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Short Communication

A note on the diophantine equation $x^2 = y^2 + 3z^2$

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ABSTRACT

In the study of 60-degree and 120-degree triangles, one encounters the Diophantine equations of the form $x^2 = y^2 + 3z^2$. This paper considers the characteristics of the solution of the Diophantine equation. More specifically, it is shown that the equation has solutions of the form $x \equiv p = 3n+1$ for some integer n (>0), where p is a prime with $7 \le p \le 199$.

Introduction

Diophantine equations, named after the famous Greek philosopher Diophantus, are equations involving two or more variables for which integer solutions are sought. The well-known Diophantine equation is $x^2 = y^2 + 3z^2$, which is related to the Pythagorean (right-angled) triangle. In many practical problems, Diophantine equations arise quite naturally.

One such problem is introduced below. The concept of the *S*-related triangles was introduced by Sastry (2000), which was subsequently extended by Ashbacher (2000) to the *Z*-related triangles, where $S(n)$ is the Smarandache function and $Z(n)$ is the pseudo-Smarandache function. Of particular interest are the 60-degree and 120-degree *S*-related and *Z*-related triangles. It has been shown in Majumdar (2010) that if ΔABC is a triangle with sides $AB=c$, $BC=a$, $CA=b$ and $\angle BAC = 60^0$, then a, b, c satisfy the Diophantine equation

$$4a^2 = (2c-b)^2 + 3b^2, \quad (1)$$

while if $\angle BAC = 120^0$, the equation satisfied by a, b and c is

$$4a^2 = (2c+b)^2 + 3b^2. \quad (2)$$

Clearly, if (a_0, b_0, c_0) is a solution of the Diophantine equation (1), so also are (a_0, c_0, b_0) and $k(a_0, b_0, c_0)$ for any constant $k>0$; and if (a_0, b_0, c_0) is a solution of (2), so also are (a_0, c_0, b_0) and $k(a_0, b_0, c_0)$ for any constant $k>0$. In Majumdar (2010), the following result has been proved.

Lemma 1: If (a_0, b_0, c_0), $c_0 > b_0$, is a solution of the Diophantine equation (1), then $(a_0, b_0, c_0 - b_0)$ is a solution of the Diophantine equation (2).

By virtue of Lemma 1, it is sufficient to consider the primitive solutions of the Diophantine equation (1).

The result below, due to Majumdar (2010), shows that the Diophantine equation (1) possesses solutions in pairs.

Lemma 2: If (a_0, b_0, c_0), $c_0 > b_0$, is a solution of the Diophantine equation (1), then $(a_0, c_0 - b_0, c_0)$ is its second (independent) solution.

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Note that each of the two Diophantine equations (1) and (2) is of the form

$$x^2 = y^2 + 3z^2. \quad (3)$$

If a solution of the Diophantine equation (3) is known, the corresponding solution of the Diophantine equation (1) may be obtained by using the following lemma, due to Majumdar (2018).

Lemma 3: If (x_0, y_0, z_0), is a solution of the Diophantine equation (3) and (a_0, b_0, c_0) is a solution of the Diophantine equation (1), then

$$a_0 = \frac{x_0}{2},\ b_0 = \frac{y_0+z_0}{2},\ c_0 = z_0.$$

Note that, Lemma 3 is applicable to the solution of (3) with even x_0.

If two independent solutions of the Diophantine equation (3) are known, using the result below, due to Majumdar (2018), two more independent solutions of (3) can be obtained.

Lemma 4: If (x_0, y_0, z_0) and (x_1, y_1, z_1) are two independent solutions of the equation (3), then $(x_0 x_1, |y_0 y_1 - 3z_0 z_1|, y_0 z_1 + y_1 z_0)$ and $(x_0 x_1, y_0 y_1 + 3z_0 z_1, |y_0 z_1 - y_1 z_0|)$ are its two more independent solutions.

The Diophantine equation (3) is a particular case of the more general equation

$$Ax^p + By^q + Cz^r = 0,\ \gcd(x, y, z) = 1, \quad (4)$$

(where A, B and C are known integers with $ABC \neq 0$), considered by Darmon and Granville (1995), who proved that the equation (4) has only a finite number of solutions in the hyperbolic case (when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$). The problem was taken up by Beukers (1998), who considered the spherical case (that is, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$) to some extent. Some particular cases of the equation (4) have been treated by Poonen (1998) and Bruin (1999, 2005). However, none of these papers considered the equation of the form (3). On the other hand, Dickson (1971, Chap. XIII) contains an outline of the solution as follows: Letting

$$\frac{x+y}{3z} = \frac{z}{x-y} = t,$$

it follows that t must divide z. Letting $z = 2\lambda t$, the following solution is obtained:

$$x = \lambda(3t^2 + 1),\ y = \lambda(3t^2 - 1),\ z = 2\lambda t, \quad (5)$$

where λ and t are any positive integers.

However, (5) does not give all the primitive solutions of the Diophantine equation (3), but it shows that (3) has solutions of the form $x = 3n + 1$, $n \geq 1$. With this in mind, let the integer N be of the form $N = 3n + 1$, $n \geq 1$. Note that the number $4N^2$ can be expressed in the following twenty-two equivalent forms:

$$4N^2 = 4(3n + 1)^2$$
$$= 1^2 + 3(12n^2 + 8n + 1) \quad \text{(i)}$$
$$= 2^2 + 3.4(3n^2 + 2n) \quad \text{(ii)}$$
$$= (6n + 1)^2 + 3(4n + 1) \quad \text{(iii)}$$
$$= (6n - 1)^2 + 3(12n + 1) \quad \text{(iv)}$$
$$= (6n - 2)^2 + 3\times 4^2 n \quad \text{(v)}$$
$$= (6n - 10)^2 + 3.4^2(3n - 2) \quad \text{(vi)}$$
$$= (6n - 25)^2 + 3.3^2(12n - 23) \quad \text{(vii)}$$
$$= (6n - 14)^2 + 3.8^2(n - 1) \quad \text{(viii)}$$
$$= (6n - 47)^2 + 3.7^2(4n - 15) \quad \text{(ix)}$$
$$= (6n - 46)^2 + 3.8^2(3n - 11) \quad \text{(x)}$$
$$= (6n - 62)^2 + 3.16^2(n - 5) \quad \text{(xi)}$$

$= (6n - 23)^2 + 3.5^2(4n - 7)$ (xii)

$= (6n - 73)^2 + 3.5^2(12n - 71)$ (xiii)

$= (6n - 119)^2 + 3(484n - 4719)$ (xiv)

$= (6n - 98)^2 + 3.20^2(n - 8)$ (xv)

$= (6n - 145)^2 + 3.7^2(12n - 143)$ (xvi)

$= (6n - 167)^2 + 3(676n - 9295)$ (xvii)

$= (6n - 106)^2 + 3.4^2(27n - 234)$ (xviii)

$= (6n - 254)^2 + 3.32^2(n - 21)$ (xix)

$= (6n - 241)^2 + 3.9^2(12n - 239)$ (xx)

$= (6n - 194)^2 + 3.28^2(n - 16)$ (xxi)

$= (6n - 287)^2 + 3(1156n - 27455)$ (xxii)

For example, when $n = 2$, then $12n^2 + 8n = 8^2$, $4n + 1 = 3^2$, $12n + 1 = 5^2$. Thus, 14^2 can be written in the following three equivalent ways (each of the form $x^2 = y^2 + 3z^2$):

$14^2 = 2^2 + 3.8^2 = 13^2 + 3.3^2 = 11^2 + 3.5^2$.

Again, when $n = 4$, $12n^2 + 8n + 1 = 15^2$, $12n + 1 = 7^2$, $16n = 8^2$, so that

$26^2 = 1^2 + 3.15^2 = 23^2 + 3.7^2 = 22^2 + 3.8^2$.

Now, when $n = 6$, we have, from (iii), (vi) and (vii)

$38^2 = 37^2 + 3.5^2 = 26^2 + 3.16^2 = 11^2 + 3.21^2$.

when $n = 10$, from (iv), (viii) and (ix),

$62^2 = 59^2 + 3.11^2 = 46^2 + 3.24^2 = 13^2 + 3.35^2$,

with $n = 12$, from (iii), (vii) and (x),

$74^2 = 73^2 + 3.7^2 = 47^2 + 3.33^2 = 26^2 + 3.40^2$.

When $n = 14$, from (iv), (xi) and (xii)

$86^2 = 83^2 + 3.13^2 = 22^2 + 3.48^2 = 61^2 + 3.35^2$,

with $n = 20$, from (iii), (x) and (xiii),

$122^2 = 121^2 + 3.9^2 = 74^2 + 3.56^2 = 47^2 + 3.65^2$.

When $n = 22$, from (vi), (xii) and (xiv), we get

$134^2 = 122^2 + 3.32^2 = 109^2 + 3.45^2 = 13^2 + 3.77^2$.

When $n = 24$, (iv), (ix) and (xv) gives

$146^2 = 143^2 + 3.17^2 = 97^2 + 3.63^2 = 46^2 + 3.80^2$,

when $n = 26$, from (vii), (viii) and (xvi), we get

$158^2 = 131^2 + 3.51^2 = 142^2 + 3.40^2 = 11^2 + 3.91^2$.

When $n = 32$, from (ii), (vii) and (xii), we have

$194^2 = 2^2 + 3.112^2 = 167^2 + 3.57^2 = 169^2 + 3.55^2$,

with $n = 34$, from (vi), (ix) and (xvii),

$206^2 = 194^2 + 3.40^2 = 157^2 + 3.77^2 = 37^2 + 3.117^2$,

with $n = 36$, from (v), (xiii) and (xvi), we get

$218^2 = 214^2 + 3.24^2 = 143^2 + 3.95^2 = 71^2 + 3.119^2$,

with $n = 42$, from (iii), (xvi) and (xviii),

$254^2 = 253^2 + 3.13^2 = 107^2 + 3.133^2 = 146^2 + 3.120^2$.

When $n = 46$, we get from (vii), (ix) and (xix),

$278^2 = 251^2 + 3.69^2 = 229^2 + 3.91^2 = 22^2 + 3.160^2$.

When $n = 50$, (viii), (xiii) and (xx) give

$302^2 = 286^2 + 3.56^2 = 227^2 + 3.115^2 = 59^2 + 3.171^2$,

when $n = 52$, (iv), (xiv) and (xxi) give

$314^2 = 311^2 + 3.25^2 = 193^2 + 3.143^2 = 118^2 + 3.168^2$,

when $n = 54$, from (vii), (xi) and (xxii), we get

$326^2 = 299^2 + 3.75^2 = 262^2 + 3.112^2 = 37^2 + 3.187^2$.

When $n = 60$, from (i), (ix) and (x), we get

$362^2 = 1^2 + 3.209^2 = 313^2 + 3.105^2 = 314^2 + 3.104^2$.

When $n = 64$, (v), (xvi) and (xx) give

$386^2 = 382^2 + 3.32^2 = 239^2 + 3.175^2 = 143^2 + 3.207^2$,

and when $n = 66$, from (vi), (xiv) and (xxii),

$398^2 = 386^2 + 3.56^2 = 277^2 + 3.165^2 = 109^2 + 3.221^2$.

The above analysis shows that, for any prime p of the form $3n + 1$ with $p < 200$, there is a solution of the Diophantine equation (3); moreover, all such solutions are obtained from the 22 identities in (i) – (xxii), listed above. These solutions are presented in Table 1 below.

Table 1. Solutions to $x^2 = y^2 + 3z^2$ $2 \leq x < 200$ is a prime.

x	y	z	x	y	z
7	1	4	103	97	20
13	11	4	109	107	12
19	13	8	127	73	60
31	23	12	139	11	80
37	13	20	151	143	28
43	11	24	157	59	84
61	37	28	163	131	56
67	61	16	181	157	52
73	23	40	193	191	16
79	71	20	199	193	28
97	1	56			

The following result can be established.

Lemma 5: Let (x_0, y_0, z_0) be a solution of the Diophantine equation (3). Then, its two more solutions are ($2x_0$, $|y_0 - 3z_0|$, $y_0 + z_0$) and ($2x_0$, $y_0 + 3z_0$, $|y_0 - z_0|$).

Proof: Since $2^2 = 1^2 + 3.1^2$, the result follows from Lemma 4.

Lemma 5 explains why corresponding to $x = 2p$, the Diophantine equation (3) has three independent solutions.

Of particular interest is the identity below, obtained from (v) with $n = 16$:

$$4.49^2 \equiv 4(7^2)^2 = 98^2 = 94^2 + 3.16^2.$$

Therefore, when $x = 98$, in addition to the three independent solutions (98, 14, 56), (98, 91, 21) and (98, 77, 35), the Diophantine equation (3) possesses the fourth independent solution (98, 94, 16). This example is a particular case of the more general result below, due to Majumdar (2018).

Lemma 6: Let (x_0, y_0, z_0) be a solution of the Diophantine equation (3). Then, in addition to the solution (x_0^2, $x_0 y_0$, $x_0 z_0$), the second solution is (x_0^2, $|y_0^2 - 3z_0^2|$, $2y_0 z_0$).

Applying Lemma 6 again to the solution (x_0^2, $x_0 y_0$, $x_0 z_0$), the resulting solution is found to be x_0^2(x_0^2, $|y_0^2 - 3z_0^2|$, $2y_0 z_0$). Thus, applying Lemma 6 a second time, only three solutions are obtained corresponding to x = x_0^4. For example, starting with the solution (7, 1, 4) of the Diophantine equation (3), applying Lemma 6, we get the two independent solutions (7^2, 7, 28) and (7^2, 47, 8). Then, applying Lemma 6 once more, we get three independent solutions, namely, (49^2, 343, 1372), (49^2, 2303, 392) and (49^2, 2017, 752). Continuing, one may get more and more independent solutions of the Diophantine equation (3).

With $n = 30$ in the identities (iii), (xi) and (xviii), we get

$$4.91^2 \equiv 4(7\times13)^2 = 182^2 = 181^2 + 3.11^2$$
$$= 118^2 + 3.80^2$$
$$= 74^2 + 3.96^2.$$

Thus, given the two independent solutions (7, 1, 4) and (13, 11, 4), in addition to the six independent solutions (182, 26, 104), (182, 169, 39), (182, 143, 65), (182, 7, 105), (182, 161, 49) and (182, 154, 56), the Diophantine equation (3) possesses three more solutions, namely, (182, 181, 11), (182, 118, 80) and (182, 74, 96). It may be mentioned here that, the last two solutions may also be obtained from Lemma 4. Note that, corresponding to $x = 7 \times 13 = 91$, there are only four independent solutions of the Diophantine equation (3), namely, (91, 13, 52), (91, 77, 28), (91, 59, 40) and (91, 37, 48). Then, considering the solution (19, 13, 8) and applying Lemma 4, eight independent solutions are found. They are (1729, 1079, 780), (1729, 1417, 572), (1729, 329, 980), (1729, 1673, 252), (1729, 193, 992), (1729, 1727, 48), 1729, 671, 920) and (1729, 1633, 328). Adding to them the solutions (1729, 247, 988), (1729, 1463, 532), (1729, 1121, 760) and (1729, 703, 912), in total 12 solutions are found corresponding to $x = 7 \times 13 \times 19$. Continuing the process, more and more solutions of the Diophantine equation (3) may be obtained.

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