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Short Communication

On some properties of the Diophantine equations $a^2 = b^2 \pm bc + c^2$

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ABSTRACT

This paper gives some properties related to the Diophantine equations $a^2 = b^2 \pm bc + c^2$. It has been shown that, in some cases, each of the two Diophantine equations has multiple solutions. The formulas for multiple solutions are derived.

The concept of the Smarandache function related triangles was introduced by Sastry (2000). Denoting by $T(a, b, c)$ the triangle ΔABC with sides $BC = a$, $CA = b$, $AB = c$, we have the following definition.

Definition 1: Two triangles $T(a, b, c)$ and $T(a', b', c')$ are said to be Smarandache function related if

$$S(a) = S(a'),\ S(b) = S(b'),\ S(c) = S(c'),$$

where the arithmetic function $S(n)$, called the Smarandache function, is defined as follows :

$$S(n) = min\{m : n \text{ divides } m!\},\ n \geq 1,$$

with

$$S(1) = 1.$$

Later, the idea was extended to the case of the pseudo Smarandache function, $Z(n)$, by Ashbacher (2000), where the function $Z(n)$, due to Kashihara (1996), is defined as follows :

$$Z(n) = min\{m : n \text{ divides } \frac{m(m+1)}{2}\},\ n \geq 1.$$

Of particular interest is the study of the 60 degrees and 120 degrees Smarandache function related and pseudo Smarandache function related triangles. It has been shown by Majumdar (2010) that the triangle ΔABC with $\angle CAB = 60^o$ satisfies the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \qquad (1)$$

and the triangle with $\angle CAB = 120^o$ satisfies the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2. \qquad (2)$$

Throughout this paper, a solution of the Diophantine equation (1) or (2) would be represented by the ordered triplet (a, b, c) with $c > b$. Thus, for example, (7, 3, 8) is a solution of equation (1), and (7, 3, 5) is a solution of equation (2). Note that, for either of the Diophantine equations (1) and (2), if (a, b, c) is a solution, so also is (ka, kb, kc) for any (positive) constant k. Two solutions would be called independent if and only if one is not a constant multiple of the other. For example, (7, 3, 8) and (7, 5, 8) are two independent solutions of the equation (1).

From (1) and (2), we see that they are particular cases of the Diophantine equation

$$x^2 = y^2 + 3z^2. \qquad (3)$$

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However, as has been pointed out by Majumdar (2018), neither (1) nor (2) is equivalent to the equation (3). For example, corresponding to $x = 14$, there are three solutions of the Diophantine equation (3), namely, (14, 13, 3), (14, 11, 5) and (14, 2, 8), whereas corresponding to $a = 7$, the Diophantine equation (1) has two solutions and (2) has only one solution. For more on equation (3), the reader is referred to the recent paper of Majumdar (2020).

Note that, the Diophantine equations (1) and (2) are equivalent to the following two Diophantine equations respectively

$$a^2 = b^2 - bc + c^2, \quad (4)$$

$$a^2 = b^2 + bc + c^2. \quad (5)$$

In connection with the Diophantine equation (4), we have the following five results.

Lemma 1: Let in the Diophantine equation (4), $c > b$ (> 0). Then, $c > a$.

Proof: If $c > b$ (> 0) then

$$a^2 = b^2 - bc + c^2 < b^2 - b^2 + c^2 = c^2.$$

Lemma 2: Let (a, b, c) be a solution of the Diophantine equation (4) (with $c > b$). Then, $(a, c - b, c)$ is also its solution.

Proof: See Lemma 5.1.4 in Majumdar (2010).

Lemma 3: Let (a, b, c) be a solution of the Diophantine equation (4) (with $c > b$). Then, $(a^2, c^2 - b^2, b(2c - b))$ is also its solution.

Proof: A cumbersome calculation shows that

$$(c^2 - b^2)^2 - b(c^2 - b^2)(2c - b) + b^2(2c - b)^2$$
$$= (b^2 - bc + b^2)^2 = (a^2)^2.$$

The above identity proves the lemma.

Lemma 4: Let (a, b, c) and (A, B, C) be two independent solutions of the Diophantine equation (4). Then, $(aA, cC - bB, b(C - B) + cB)$ and $(aA, c(C - B) + bB, bC - cB)$ are also its two solutions.

Proof: Tedious calculations give

$$(cC - bB)^2 - (cC - bB)[b(C - B) + cB] +$$
$$[b(C - B) + cB]^2$$
$$= (b^2 - bc + b^2)(B^2 - BC + C^2)$$
$$= [c(C - B) + bB]^2 -$$
$$[c(C - B) + bB](bC - cB) + (bC - cB)^2.$$

The identity above establishes the lemma.

Lemma 5: Let (a, b, c) be a solution of the Diophantine equation (4) with $c < 0$. Then, $(a, b, b - c)$ is also a solution of (4).

Proof: follows from the fact that

$$b^2 - b(b - c) + (b - c)^2 = b^2 - bc + c^2.$$

From the solution (7, 3, 8) of the Diophantine equation (4), we get the second solution (7, 5, 8) by Lemma 2. Using Lemma 3 to the solution (7, 3, 8), we get (7^2, 55, 39) as another solution of the equation (4). Again, using Lemma 4 to the two solutions (7, 3, 8) and (13, 7, 15) of the equation (4), we get the solutions (7×13, 99, 80) and (7×13, 85, –11) respectively, where the final solution gives, by Lemma 5, the solution (7×13, 96, 11).

The following four lemmas deal with the Diophantine equation (5).

Lemma 6: Let (a, b, c) be a solution of the Diophantine equation (4) (with $c > b$). Then, $(a, b, c - b)$ is a solution of the Diophantine equation (5).

Proof: See Lemma 5.1.5 in Majumdar (2010).

Lemma 7: Let (a, b, c) be a solution of the Diophantine equation (5) (with $c > b$). Then, (a^2, $c^2 - b^2$, $b(2c + b)$) is also its solution.

Proof: It is straightforward to deduce that

$$(c^2 - b^2)^2 + b(c^2 - b^2)(2c + b) + b^2(2c + b)^2$$
$$= (b^2 + bc + b^2)^2 = (a^2)^2,$$

which, in turn, proves the lemma.

Lemma 8: Let (a, b, c) and (A, B, C) be two independent solutions of the Diophantine equation (5). Then, (aA, $cC - bB$, $b(C + B) + cB$) and (aA, $bC - cB$, $c(C + B) + bB$) are also its two solutions.

Proof: To prove the lemma, we need to show that

$$(cC - bB)^2 + (cC - bB)[b(C + B) + cB]$$
$$+ [b(C + B) + cB]^2$$
$$= (b^2 + bc + b^2)(B^2 + BC + C^2)$$
$$= (bC - cB) + (bC - cB)[c(C + B) + bB]$$
$$+ [c(C + B) + bB]^2.$$

A time-consuming simplification would verify the above identity.

Lemma 9: Let (a, b, c) be a solution of the Diophantine equation (5) with $b < 0$. Then, (a, $-b$, $c + b$) is also a solution of (4).

Proof: Since

$$b^2 + b(c + b) + (c + b)^2 = b^2 + bc + c^2,$$

the lemma follows.

Since (7, 3, 8) is a solution of the Diophantine equation (4), by Lemma 6, (7, 3, 5) is a solution of the Diophantine equation (5). It may be mentioned here that the solution (7, 5, 8) of equation (4) gives the same solution to the equation (5). Now, starting with the solution (7, 3, 5) of the equation (5), we get, using Lemma 7, the solution (7^2, 16, 39) of the same equation.

Again, applying Lemma 8 to the two solutions (7, 3, 5) and (13, 7, 8) of the equation (5), we get its two new solutions, (7×13, 19, 80) and (7×13, -11, 96), where the last solution gives, by Lemma 9, the solution (7×13, 11, 85).

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