 25 October 2020

Revised: 29 October 2020

Accepted: 15 December 2020

Keywords: Bottleneck Tower of Hanoi, Divine rule, Bottleneck Reve's puzzle

ABSTRACT

This paper considers two variants of the bottleneck Tower of Hanoi problems with n (≥1) discs and the bottleneck size b (≥2), which allows violation of the "divine rule" (at most) once. Denoting by $MB_3(n,b)$ the minimum number of moves required to solve the new variant of the bottleneck Tower of Hanoi problem, an explicit form of $MB_3(n,b)$ is found. Also, $MB_4(n,b)$ denotes the minimum number of moves required to solve the new variant of the bottleneck Reve's puzzle, a closed-form expression of MB_4(n,b) is derived.

Introduction

The Tower of Hanoi puzzle, due to the famous French puzzlist and number theorist Lucas (1883), is as follows: Given are n (≥1) discs d_1, d_2, ..., d_n of increasing sizes, and three pegs, S, P and D. At the beginning of the game, the discs all rest on the *source peg*, S, in a tower in increasing order, from top to bottom. The objective is to shift this tower of n discs to the *destination peg*, D, in a minimum number of moves, where each move can shift only the topmost disc from one peg to another, under the "divine rule," which requires that, during the transfer process of the discs, no disc could ever be placed on top of a smaller one. It is well-known that the (minimum) number of moves required to solve this problem is 2^n-1.

One generalization of the above problem is the bottleneck Tower of Hanoi problem, posed by Wood (1983), and later solved by Poole (1992).

Given any collection C of any number of discs, d_i, d_j, ..., d_m, the *narrowness* of C, denoted by $N(C)$, is the label-index of the smallest disc in C, that is,

$$N(C)=\min\{i: D_i \in C\};\ N(\emptyset)=\infty$$

(Ø being the empty set).

The bottleneck Tower of Hanoi problemis as follows: Given are three pegs, S, P and D, and n(≥1) discs of increasing sizes, d_1, d_2, ..., d_n, resting on the *source peg*, S, in a tower in *standard position* (with the largest disc at the bottom, the second largest disc above it, and so on, with the smallest disc at the top).

The objective is to shift this tower of n discsfrom the peg S to the destination peg, D, in astandard position, in a minimum number of moves, under the following two conditions:Condition 1: Only the topmost disc can be shifted from one peg to another in a single move,Condition 2: A disc Di may not be placed on a tower of discs T if

$$i>N(T)+b-1,$$

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where b (≥ 1) is a pre-assigned integer, called the *bottleneck size*.

Any arrangement of the n (≥ 1) discs on the hree pegs that can be obtained without violating condition (2) above is called a *legal position*.

Let $g_3(n, b)$ denote the minimum number of moves required to transfer the tower of n discs from its starting position (not necessarily standard) to a legal (but not necessarily standard) position on another peg, and let $M_3(n, b)$ denote the minimum number of moves required to solve the bottleneck Tower of Hanoi problem. The following result, giving the recurrence relations satisfied by $g_3(n, b)$ and $M_3(n, b)$, is due to Poole (1992).

Lemma 1: For any n (≥ 1) and b (≥ 1),

(1) $g_3(n, b) = 2g_3(n-b, b) + b,\ n \geq b,$

(2) $M_3(n, b) = 2g_3(n-1, b) + 1,\ n \geq 1,$

with

$g_3(n, b) = n$ for all $0 \leq n \leq b,$

$M_3(0, b) = 0$ for all $b \geq 1.$

The solution of the bottleneck Tower of Hanoi problem, due to Poole (1992), is given below.

Theorem 1: Given n (≥ 1) and b (≥ 1), let

$n = bq + r,$

where,

$q \in \{1, 2, \ldots\},\ 0 \leq r < b.$

Then,

(1) $g_3(n, b) = (b + r)2^q - b,$

(2) $M_3(n, b) = \begin{cases} (2b-1)(2^q-1), & \text{if } r = 0 \\ (b+r-1)2^{q+1} - 2b + 1, & \text{if } r \neq 0 \end{cases}$

The 4-peg generalization of the Tower of Hanoi problem is commonly known as the Reve's puzzle due to the English puzzlist Dudeney (1958). The problem may be stated as follows: Given are four pegs, S, P_1, P_2 and D, and a tower of n (≥ 1) discs (of varying sizes) on the source peg S, in small-on-large ordering. The objective is to move this tower to the peg D, using the auxiliary pegs P_1 and P_2, in a minimum number of moves, where each action shifts the topmost disc from one peg to another under the "divine rule" that no disc can ever be placed on top of a smaller one.

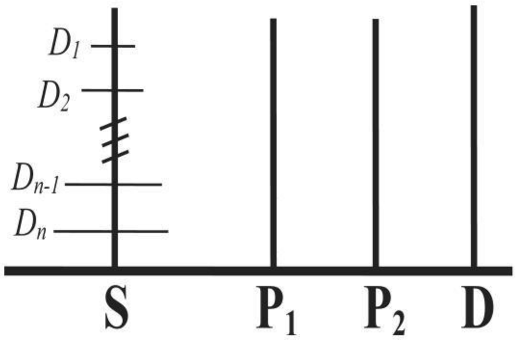

Fig. 1. **Initial state in the Reve's puzzle.**

Let $M_4(n)$ be the minimum number of moves required to solve the Reve's puzzle with n (≥ 1) discs. Then, the dynamic programming equation satisfied by $M_4(n)$ is as follows:

$$M_4(n) = \min_{0 \leq k \leq n-1} \{2M_4(k) + 2^{n-k} - 1\},\ n \geq 1, \quad (1)$$

$$M_4(0) = 0. \quad (2)$$

The complete solution of the recurrence relation (1) is given in the theorem below.

***Theorem 2*:** Let $n = \frac{s(s+1)}{2} + R$ for some integers s (≥ 1) and R ($0 \leq R \leq s$). Then, $M_4\left(\frac{s(s+1)}{2}\right)$ is attained at the unique point $k = \frac{s(s-1)}{2}$, and for $R \neq 0$, $M_4\left(\frac{s(s+1)}{2} + R\right)$ is attained exactly at the two points $k = n - s$, $n - s - 1$; moreover, in either case,

$$M_4\left(\frac{s(s+1)}{2} + R\right) = 2^s(s + R - 1) + 1.$$

For further details on Reve's puzzle, the reader is referred to Majumdar (1994).

The bottleneck Reve's puzzle was proposed by Majumdar (1996), who gave a scheme to derive the dynamic programming equation for the optimal value function. Based on some local-value relationships, Majumdar and Halder (1996) presented a recurrence relation to calculate the optimal value function as well as the optimal partition numbers recursively in n. It may be noted that, in Theorem 1, there is no restriction on the number of violations of the "divine rule", so long as Condition 2 is not violated. But what happens if the violation of the "divine rule" is allowed only once? To solve this new version of the problem, let $MB_3(n, b)$ be the minimum number of moves required to solve the restricted bottleneck Tower of Hanoi problem with n (≥1) discs and bottleneck size b (≥2). Then, the closed-form expression of $MB_3(n)$ is given below.

***Theorem 3*: *For any b* (≥2),**

$MB_3(n) = 2^{n-b+1} + 2b - 3,\ n \geq b,$

$MB_3(n) = 2n - 1,\ 1 \leq n \leq b,$

***Proof*:** To find $MB_3(n, b)$, $n \geq b$, the scheme below is employed:

Step 1: Move the topmost $n - b - 1$ discs $d_1, d_2, \ldots, d_{n-b-1}$, (from the peg S) to the peg D, in (minimum) $2^{n-b-1} - 1$ number of moves,

Step 2: Form the inverted tower with the b discs, $d_{n-b}, d_{n-b+1}, \ldots, d_{n-1}$,

Step 3: Now, transfer the discs $d_1, d_2, \ldots, d_{n-b-1}$ (from the peg D) to the peg P, in (minimum) $2^{n-b-1} - 1$ number of moves,

Step 4: Shift the largest disc, d_n, (from the peg S) to the peg D,

Step 5: Move the discs $d_1, d_2, \ldots, d_{n-b-1}$ (from the peg P) to the peg S, in (minimum) $2^{n-b-1} - 1$ number of moves,

Step 6: Transfer the b discs on the peg P, one-by-one, to the peg D,

Step 7: Finally, shift the discs $d_1, d_2, \ldots, d_{n-b-1}$ (from the peg S) to the peg D, in (minimum) $2^{n-b-1} - 1$ number of moves, to complete the tower on the peg D.

The total number of moves involved is

$$2(2^{n-b-1} - 1 + b + 2^{n-b-1} - 1) + 1$$
$$= 2^{n-b-1} + 2b - 3.$$

When $1 \leq n \leq b$, the transfer process can be done linearly in $2n$–1 move by first forming an inverted tower of the smallest n–1 disc on the peg P, then moving the largest disc to the peg D, and finally, shifting the n–1 disc, one-by-one, to the peg D.

All these complete the proof of the theorem. Next, let $MB_4(n, b)$ be the minimum number of moves required to solve the restricted bottleneck Reve's puzzle with n (≥1) discs and bottleneck size b (≥2), when the violation of the "divine rule" is allowed (at most) once. Then, we have the following theorem.

***Theorem 4*: *For any b* (≥ 2),**

$MB_4(n, b) = M_4(n - b + 1) + 2(b - 1),\ n \geq b,$

$MB_4(n, b) = 2n - 1,\ 1 \leq n \leq b.$

***Proof*:** To find the dynamic programming equation satisfied by $MB_4(n, b)$, $n \geq b$, the scheme below is followed:

Step 1: Move the topmost k discs from the peg S, to some auxiliary peg, say, P_1, using the available four pegs, in (minimum) $M_4(k)$ moves,

Step 2: Shift the remaining $n - k$ discs, from the peg S to the peg D, in (minimum) $MB_3(n - k, b) = 2^{n-k-b+1} + 2b - 3$ number of moves,

Step 3: Transfer the k discs (from the peg P_1) to the peg D, in (minimum) $M_4(k)$ moves.

The total number of moves involved in the above three steps is

$$2M_4(k)+2^{n-k-b+1}+2b-3,$$

where k is determined so as to minimize the above expression. Therefore, $MB_4(n, b)$ satisfies the following recurrence relation

$$MB_4(n,b)=\min_{0\le k\le n-1}\left\{2M_4(k)+2^{n-k-b+1}+2b-3\right\},$$

which gives the desired expression by virtue of (1). The remaining part is evident from the proof of Theorem 3.

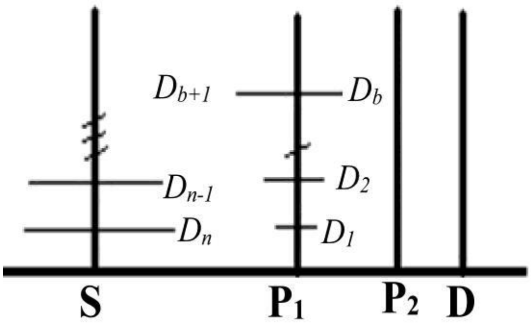

Fig. 2. A legal position in the bottleneck Reve's puzzle with.

***Theorem 5*: *For any* $b \ge 2$ fixed,**

$$MB_4\left(\frac{s(s+1)}{2}+R+b\right)=2^s(s+R)+2b-1, \quad s\ge 1,\ 0\le R\le s,$$

which is attained at the two points

$$k=\frac{s(s-1)}{2}+R,\ \frac{s(s-1)}{2}+R-1.$$

***Proof*:** Follows from Theorem 2.

Note from Theorem 4 that $MB_4(n, 1) = M_4(n)$, that is, when $b = 1$, the restricted bottleneck Reve's puzzle reduces to the Reve's puzzle.

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