APPROXIMATE SOLUTION OF A FOURTH ORDER WEAKLY NON-LINEAR DIFFERENTIAL SYSTEM WITH STRONG DAMPING AND SLOWLY VARYING COEFFICIENTS BY UNIFIED KBM METHOD

M. ALHAZ UDDIN[*] AND M. A. SATTAR

Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh

ABSTRACT

The unified Krylov-Bogoliubov-Mitropolskii (KBM) method is used for determining the analytical approximate solution of a fourth order weakly nonlinear differential system with strong damping and slowly varying coefficients when a pair of eigen-values of the unperturbed equation is a multiple (approximately or perfectly) of the other pair or pairs. In a damped case, one of the natural frequencies of the linearized equation may be a multiple of the other. The analytical first order approximate solution for different initial conditions shows a good coincidence with those obtained by the numerical procedure. The method is illustrated by an example.

Key words**:** Perturbation method, Weak nonlinearity, Oscillatory process, Strong damping, Varying coefficients

INTRODUCTION

The method of KBM is convenient and one of the widely used techniques to obtain the analytical approximate solutions of nonlinear differential systems. It is perhaps noteworthy that because of importance of physical process involving damping, Popov (1956) extended this method to damped oscillatory systems. Murty *et al.* (1971) used Popov's method to obtain over-damped solutions of nonlinear differential equations, which were the basis of unified theory of Murty (1969). Later this method has been extended to damped oscillatory and purely non oscillatory systems with slowly varying coefficients by Bojadziev and Edwards (1981). Arya and Bojadziev (1981) have studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Feshchenko *et al.* (1966) have presented a brief way to determine KBM (Krylov *et al.* 1947, Bogoliubov *et al.* 1961)) solution (first order) of a second or third order nonlinear differential system. Arya and Bojadziev (1980) have also studied a system of second order nonlinear hyperbolic partial differential equation with slowly varying coefficients. Alam (1997) has investigated a unified KBM method for solving nonlinear system of order $n \geq 2$. Further, Alam (2002) has investigated a unified KBM method for solving of second and third nonlinear systems with constant coefficients. In another paper, Alam (2003) has also investigated a unified KBM method for solving nonlinear system of order $n \geq 3$ with slowly varying coefficients. Recently Alam and

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[*] Author for correspondence. <alhazuddin@yahoo.com>.

Sattar (2004) have also presented an asymptotic method for third order nonlinear system with slowly varying coefficients. Recently Akbar *et al.* (2006) have studied a fourth order nonlinear differential equation with constant coefficients. Most of the authors have studied the second and third order nonlinear differential systems for both constant and varying coefficients to obtain the analytical first order approximate solutions. The complicated and no less important case of a fourth order nonlinear differential equation with strong damping and slowly varying coefficients has remained almost untouched. The main goal of this paper is to fill this gap.

METHOD

Let us consider a fourth order weakly nonlinear ordinary differential equation with slowly varying coefficients in the following form:

$$x^{(4)} + c_1(\tau)\dddot{x} + c_2(\tau)\ddot{x} + c_3(\tau)\dot{x} + c_4(\tau)x = \varepsilon f(x,\dot{x},\ddot{x},\dddot{x},\tau) \tag{1}$$

where the over dots represent the time derivatives, ε is a small positive parameter which measures the strength of the nonlinearity, $\tau = \varepsilon t$ is the slowly varying time, $c_j(\tau) \geq 0$, $j = 1, 2, 3, 4$ are slowly varying coefficients and f is a given nonlinear function. The coefficients are slowly varying in the sense that their time derivatives are proportional to ε (Alam 2003).

By setting $\varepsilon = 0, \tau = \tau_0 = \text{constant}$ in Eq. (1), we obtain the solution of the unperturbed equation. We assume that the unperturbed equation of Eq. (1) has four eigenvalues $\lambda_j(\tau_0)$, $j = 1, 2, 3, 4$; where $\lambda_j(\tau_0)$ are constants, but if $\varepsilon \neq 0$ then $\lambda_j(\tau)$ are varying slowly with time t. The solution of the linearized equation of Eq. (1) has the following form:

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{\lambda_j(\tau_0)t}, \tag{2}$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

Now we are going to choose a solution of Eq. (1) that reduces to Eq. (2) as a limit $\varepsilon \to 0$ in the following form according to the KBM (Krylov *et al.* 1947, Bogoliubov *et al.* 1961) method:

$$x(t,\varepsilon) = \sum_{j=1}^{4} a_j(t) + \varepsilon u_1(a_1,a_2,a_3,a_4,\tau) + \varepsilon^2 u_2(a_1,a_2,a_3,a_4,\tau) + \varepsilon^3 \cdots, \tag{3}$$

where u_1 is a function of a_j, $j = 1, 2, 3, 4$ and each a_j satisfies the following first order differential equation:

$$\dot{a}_j = \lambda_j a_j + \varepsilon A_j(a_1,a_2,a_3,a_4,\tau) + \varepsilon^2 B_j(a_1,a_2,a_3,a_4,\tau) + \varepsilon^3 \cdots. \tag{4}$$

Confining only to the first few terms, 1, 2, 3… in the series expansions of Eq. (3) and Eq. (4), we evaluate the functions u_1, u_2, $\cdots$ and A_j, B_j, $\cdots$, $j = 1, 2, 3, 4$ such that each $a_j(t)$ appearing in Eq. (3) and Eq. (4) satisfies the given differential equation (1) with an accuracy of ε^{m+1} (Alam 2003). In order to determine these functions it is assumed that the functions u_1, u_2, $\cdots$ do not contain the fundamental terms (Alam *et al.* 1997, Alam 2003, Bojadziev *et al.* 1981, Murty 1971) which are included in the series expansions (3) at order ε^0. Now differentiating Eq. (3) four-times with respect to time t and using the relations Eq. (4) and by substituting the values of $x^{(4)}$, $\dddot{x}$, $\ddot{x}$, $\dot{x}$ and x into the original Eq. (1) with the slowly varying coefficients

$$c_1(\tau) = -(\lambda_1(\tau) + \lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau)),$$

$$c_2(\tau) = \lambda_1(\tau)\lambda_2(\tau) + \lambda_1(\tau)\lambda_3(\tau) + \lambda_1(\tau)\lambda_4(\tau) + \lambda_2(\tau)\lambda_3(\tau) + \lambda_2(\tau)\lambda_4(\tau) + \lambda_3(\tau)\lambda_4(\tau),$$

$$c_3(\tau) = -(\lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau) + \lambda_1(\tau)\lambda_2(\tau)\lambda_4(\tau) + \lambda_1(\tau)\lambda_3(\tau)\lambda_4(\tau) + \lambda_2(\tau)\lambda_3(\tau)\lambda_4(\tau))$$

and $c_4(\tau) = \lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)\lambda_4(\tau)$ and by expanding the right hand side of Eq. (1) by Taylor series and by equating the coefficients of ε on both sides, we obtain the following equation:

$$\prod_{j=1}^{4}(\Omega - \lambda_j)u_1 + \sum_{j=1}^{4}(\prod_{k=1,k\neq j}^{4}(\Omega - \lambda_k)A_j) + \sum_{j=1}^{4}\frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k\lambda_j^{(2-k)})\lambda_j'a_j$$

$$= f^{(0)}(a_1, a_2, a_3, a_4, \tau) \tag{5}$$

where

$$\Omega = \sum_{j=1}^{4}\lambda_j a_j\frac{\partial}{\partial a_j}, \ \lambda_j' = \frac{d\lambda_j}{d\tau}, j=1.2,\cdots 4, f^{(0)}(a_1, a_2, a_3, a_4, \tau) = f(x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0, \tau)$$

and $\quad x_0 = \sum_{j=1}^{4}a_j.$

We have already assumed that u_1 does not contain the fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t$, $t\sin t$ and te^{-t}. Under these restrictions, we are able to solve Eq. (5) by separating this into five individual equations for the unknown functions u_1 and A_j. In general, the functions $f^{(0)}$ and u_1 are expanded in Taylor's series in the following forms:

$$f^{(0)} = \sum_{m_1=0,m_2=0,m_3=0m_4=0}^{\infty,\infty,\infty,\infty}F_{m_1,m_2,m_3,m_4}(\tau)a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}, \tag{6}$$

$$u_1 = \sum_{m_1=0,m_2=0,m_3=0m_4=0}^{\infty,\infty,\infty,\infty}U_{m_1,m_2,m_3,m_4}(\tau)a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}. \tag{7}$$

The eigen-values of the unperturbed equation can be written as $-\mu_l(\tau_0) \pm i\omega_l(\tau_0)$, where $l = 1, 2$. For the above restrictions, it guaranties that u_1 must exclude all terms

with $a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}$ of $f^{(0)}$, where $m_{2l-1} - m_{2l} = \pm 1$. Since according to the linear approximation (*i.e.* $\varepsilon \to 0$), $a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}$ becomes $e^{\omega_l t}$ when $m_{2l-1} - m_{2l} = 1$ or $e^{-\omega_l t}$ when $m_{2l-1} - m_{2l} = -1$. It is noticed that $e^{\pm \omega_l t}$ are known as the fundamental terms (Alam *et al.* 1997, Alam 2003, Bojdziev *et al.* 1981, Murty 1971). Usually these are included in equations A_j. Also, it is restricted (by Krylov *et al.* 1947, Bogoliubov *et al.* 1961) that the functions A_j are independent of the fundamental terms.

Then the equations for u_1 and A_j, $j = 1, 2, 3, 4$ are written as

$$\prod_{j=1}^{4}(\Omega - \lambda_j)u_1 = \sum_{m_{2l-1}=0, m_{2l}=0}^{\infty,\infty} F_{,m_{2l-1},m_{2l}}(\tau) a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}} \qquad m_{2l-1} - m_{2l} \neq 0, \pm 1 \tag{8}$$

and

$$(\prod_{k=1,k\neq 2l-1}^{4}(\Omega - \lambda_k))A_{2l-1} + \frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k \lambda_{2l-1}^{(2l-k-2)})\lambda_{2l-1}' a_{2l-1}$$
$$= \sum_{m_{2l-1}=0, m_{2l}=0}^{\infty,\infty} F_{m_{2l-1},m_{2l}} a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}, \quad m_{2l-1} - m_{2l} = 1, \tag{9}$$

$$(\prod_{k=1,k\neq 2l}^{4}(\Omega - \lambda_k))A_{2l} + \frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k \lambda_{2l}^{(2l-k-2)})\lambda_{2l}' a_{2l}$$
$$= \sum_{m_{2l-1}=0, m_{2l}=0}^{\infty,\infty} F_{m_{2l-1},m_{2l}} a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}, \quad m_{2l-1} - m_{2l} = -1. \tag{10}$$

To determine the particular solutions of Eqs. (8) - (10), we have replaced the operator Ω by $\sum_{j=1}^{4} m_j \lambda_j$, since we know that $\Omega(a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}) = \sum_{j=1}^{n} m_j \lambda_j (a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}})$. Hence the determination of the analytical first order approximate solution of Eq. (1) is clear to us. We notice that the solution Eq. (3) is not a standard form of KBM method and is presented in terms of some unusual variables. Therefore, the solution obtained by formula of Eq. (1) is transformed to the formal form by replacing the unusual variables by amplitudes and phases variables in the forms:

$$a_{2l-1} = \frac{1}{2} b_l e^{i\varphi_l}, \quad a_{2l} = \pm \frac{1}{2} b_l e^{-i\varphi_l}, \; l = 1, 2 \tag{11}$$

Thus the determination of the first order approximate solution of Eq. (1) is clear. The method can be carried out to higher order approximations in a similar way. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a low order, usually the first order (Krylov *et al.* 1947, Bogoliubov *et al.* 1961, Alam 2003, Akbar *et al.* 2006).

EXAMPLE

To obtain the practical working of the above method, we consider the following fourth order weakly nonlinear differential equation with slowly varying coefficients in the following form:

$$x^{(4)} + c_1(\tau)\dddot{x} + c_2(\tau)\ddot{x} + c_3(\tau)\dot{x} + c_4(\tau)x = \varepsilon x^3, \tag{12}$$

where $f(x, \dot{x}, \ddot{x}, \dddot{x}, \tau) = x^3$ and $x_0 = a_1 + a_2 + a_3 + a_4$.

Now

$$f^{(0)} = a_1^3 + a_2^3 + a_3^3 + a_4^3 + 3(a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_3 + 2a_1 a_2 a_3 + a_2^2 a_3 + a_1^2 a_4$$
$$2a_1 a_2 a_4 a_2^2 a_4 + a_1 a_3^2 + 2a_1 a_3 a_4 + a_1 a_4^2 + a_2 a_3^2 + 2a_2 a_3 a_4 + a_2 a_4^2 + a_3^2 a_4 + a_3 a_4^2). \tag{13}$$

Substituting the values of $f^{(0)}$ in Eq. (5) and according to the above restrictions, we obtain five equations for A_1, A_2, A_3, A_4 and u_1 whose solutions are respectively given by

$$A_1 = -\frac{(3\lambda_1^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}$$

$$+\frac{3a_1^2 a_2}{2\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)} + \frac{3a_1 a_3^2}{(\lambda_1 + \lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + 2\lambda_3 - \lambda_4)}$$

$$+\frac{3a_1 a_4^2}{(\lambda_1 + \lambda_4)(\lambda_1 + 2\lambda_4 - \lambda_2)(\lambda_1 + 2\lambda_4 - \lambda_3)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)},$$

$$A_2 = -\frac{(3\lambda_2^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}$$

$$+\frac{3a_1 a_2^2}{2\lambda_2(\lambda_1 + 2\lambda_2 - \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_4)} + \frac{3a_2 a_3^2}{(\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)}$$

$$+\frac{3a_2 a_4^2}{(\lambda_2 + \lambda_4)(\lambda_2 + 2\lambda_4 - \lambda_1)(\lambda_2 + 2\lambda_4 - \lambda_3)} + \frac{6a_2 a_3 a_4}{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1)},$$

$$A_3 = -\frac{(3\lambda_3^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} +$$

$$\frac{3a_1^2 a_3}{(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 - \lambda_2)(2\lambda_1 + \lambda_3 - \lambda_4)} + \frac{3a_2^2 a_3}{(\lambda_2 + \lambda_3)(2\lambda_2 + \lambda_3 - \lambda_1)(2\lambda_2 + \lambda_3 - \lambda_4)}$$

$$+\frac{3a_3^2 a_4}{2\lambda_3(2\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)},$$

$$A_4 = -\frac{(3\lambda_4^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_4' a_4}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} +$$

$$\frac{3a_1^2 a_4}{(\lambda_1 + \lambda_4)(2\lambda_1 + \lambda_4 - \lambda_2)(2\lambda_1 + \lambda_4 - \lambda_3)} + \frac{3a_2^2 a_4}{(\lambda_2 + \lambda_4)(2\lambda_2 + \lambda_4 - \lambda_1)(2\lambda_2 + \lambda_4 - \lambda_3)} \tag{14}$$

$$+\frac{3a_3 a_4^2}{2\lambda_4(\lambda_3 + 2\lambda_4 - \lambda_1)(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{6a_1 a_2 a_4}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)},$$

and

$$u_1 = C_1 a_1^3 + D_1 a_2^3 + C_2 a_3^3 + D_2 a_4^3, \tag{15}$$

where

$$C_1 = \frac{1}{2\lambda_1(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)}, \; D_1 = \frac{1}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)(3\lambda_2 - \lambda_3)},$$

$$C_2 = \frac{1}{2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)(3\lambda_3 - \lambda_4)}, \; D_2 = \frac{1}{2\lambda_4(3\lambda_4 - \lambda_1)(3\lambda_4 - \lambda_2)(3\lambda_4 - \lambda_3)}. \tag{16}$$

Now by substituting the values of A_1, A_2, A_3 and A_4 from Eq. (14) into Eq. (4), we get

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{(3\lambda_1^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}\right.$$

$$+\frac{3a_1^2 a_2}{2\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)} + \frac{3a_1 a_3^2}{(\lambda_1 + \lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + 2\lambda_3 - \lambda_4)}$$

$$\left.+\frac{3a_1 a_4^2}{(\lambda_1 + \lambda_4)(\lambda_1 + 2\lambda_4 - \lambda_2)(\lambda_1 + 2\lambda_4 - \lambda_3)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)}\right),$$

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(-\frac{(3\lambda_2^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}\right.$$

$$+\frac{3a_1 a_2^2}{2\lambda_2(\lambda_1 + 2\lambda_2 - \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_4)} + \frac{3a_2 a_3^2}{(\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)}$$

$$\left.+\frac{3a_2 a_4^2}{(\lambda_2 + \lambda_4)(\lambda_2 + 2\lambda_4 - \lambda_1)(\lambda_2 + 2\lambda_4 - \lambda_3)} + \frac{6a_2 a_3 a_4}{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1)}\right),$$

$$\dot{a}_3 = \lambda_3 a_3 + \varepsilon\left(-\frac{(3\lambda_3^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} + \right.$$

$$\frac{3a_1^2 a_3}{(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 - \lambda_2)(2\lambda_1 + \lambda_3 - \lambda_4)} + \frac{3a_2^2 a_3}{(\lambda_2 + \lambda_3)(2\lambda_2 + \lambda_3 - \lambda_1)(2\lambda_2 + \lambda_3 - \lambda_4)}$$

$$\left.+\frac{3a_3^2 a_4}{2\lambda_3(2\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)}\right),$$

$$\dot{a}_4 = \lambda_4 a_4 + \varepsilon\left(-\frac{(3\lambda_4^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_4' a_4}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} + \right.$$

$$\frac{3a_1^2 a_4}{(\lambda_1 + \lambda_4)(2\lambda_1 + \lambda_4 - \lambda_2)(2\lambda_1 + \lambda_4 - \lambda_3)} + \frac{3a_2^2 a_4}{(\lambda_2 + \lambda_4)(2\lambda_2 + \lambda_4 - \lambda_1)(2\lambda_2 + \lambda_4 - \lambda_3)} \tag{17}$$

$$\left.+\frac{3a_3 a_4^2}{2\lambda_4(\lambda_3 + 2\lambda_4 - \lambda_1)(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{6a_1 a_2 a_4}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)}\right).$$

For a damped solution of Eq. (12), we may substitute $\lambda_{1,2} = -\mu_1(\tau) \pm i\omega_1(\tau)$, $\lambda_{3,4} = -\mu_2(\tau) \pm i\omega_2(\tau)$ and using the transformation equations $a_1 = \dfrac{1}{2}ae^{i\varphi_1}$, $a_2 = \dfrac{1}{2}ae^{-i\varphi_1}$ $a_3 = \dfrac{1}{2}be^{i\varphi_2}$, $a_4 = \dfrac{1}{2}be^{-i\varphi_2}$ into Eq. (17) and by separating the real and imaginary parts, we obtain the following variational equations for the amplitudes and phases variables:

$$
\begin{aligned}
\dot{a} &= -\mu_1 a + \varepsilon(l_1 a + l_2 a^3 + l_3 ab^2 + ab^2(E_1 \cos 2\varphi_2 + F_1 \sin 2\varphi_2)), \\
\dot{\varphi}_1 &= \omega_1(\tau) + \varepsilon(m_1 + m_2 a^2 + m_3 b^2 + b^2(E_2 \cos 2\varphi_2 + F_2 \sin 2\varphi_2)), \\
\dot{b} &= -\mu_2 b + \varepsilon(p_1 b + p_2 a^2 b + p_3 b^3 + a^2 b(E_3 \cos 2\varphi_1 + F_3 \sin 2\varphi_1)), \\
\dot{\varphi}_2 &= \omega_2(\tau) + \varepsilon(q_1 + q_2 a^2 + q_3 b^2 + a^2(E_4 \cos 2\varphi_1 + F_4 \sin 2\varphi_1)),
\end{aligned}
\tag{18}
$$

and the first correction term u_1 is obtained as

$$
u_1 = a^3(c_1 \cos 3\varphi_1 + d_1 \sin 3\varphi_1) + b^3(c_2 \cos 3\varphi_2 + d_2 \sin 3\varphi_2),
\tag{19}
$$

where

$$
l_1 = -\frac{\begin{aligned}&(\omega_1'(((\mu_1-\mu_2)^2 - 5\omega_1^2 + \omega_2^2)((\mu_1-\mu_2)^2 - \omega_1^2 + \omega_2^2) + 12\omega_1^2(\mu_1-\mu_2)^2)\\&+4\mu_1'\omega_1(\mu_1-\mu_2)((\mu_1-\mu_2)^2 + \omega_1^2 + \omega_2^2))\end{aligned}}{2\omega_1((\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)},
$$

$$
m_1 = \frac{\begin{aligned}&4\omega_1\omega_1'(\mu_1-\mu_2)((\mu_1-\mu_2)^2 + \omega_1^2 + \omega_2^2) - \mu_1'(((\mu_1-\mu_2)^2 - 5\omega_1^2 + \omega_2^2)\\&((\mu_1-\mu_2)^2 - \omega_1^2 + \omega_2^2) + 12\omega_1^2(\mu_1-\mu_2)^2)\end{aligned}}{2\omega_1((\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)},
$$

$$
l_2 = -\frac{3(\mu_1((3\mu_1-\mu_2)^2 - \omega_1^2 + \omega_2^2) - 2\omega_1^2(3\mu_1-\mu_2))}{8(\mu_1^2 + \omega_1^2)((3\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)},
$$

$$
m_2 = -\frac{3\omega_1((3\mu_1-\mu_2)(5\mu_1-\mu_2) - \omega_1^2 + \omega_2^2)}{8(\mu_1^2 + \omega_1^2)((3\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)},
$$

$$
l_3 = -\frac{3(\mu_2((\mu_1+\mu_2)^2 - \omega_1^2 + \omega_2^2) - 2\omega_2^2(\mu_1+\mu_2))}{4(\mu_2^2 + \omega_1^2)((\mu_1+\mu_2)^2 + (\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2 + (\omega_1+\omega_2)^2)},
$$

$$
m_3 = -\frac{3\omega_1((\mu_1+\mu_2)(\mu_1+3\mu_2) - \omega_1^2 + \omega_2^2)}{4(\mu_2^2 + \omega_1^2)((\mu_1+\mu_2)^2 + (\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2 + (\omega_1+\omega_2)^2)},
$$

$$
l_4 = -\frac{3(\mu_2((\mu_1+\mu_2)^2 - (\omega_1+\omega_2)(\omega_1+3\omega_2)) - 2(\mu_1+\mu_2)(\omega_1+\omega_2)(\omega_1+2\omega_2))}{8(\mu_2^2 + (\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2 + (\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2 + (\omega_1+3\omega_2)^2)},
$$

$$l_4^* = \frac{3(2\mu_2(\mu_1+\mu_2)(\omega_1+2\omega_2)+(\omega_1+\omega_2)((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(\omega_1+3\omega_2)))}{8(\mu_2^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+3\omega_2)^2)},$$

$$m_4 = -l_4^*, m_4^* = l_4,$$

$$l_5 = -\frac{3(\mu_2((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(\omega_1-3\omega_2))-2(\mu_1+\mu_2)(\omega_1-\omega_2)(\omega_1-2\omega_2))}{8(\mu_2^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-3\omega_2)^2)},$$

$$l_5^* = -\frac{3(2\mu_2(\mu_1+\mu_2)(\omega_1-2\omega_2)+(\omega_1-\omega_2)((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(\omega_1-3\omega_2)))}{8(\mu_2^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-3\omega_2)^2)},$$

$$m_5 = l_5^*, m_4^* = -l_5,$$

$$p_1 = -\frac{\left(\omega_2'(((\mu_1-\mu_2)^2+\omega_1^2-\omega_2^2)((\mu_1-\mu_2)^2+\omega_1^2-5\omega_2^2)+12\omega_2^2(\mu_1-\mu_2)^2)\right.}{\left.-4\mu_2'\omega_2(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2)\right)}{2\omega_2((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$q_1 = -\frac{\left(4\omega_2\omega_2'(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2)+\mu_2'(((\mu_1-\mu_2)^2+\omega_1^2-\omega_2^2)\right.}{\left.((\mu_1-\mu_2)^2+\omega_1^2-5\omega_2^2)+12\omega_2^2(\mu_1-\mu_2)^2)\right)}{2\omega_2((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$p_2 = -\frac{3(\mu_1((\mu_1+\mu_2)^2+\omega_1^2-\omega_2^2)-2\omega_2^2(\mu_1+\mu_2))}{4(\mu_1^2+\omega_2^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)},$$

$$q_2 = -\frac{3\omega_2((\mu_1+\mu_2)(3\mu_1+\mu_2)+\omega_1^2-\omega_2^2)}{4(\mu_1^2+\omega_2^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)},$$

$$p_3 = -\frac{3(\mu_2((\mu_1-3\mu_2)^2+\omega_1^2-\omega_2^2)+2\omega_2^2(\mu_1-3\mu_2))}{8(\mu_2^2+\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$q_3 = -\frac{3\omega_2((\mu_1-3\mu_2)(\mu_1-5\mu_2)+\omega_1^2-\omega_2^2)}{8(\mu_2^2+\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$p_4 = -\frac{3(\mu_1((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(3\omega_1+\omega_2))-2(\mu_1+\mu_2)(\omega_1+\omega_2)(2\omega_1+\omega_2))}{8(\mu_1^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$p_4^* = \frac{3(2\mu_1(\mu_1+\mu_2)(2\omega_1+\omega_2)+(\omega_1+\omega_2)((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(3\omega_1+\omega_2)))}{8(\mu_1^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$q_4 = -p_4^*, q_4^* = p_4,$$

$$p_5 = -\frac{3(\mu_1((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(3\omega_1-\omega_2))-2(\mu_1+\mu_2)(\omega_1-\omega_2)(2\omega_1-\omega_2))}{8(\mu_1^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1-\omega_2)^2)},$$

$$p_5^* = \frac{3(2\mu_1(\mu_1+\mu_2)(2\omega_1-\omega_2)+(\omega_1-\omega_2)((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(3\omega_1-\omega_2)))}{8(\mu_1^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1-\omega_2)^2)},$$

$$q_5 = p_5^*, q_5^* = -p_5,$$

$$E_1 = l_4 + l_5, \; F_1 = l_4^* + l_5^*, \;\; E_2 = m_4 + m_5, \; F_2 = m_4^* + m_5^*,$$
$$E_3 = p_4 + p_5, \; F_3 = p_4^* + p_5^*, \;\; E_4 = q_4 + q_5, \; F_4 = q_4^* + q_5^*, \tag{20}$$

and

$$c_1 = \frac{(\mu_1^2-2\omega_1^2)((3\mu_1-\mu_2)^2-9\omega_1^2+\omega_2^2)-18\mu_1\omega_1^2(3\mu_1-\mu_2)}{16(\mu_1^2+\omega_1^2)(\mu_1^2+4\omega_1^2)((3\mu_1-\mu_2)^2+(3\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$d_1 = -\frac{3\omega_1(\mu_1((3\mu_1-\mu_2)^2-9\omega_1^2+\omega_2^2)+2(3\mu_1-\mu_2)(\mu_1^2-2\omega_1^2))}{16(\mu_1^2+\omega_1^2)(\mu_1^2+4\omega_1^2)((3\mu_1-\mu_2)^2+(3\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$c_2 = \frac{(\mu_2^2-2\omega_2^2)((\mu_1-3\mu_2)^2+\omega_1^2-9\omega_2^2)+18\mu_2\omega_2^2(\mu_1-3\mu_2)}{16(\mu_2^2+\omega_2^2)(\mu_2^2+4\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-3\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+3\omega_2)^2)},$$

$$d_2 = -\frac{3\omega_2(\mu_2((\mu_1-3\mu_2)^2+\omega_1^2-9\omega_2^2)-2(\mu_1-3\mu_2)(\mu_2^2-2\omega_2^2))}{16(\mu_2^2+\omega_2^2)(\mu_2^2+4\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-3\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+3\omega_2)^2)}. \tag{21}$$

Thus the analytical first order approximate solution (improved) of Eq. (12) is obtained by $x(t,\varepsilon) = a\cos\varphi_1 + b\cos\varphi_2 + \varepsilon u_1,$ (22)

where the amplitudes a, b and phases φ_1, φ_2 are the solutions of Eq. (18) and u_1 is given by Eq. (19).

RESULTS AND DISCUSSION

A standard form of KBM method is presented to obtain the analytical approximate solution of a fourth order nonlinear differential equation with strong damping and slowly varying coefficients with small nonlinearity. The KBM method was originally developed for obtaining the periodic solutions of second-order nonlinear systems by Krylov and Bogoliubov (1947) and later it was amplified and justified by Bogoliubov and Mitropolskii (1961). The method is not only limited to second-order nonlinear problems, but also useful in third-order (Alam 2002, 2003) and fourth order (Akbar 2006) nonlinear systems. A general solution has been found for the damped nonlinear differential equation with slowly varying coefficients based on the unified KBM (Alam *et al.* 1997, Alam 2002, 2003, Bogoliubov *et al.* 1961, Krylov *et al.* 1947) method.

We have solved four simultaneous differential equations for amplitude(s) and phase(s) variables and a partial differential equation for u_1 involving four independent variables of amplitudes and phases. Also we are able to solve all the equations of A_j, $j = 1, 2, 3, 4$ and u_1 by a unified formula. In a particular case, we are forced to assume that $\mu_l(\tau)$, $l = 1, 2$ are constants, $\omega_1(\tau) = 2\,\omega_2(\tau)$ and $\omega_2(\tau) = \omega_0\,e^{-h\tau}$ are varying slowly with time t, where ω_0 and h are constants. Figures are drawn to compare between the analytical first order approximate solutions obtained by the perturbation method and those obtained by the numerical procedure (fourth-order *Runge-Kutta* method) for several damping effects. Moreover, this method is able to give the required results when the coefficients of the given nonlinear differential equation become constants $(h = 0)$. From the Figs. (1)- (2), it is seen that the new analytical approximate solution shows a good agreement with the corresponding numerical solution (considered to be exact).
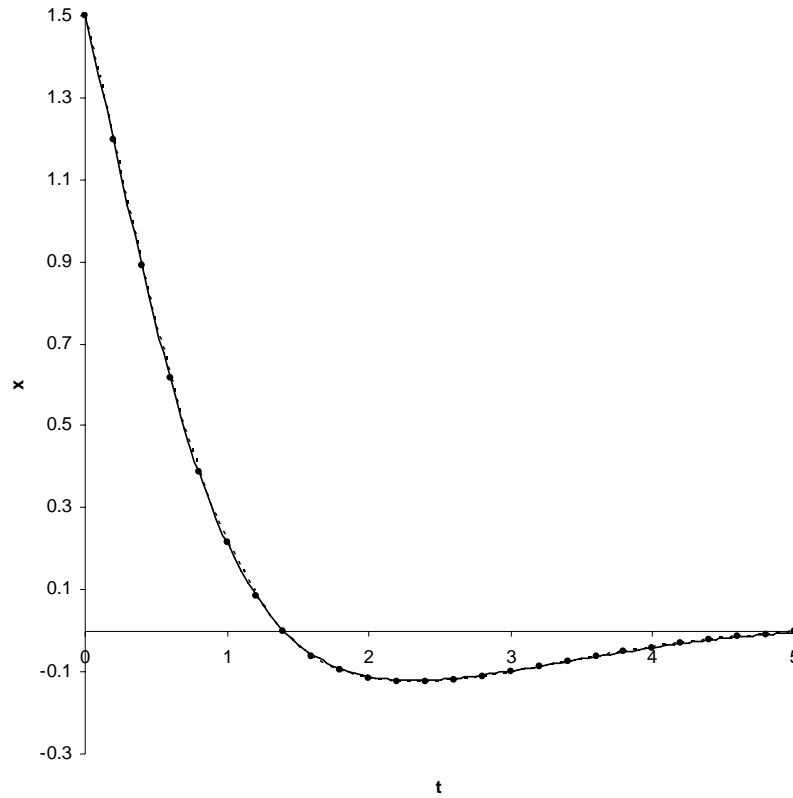

Fig. 1. First approximate solution (denoted by $-\bullet-$) of Eq. (13), with the initial conditions $[x(0)$ $= 1.49992,\ \dot{x}(0) = -1.43242,\ \ddot{x}(0) = -1.43652,\ \dddot{x}(0) = 8.767772]$ or $a_0 = 1.5$, $\varphi_1 = 0$, $b = 1.0$ $\varphi_2 = 0$, with $\mu_1 = 1.5$, $\mu_2 = 0.75$, $\omega_0 = 1.0$, $h = 0.5$, $\varepsilon = 0.1$, $\omega_1 = 2\omega_2$, $\omega_2 = \omega_0 e^{-hr}$, $\tau = \varepsilon t$ and $f = x^3$. Corresponding numerical solution is denoted by – (solid line).
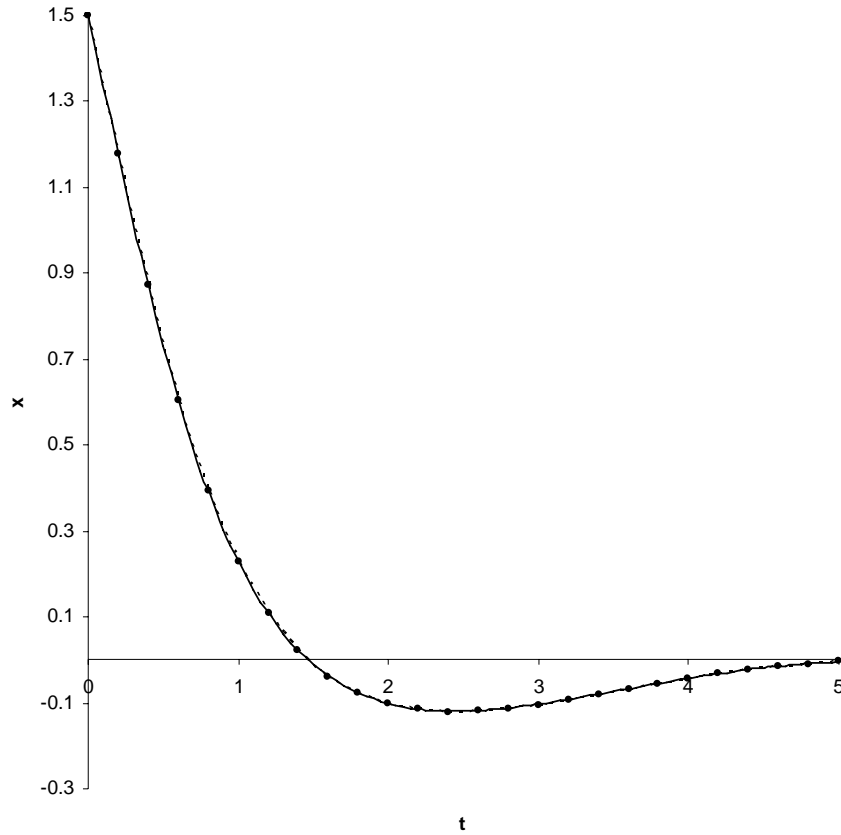
Fig. 2: First approximate solution (denoted by $-\bullet-$) of Eq. (13), with the initial conditions [$x(0) =$ 1.5000, $\dot{x}(0) - 1.55988$, $\ddot{x}(0) = -1.01827$, $\dddot{x}(0) = 9.2714$] or $a_0 = 0.5$, $\varphi_1 = 0$, $b = 1.0$, $\varphi_2 = 0$, with $\mu_1 = 1.75$, $\mu_2 = 0.75$, $\omega_0 = 1.0$, $h = 0.5$, $\varepsilon = 0.1$, $\omega_1 = 2\omega_2$, $\omega_2 = \omega_0 e^{-hr}$, $\tau = \varepsilon t$ and $f = x^3$. Corresponding numerical solution is denoted by — (solid line).

CONCLUSION

A unified KBM (Alam 2002, 2003, Bogoliubov *et al.* 1961, Krylov *et al.* 1947) method is presented to obtain the analytical approximate solution of a fourth order nonlinear differential system with strong damping and slowly varying coefficients with small nonlinearity. The later form of the solution of Eq. (12) is presented in terms of amplitudes and phases variables. This form is very important in physical problems, since amplitudes and phases characterize the oscillating processes. Moreover the variational equations of amplitudes and phases are important to investigate the stability of a differential system. In general, the variational equations for the amplitudes and phases, namely Eq. (18) is solved numerically. In this case, the perturbation method facilitates the numerical method. The variables a, φ_1, b and φ_2 change slowly with time t. So it

requires the numerical calculation of a few numbers of points. On the contrary, a direct attempt to solve Eq. (12) dealing with some harmonic terms in the solution Eq. (22), requires the numerical calculation of a great number of points.

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