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**Short Communication**

# On the Diophantine equations $z^2 = k(k^2 + 3)$ and $z^2 = k(k^2 + 12)$

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**ABSTRACT**

This paper provides an analytical method of finding all the (positive, integral) solutions of the Diophantine equation $z^2 = k(k^2+3)$. We also prove analytically that the Diophantine equation $z^2 = k(k^2+12)$ has no positive, integer solution.

In studying the Diophantine equation $x^8+y^3 = z^4$, Cenberci and Peker (2012) faced the two Diophantine equations $z^2 = k(k^2+3)$ and $z^2 = k(k^2+12)$. To find the solutions of these equations, Cenberci and Peker (2012) took the advantage of GP/Pari (2019).

This paper uses an analytical approach to find all the solutions of the Diophantine equation $z^2 = k(k^2+3)$. This is given in Theorem 1.

*Theorem 1:* The only (positive, integral) solutions of the Diophantine equation

$$z^2 = k(k^2+3) \tag{1}$$

are $(k, z) = (1, 2), (3, 6), (12, 42)$.

*Proof:* From (1), we note that $k$ divides $z^2$, so that

$$z^2 = ak \text{ for some integer } a \geq 1.$$

Then, the Diophantine equation (1) takes the form

$$k^2 = a - 3. \tag{2}$$

From (2), we note that $a$ and $k$ are of opposite parity (if $a$ is odd then $k$ is even, and vice versa). Making use of (2), we get from (1),

$$z^4 = a^2 k^2 = a^2(a - 3). \tag{3}$$

To solve the Diophantine equation (3), we consider separately the three cases that may result:

*Case 1:* 3 does not divide $a$.

In this case, (3) has a solution if

$$a = m^2,\ a - 3 = n^4$$

for some integers $m \geq 1$ and $n \geq 1$, so that

$$(m - n^2)(m + n^2) = 3,$$

for which the only possibility is

$$m - n = 1,\ m + n = 3,$$

which gives $m = 2$, $n = 1$, so that

$$a = 4,\ k = 1,\ z = 2.$$

We thus get the first set of the solution mentioned in the theorem.

*Case 2:* 3 divides $a$.

In this case,

$$a = 3b \text{ for some integer } b \geq 1.$$

We then rewrite (3) as

$$z^4 = 3^3 b^2(b - 1). \tag{4}$$

The Diophantine equation (4) has a solution if, for some integers $u \geq 1$, $v \geq 1$,

$$b = u^2,\ b - 1 = 3v^4. \tag{5}$$

From (5), we get

$$(u - 1)(u + 1) = 3v^4. \tag{6}$$

We now consider all the possible cases of (6).

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*Case A:* $u + 1 = v^4$, $u - 1 = 3$.

Clearly, in this case, there is no solution.

*Case B:* $u + 1 = v^3$, $u - 1 = 3v$.

In this case, since

$$2 = v(v^2 - 3),$$

the only solution is $v = 2$ (so that $u = 7$).

*Case C:* $u + 1 = 3v^4$, $u - 1 = 1$.

In this case, $v = 1$ ($u = 7$) is the only solution.

*Case D:* $u + 1 = 3v^3$, $u - 1 = v$.

Here, since

$$2 = v(3v^2 - 1),$$

The only solution is $v = 1$.

*Case E*: $u + 1 = 3v^2$, $u - 1 = v^2$.

In this case, the only solution is $v = 1$.

Thus, in Case 2, corresponding to the Cases B – C, we get two more solutions of the Diophantine equation (1), namely,

$$(k, z) = (3, 6), (12, 42).$$

*Case 3:* When, in (4),

$$b - 1 = 3b^2 w^4$$

for some integer $w \geq 1$.

However, since in the resulting quadratic equation in $b$, that is, in $3b^2 w^4 - b + 1 = 0$,

$$\text{discriminant} = 1 - 12w^4 < 0.$$

Thus, this case cannot occur.

All these complete the proof of the theorem.

Next, we focus our attention to the equation $z^2 = k(k^2 + 12)$. For this equation, we have the following result.

*Theorem 2:* The Diophantine equation

$$z^2 = k(k^2 + 12) \tag{7}$$

possesses only the trivial solution (0, 0).

*Proof:* We consider the following two cases separately.

*Case 1: k* divides *z*, so that

$$z = ck \text{ for some integer } c \geq 1.$$

Then, (7) becomes, after rearranging the terms,

$$k^2 - c^2 k + 12 = 0.$$

So that the above quadratic equation in $k$ has a solution, the discriminant must be a perfect square. Thus,

$$c^4 - 48 = d^2 \text{ for some integer } d \geq 1.$$

Rewriting the above equation as

$$(c^2 - d)(c^2 + d) = 48,$$

and noting that $c^2 + d$ and $c^2 - d$ must be of the same polarity (that is, both even or both odd), we need only consider the following three possibilities :

*Case I:* $c^2 + d = 24$, $c^2 - d = 2$,

*Case II:* $c^2 + d = 12$, $c^2 - d = 4$,

*Case III:* $c^2 + d = 8$, $c^2 - d = 6$.

It is now an easy exercise to verify that none of the above systems of equations has a solution.

Case 2: $k$ divides $z^2$, so that

$$z^2 = ek \text{ for some integer } e \geq 1.$$

Then, (7) becomes, after simplification,

$$k^2 = e - 12,$$

so that

$$z^4 = e^2(e - 12). \tag{8}$$

Now, the Diophantine equation (8) has a solution if

$$e = f^2,\ e - 12 = g^4,$$

for some integers $f \geq 1$, $g \geq 1$. Then, we must have

$$(f - g^2)(f + g^2) = 12.$$

Noting that $f - g^2$ and $f + g^2$ must have the same polarity, we have only one possibility, namely,

$$f - g^2 = 2, f + g^2 = 6.$$

The above system gives $g^2 = 2$, which has no integer solution.

Thus, the Diophantine equation (7) has no (positive, integer) solution, which we intended to show.

This paper gives the solution of two elliptic curves (see, for example, Silverman (1986), of the form

$$z^2 = k(k^2 + n),$$

with $n = 3, 12$. Of the two equations, one has a finite number of (positive, integer) solutions, while the other has no solution. It would be an interesting problem to find all the solutions of the above equation, when $n$ is any positive integer.

We conclude the paper with the following result.

*Theorem 3:* In the Diophantine equation

$$x^8 + y^3 = z^{2k}, \tag{9}$$

where gcd $(x, y, z) = 1$, and $k$ ($\geq 1$) is an integer, $z^k - x^4$ does not divide $y$.

*Proof:* First we rewrite the equation (9) as

$$y^3 = (z^k - x^4)(z^k + x^4).$$

Now, assuming that $z^k - x^4$ divides $y$, we can have the following two cases only:

*Case 1:* $z^k + x^4 = y^3, z^k - x^4 = 1$.

Clearly, this case cannot happen, noting that $z^k - x^4 = 1$ if and only if $z = 1, x = 0$.

*Case 2:* $z^k + x^4 = y^2, z^k - x^4 = y$.

Here, if $y$ divides $z$, then $y$ must also divide $x$, and we reach a contradiction (to the fact that $x$ and $z$ are relatively prime). Thus, $y$ divides neither $z$ nor $x$. Now, from the second equation, $y$ divides $z^k - x^4$, so that writing

$$z^k + x^4 = (z^k - x^4) + 2x^4,$$

we see that $y$ does not divide $z^k + x^4$, which is contrary to our assumption.

Because of Theorem 3, we see that, in studying the Diophantine equation (9), we have to look for a solution(s) of the following system of Diophantine equations:

$$z^k + x^4 = m^3, z^k - x^4 = n^3.$$

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