J. Bangladesh Acad. Sci. 45(2); 231-240: December 2021 DOI: 10.3329/jbas.v45i2.56654

Journal of Bangladesh Academy of Sciences

Journal homepage: http://www.bas.org.bd/publications/jbas.html

Research Article

An analytical technique for handling forced Van der Pol vibration equation

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ARTICLE INFO

Article History

Received: 01 December 2021
Revised: 20 December 2021
Accepted: 20 December 2021

Keywords: Van der Pol oscillator, Forced vibration, Harmonic balance method, Periodic solutions.

ABSTRACT

Based on the modified harmonic balance method, an analytical method has been developed for handling the forced Van der Pol vibration equation. Usually, a system of nonlinear algebraic equations arises within the unfamiliar coefficients in several harmonics terms and the frequency of the forcing term. A numerical technique has been applied to handle those nonlinear algebraic equations in the classical harmonic balance method. In our study, a system of linear algebraic equations is calculated with the aid of a single nonlinear one. The solutions attained by the suggested scheme have been likened to the results acquired by the well-known Runge-Kutta method, and these results display very nice harmony with the result obtained by the mentioned method. Also, it is noticed that the proposed technique is straightforward and gives the desired results in the whole solution domain.

Introduction

Van der Pol equation is known as the self-excited system with negative damping, and this equation is important to scientists, physicists, and engineers. It is noticed that a self-excited oscillator is a system that has some external energy sources. Nowadays, many researchers are interested in handling the forced Van der Pol equation owing to its various applications in science, technologies, and human activities. Hence nonlinear operations are one of the most critical challenges to researchers and scientists. But it is too difficult to control the nonlinear systems since their properties change rapidly due to some little variation of the system parameters and time. The standard theories and solution procedures for linear differential equations are widely developed. There are no general theories and solution techniques for handling the nonlinear differential equations. But most of the physical, engineering and real-life oscillatory systems have occurred in terms of nonlinear differential equations. To handle these oscillatory systems, physicists and engineers use linear approximation techniques. But such linearization is not always possible. In such cases, the original nonlinear oscillatory systems must be solved straightway. Several researchers have developed analytical techniques to handle nonlinear oscillatory systems using various methods (Belendez et al., 2012; Kovacic and Mickens, 1986; Krylov and Bogoliubov, 1947; Mondal et al., 2019; Nayfeh, 1981; Liu, 2005; Lim and Lai, 2006; Guo and Leung, 2010; Guo and Ma, 2014; He, 1998, 1999, 2006; Uddin al et., 2011, 2012, 2015; Mishara et al., 2016; Khan, 2019; Yeasmin et al., 2020; Uddin and Sattar, 2010; Ullah et al., 2021). Perturbation techniques (Kovacic and Mickens, 1986; Krylov and Bogoliubov, 1947; Nayfeh, 1981; Uddin and Sattar, 2010), which are extensively applied tools. Most of these techniques were originally developed to handle weakly nonlinear dynamical

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systems in the presence of small parameters. According to these techniques, the solutions are expanded analytically into a power series of small parameters. The coefficients of the series of several harmonics are attained as a group of nonlinear linear algebraic equations. Moreover, many nonlinear differential systems arise in science and engineering without perturbation parameters. There exists little accuracy to the results attained by the perturbation techniques.

To overcome these limitations, several approximation methods are established to handle strongly nonlinear oscillatory systems, including the modified Lindstedt-Poincare method (Cheung et al., 1991; Liu, 2005; Nayfeh, 1981), harmonic balance method (Alam et al., 2016; Lim and Lai, 2006; Mickens, 1986; Rahman et al., 2010), residual harmonic balance method (Guo and Ma, 2014) and residual mass harmonic balance method (Ju, and Xue, 2015), iterative harmonic balance method (Guo and Leung, 2010), amplitude and frequency formulation (EI-Naggar and Ismail, 2012) and symmetry analysis method (Khan and Mirzabeigy, 2014), Homotopy perturbation method (He, 1998, 1999, 2006; Uddin et al., 2011, 2012). Furthermore, the energy balancing technique is applied to solve strongly nonlinear differential systems (Molla et al., 2018, Mehdipour et al., 2010; Babazadeh et al., 2008, Molla et al., 2017; Molla and Alam, 2017). A clear analysis of some newly exhibited approximate techniques can be attained in (Alam et al., 2007; Liu et al., 2007; Shen et al., 2014, Zhang and Gu, 2010; Barro, 2016; Casaleiroa et al., 2014; Mondal et al., 2019). Rahman et al. (2010) applied the harmonic balance method for solving Van der Pol oscillator without external force.

Mondal et al. (2019) developed a new analytical method for handling the Van der Pol oscillator based on the harmonic balance method. Khan (2019) presented an analytical technique to solve Van der Pol equation by using the homotopy perturbation method. Yeasmin et al. (2020) have introduced a technique for handing free vibration quadratic nonlinear dynamical systems based on the harmonic balance method. However, these methods have not been greatly improved to attain approximate solutions. Any given method is not appropriate for all nonlinear dynamical systems. The individual methods are suitable for particular nonlinear problems. Some procedures are long and difficult to handle the systems. Also, the derivations and calculations of these procedures are very tedious work. Recently, Ullah et al. (2021) have developed a modified harmonic balance method for handling damped forced Duffing oscillators in the presence of cubic and quadratic nonlinearities.

In this article, a modified harmonic balance approach is developed to handle the forced Van der Pol equation. The benefit of the mentioned technique is that a group of algebraic linear equations is handled by using a nonlinear one. As a result, it requires less computational attempt and time than the other existing harmonic balance methods.

The Method

Consider a strongly nonlinear dynamical system (Mondal et al., 2019, Khan, 2019, Liu et al., 2007, Shen et al., 2014) with a periodic forcing term in the form

$$\ddot{y} + \omega_0^2 y - \epsilon g(y, \dot{y}) = P sin(\omega t) \quad (1)$$

where $y(t)$ is the deformation or displacement of the system, dots represent derivative w. r. to t, ω_0 is the natural frequency, $g(y, \dot{y})$ is a certain nonlinear function of y and $\dot{y}$, $\in$ is a positive parameter which is not necessarily small and denotes the strength of the damping in Van der Pol equation, P is the amplitude of the exciting force and ω is the forcing frequency.

The trial solution of Eq. (1) is assumed as (Ullah et al., 2021)

$$y = \alpha \cos(\omega t) + \beta\, sin(\omega t) + \gamma\, \cos(3\omega t) + \delta \sin(3\omega t) + \cdots \quad (2)$$

where α, β, γ and δ are unknown constants of the truncated Fourier series. Inserting this trial solution in Eq. (1) and spreading $g(y, \dot{y})$ using truncated Fourier series and equating the coefficients of like harmonics, the following system of nonlinear algebraic equations are attained

$$\beta(-\omega^2+\omega_0^2)+\alpha \in \omega + \in S_1(\alpha,\beta,\gamma,\delta,\cdots)=P \quad (3)$$

$$\alpha(-\omega^2+\omega_0^2)-\beta \in \omega + \in C_1(\alpha,\beta,\gamma,\delta,\cdots)=0 \quad (4)$$

$$\delta(-9\omega^2+\omega_0^2)+3\gamma \in \omega + \in S_3(\alpha,\beta,\gamma,\delta,\cdots)=0 \quad (5)$$

$$\gamma(-9\omega^2+\omega_0^2)-3\delta \in \omega + \in C_3(\alpha,\beta,\gamma,\delta,\cdots)=0 \quad (6)$$

Now omitting ω^2 from the Eqs. (4)- (6) by using Eq. (3), and deleting the terms whose response are small, then we get

$$\omega^2=\omega_0^2+\in S_1(\alpha,\beta,\gamma,\delta,\cdots)-P/\beta \quad (7)$$

$$-\beta \in \omega + \in C_1(\alpha,\beta,\gamma,\delta,\cdots)+\alpha \in S_1(\alpha,\beta,\gamma,\delta,\cdots)+P\alpha/\beta=0 \quad (8)$$

$$-8\omega_0^2\delta+\delta \in S_1(\alpha,\beta,\gamma,\delta,\cdots)+\in S_3(\alpha,\beta,\gamma,\delta,\cdots)+3\gamma \in \omega+9P\delta/\beta=0 \quad (9)$$

$$-8\omega_0^2\gamma+\gamma \in S_1(\alpha,\beta,\gamma,\delta,\cdots)+\in C_3(\alpha,\beta,\gamma,\delta,\cdots)-3\delta \in \omega+9\gamma P/\beta=0 \quad (10)$$

Now eliminating ω from the Eqs. (9) and (10) by using Eq. (8), and taking $O(1)$ of γ, δ and removing the expressions with small effect on the systems. As a result system of equations in γ, δ are attained. Simplifying these two equations, γ, δ are expressed in terms of α, β. Finally, putting γ, δ into Eq. (8) and α is expanding in a power series small parameter $\xi(\in, \omega, p)$

$$\alpha=\mu_0+\mu_1\xi+\mu_2\xi^2+\mu_3\xi^3+\cdots \quad (11)$$

where $\mu_0, \mu_1, \mu_2, \ldots$ are function of β. Finally, substituting γ, δ, and α into Eq. (3) and calculating, β is attained. Consequently, the values of α, γ, and δ are obtained.

Example

Consider the following forced Van der Pol equation (Mondal et al., 2019, Khan 2019, Liu et al., 2007, Shen et al., 2014)

$$\ddot{y}+y-\in (1-y^2)\dot{y}=P sin(\omega t) \quad (12)$$

where $g(y,\dot{y})=(1-y^2)\dot{y}$. According to the truncated Fourier series, the guess solution of Eq. (1) is given by (Ullah et al., 2021)

$$y=\alpha\cos(\omega t)+\beta\, sin(\omega t)+\gamma\cos(3\omega t)+\delta\sin(3\omega t) \quad (13)$$

Inserting Eq. (13) in Eq. (12) and taking the coefficients of similar harmonics terms and removing the parts which play a small effect on the system, then the following algebraic equations are generated:

$$\beta(1-\omega^2)+\alpha \in \omega+\frac{1}{4}\alpha\epsilon\omega\left(-\alpha^2-\beta^2-\alpha\gamma+\frac{\beta^2\gamma}{\alpha}-2\gamma^2-2\beta\delta-2\delta^2\right)=\mathrm{P} \quad (14)$$

$$(1-\omega^2)-\beta\epsilon\omega+\frac{1}{4}\beta\epsilon\omega\left(\alpha^2+\beta^2-\beta\delta+\frac{\alpha^2\delta}{\beta}+2\gamma^2-2\alpha\gamma+2\delta^2\right)=0 \quad (15)$$

$$\delta(1-9\omega^2)+3\gamma\epsilon\omega-\frac{3}{4}\gamma\epsilon\omega\left(\gamma^2+\delta^2+2\alpha^2+2\beta^2+\frac{\alpha^3}{3\gamma}-\frac{\alpha\beta^2}{\gamma}\right)=0 \quad (16)$$

$$\gamma(1-9\omega^2)-3\delta \in \omega+\frac{3\delta\in\omega}{4}\left(\gamma^2+\delta^2+2\beta^2+2\alpha^2+\frac{\alpha^2\beta}{\delta}-\frac{\beta^3}{3\delta}\right)=0 \quad (17)$$

Terminating ω^2 from Eqs. (15)- (17) by using Eq. (14), and deleting the terms whose responses are small, we get

$$4P\alpha+\left(\alpha^4+\alpha^3\gamma-3\alpha\beta^2\gamma+\beta^2(-4+\beta^2-\beta\delta)+\alpha^2(-4+2\beta^2+3\beta\delta)\right)\epsilon\omega=0 \quad (18)$$

$$36P\delta+\epsilon\omega(3\beta^3(\alpha-2\gamma)+9\alpha(-4+\alpha^2)\delta+9\alpha\beta^2\delta)-\beta(32\delta+(\alpha^3-12\gamma+6\alpha^2\gamma)\epsilon\omega)=0 \quad (19)$$

$$36P\gamma+\epsilon\omega(6\beta^3\delta-\beta^4+9\alpha(-4+\alpha^2)\gamma\; 3\alpha\beta^2(\alpha+3\gamma))-2\beta(16\gamma-3(-2+\alpha^2)\delta\epsilon\omega)=0 \quad (20)$$

Now using Eq. (18), terminating ω from Eqs. (19) and (20) and taking linear terms of γ, δ only and then excepting the terms whose responses are small, we attained

$$4\beta\left(-8\left(\alpha^4+\beta^2(-4+\beta^2)+2\alpha^2(-2+\beta^2)\right)\delta+P(\alpha^4-3\alpha^2\beta(\beta-3\delta)+9\beta(-4+\beta^2)\delta)\right)=0 \quad (21)$$

$$-4\beta\left(8\left(\alpha^4+\beta^2(-4+\beta^2)+2\alpha^2(-2+\beta^2)\right)\gamma - P\beta(-3\alpha^3+\alpha\beta^2+9\alpha^2\gamma+9(-4+\beta^2)\gamma)\right)=0 \quad (22)$$

Solving Eq. (10a) and Eq. (10b), γ and δ are calculated as:

$$\gamma=\frac{P\alpha\beta(-3\alpha^2+\beta^2)}{(-4+\alpha^2+\beta^2)(8\alpha^2-9P\beta+8\beta^2)},$$

$$\delta=\frac{P\alpha^2(\alpha^2-3\beta^2)}{(-4+\alpha^2+\beta^2)(8\alpha^2-9P\beta+8\beta^2)} \quad (23)$$

Puting γ and δ into Eq. (18), then expanding α in power series of the small parameter ξ, we get

$$\alpha=\mu_0+\mu_1\xi+\mu_2\xi^2+\mu_3\xi^3 \quad (24)$$

where $\xi=\frac{\epsilon\omega}{P}$, $\mu_0=\frac{\beta^2\epsilon\omega}{P}$, $\mu_1=\frac{\beta^4\epsilon^2\omega^2}{P^2}$,

$$\mu_2=\frac{2\beta^6\epsilon^3\omega^3}{P^3},\ \mu_3=\frac{5\beta^8\epsilon^4\omega^4}{P^4} \quad (25)$$

Finally, after substituting the values of δ and α into Eq. (14), the values of β are computed. Consequently the values of α, γ and δ are obtained.

Results and Discussion

In this study, a new modified analytical approach has been exhibited for obtaining the periodic solution of the forced Van der Pol oscillator. The solutions determined by the present technique are compared with the corresponding numerical (considered to be exact) solutions to justify the validity and accuracy of the proposed technique. The solutions curves attained by the present method and a numerical method are shown graphically in Figs. 1(a)-1(g) for the forced Van der Pol equation in presence of several damping and various values of the system parameters. Also, the phase planes are drawn for various values of the system parameters in Figs. 2(a)-2(d). Geometric representation is vital to visualize the system's behavior of the physical systems. From these figures, it is noticed that the obtained results agree nicely with those results determined by the numerical method.

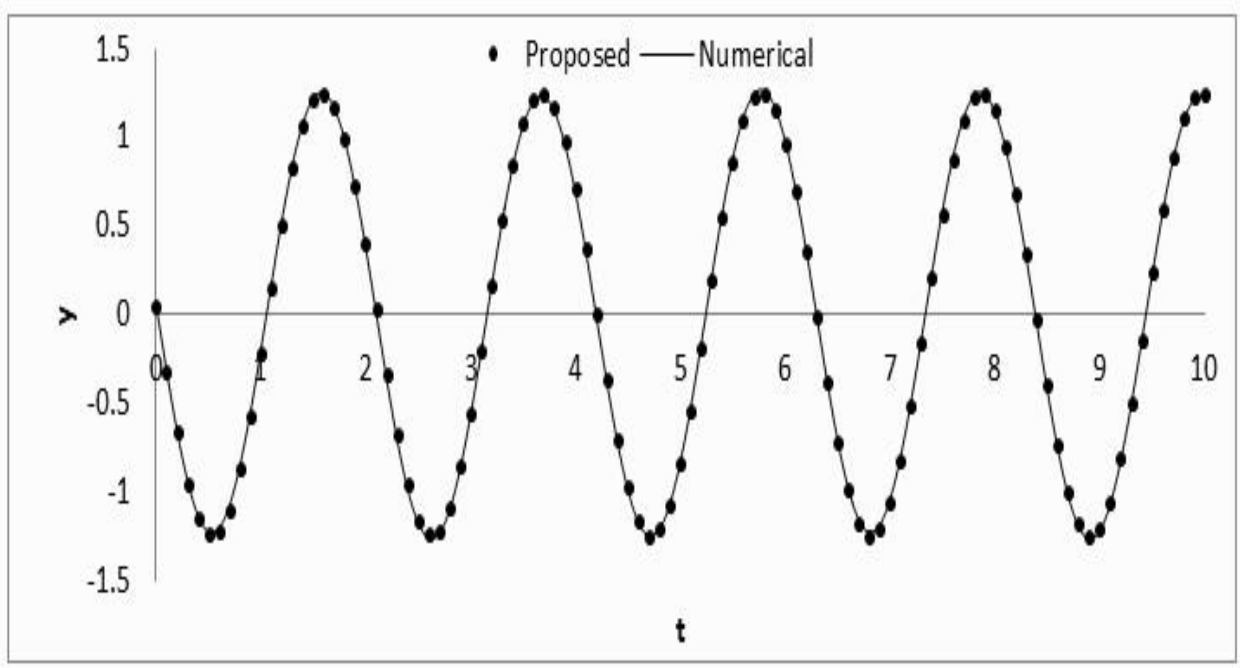

Fig. 1(a). Comparison between the time versus solution of Eq. (12) is attained by the presented technique (represented by dots) and numerical technique when $\omega=3,\ \epsilon=0.1,\ P=10$.

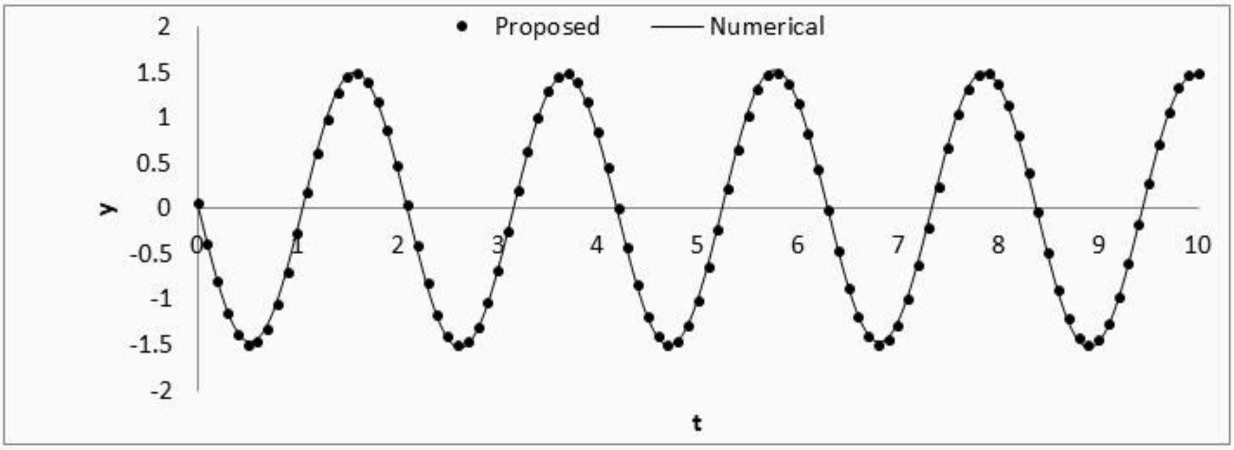

Fig. 1(b). Comparison between the time versus solution of Eq. (12) is attained by the presented technique (represented by dots) and numerical technique when $\omega = 3,\ \epsilon = 0.1,\ P = 20$.

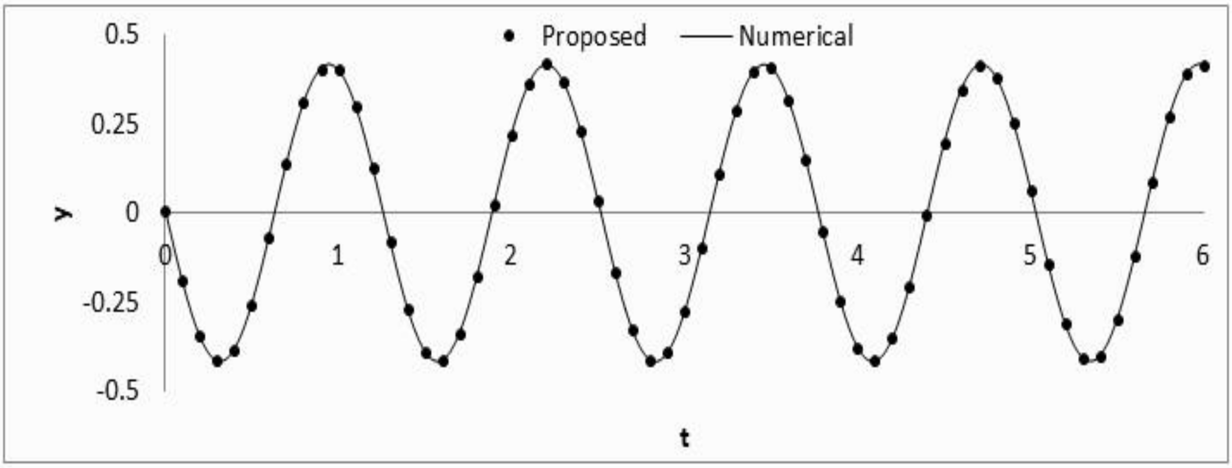

Fig. 1(c). Comparison between the time versus solution of Eq. (12) is attained by the presented technique (represented by dots) and numerical technique when $\omega = 3,\ \epsilon = 0.1,\ P = 12$.

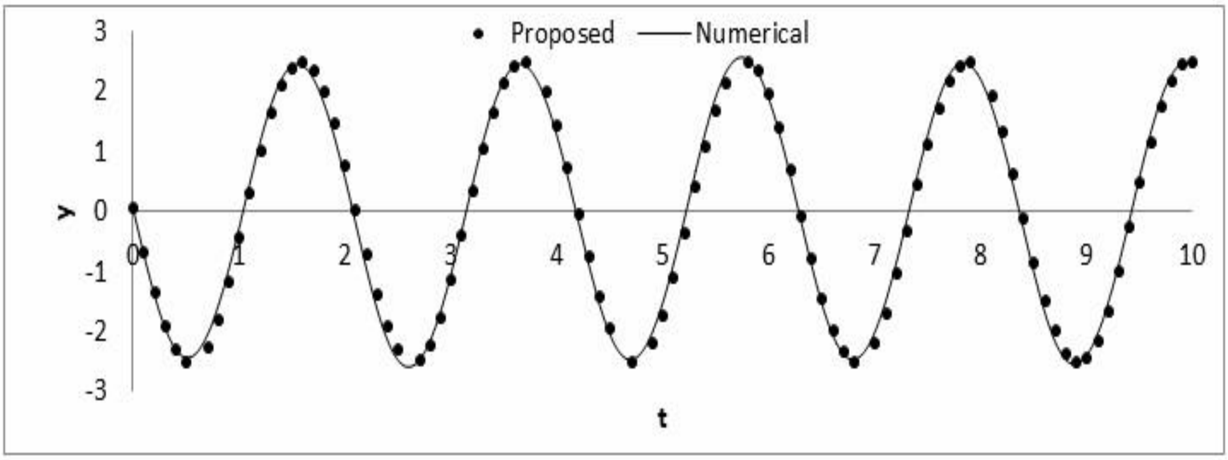

Fig. 1(d). Comparison between the time versus solution of Eq. (12) is attained by the presented technique (represented by dots) and numerical technique when $\omega = 5,\ \epsilon = 0.1,\ P = 10$.

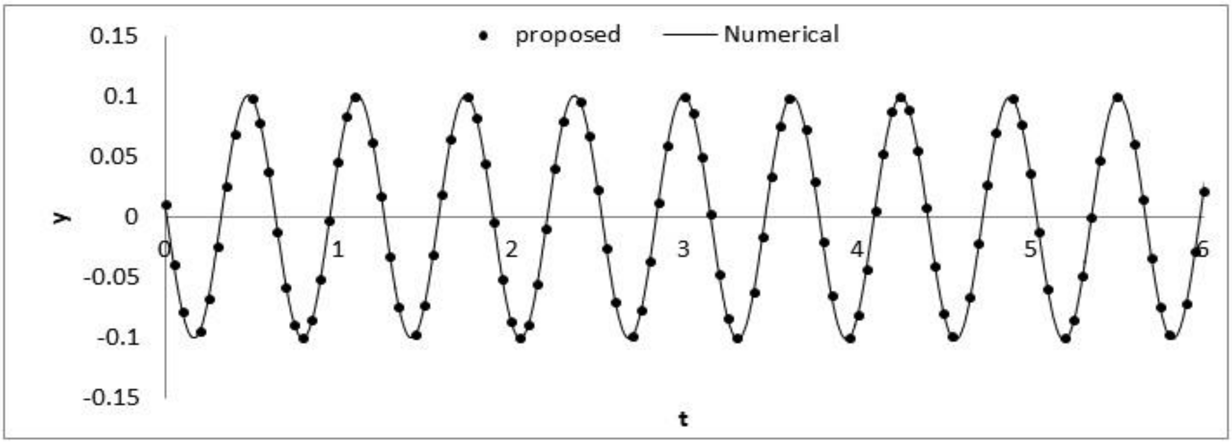

Fig. 1(e). Comparison between the time versus solution of Eq. (6) is attained by the presented technique (represented by dots) and numerical technique when $\omega = 12,\ \epsilon = 0.5,\ P = 12$.

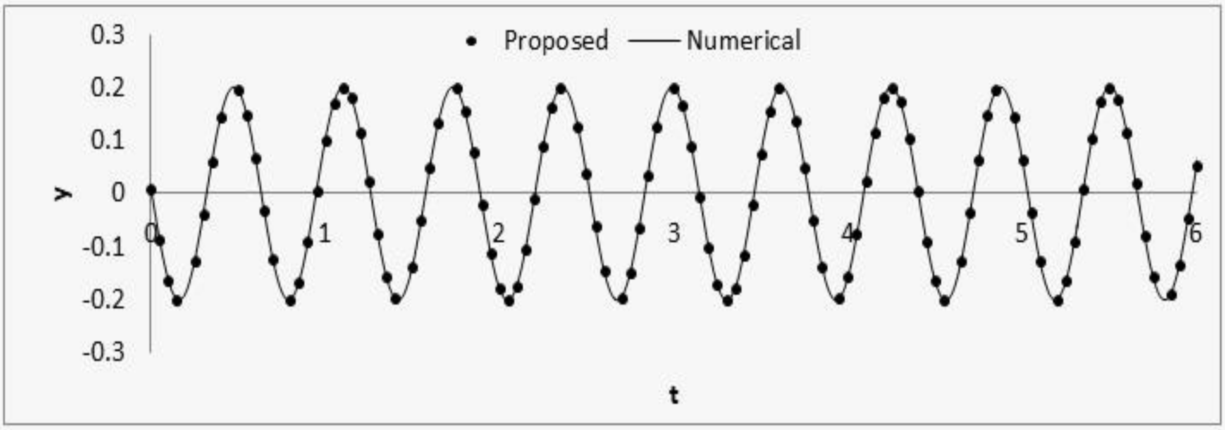

Fig. 1(f). Comparison between the time versus solution of Eq. (12) is attained by the presented technique (represented by dots) and numerical technique when $\omega = 10,\ \epsilon = 1,\ P = 10$.

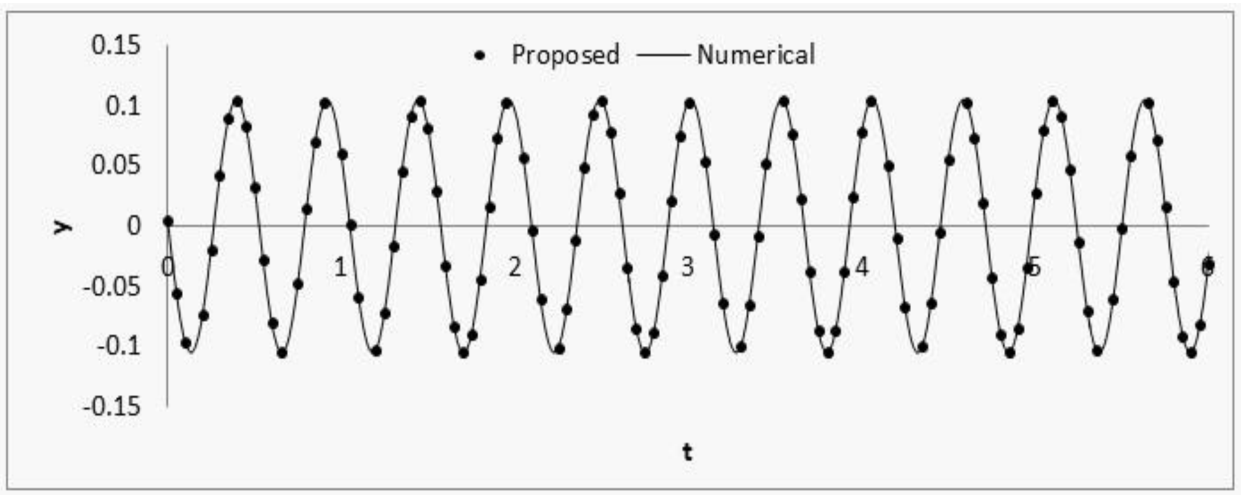

Fig. 1(g). Comparison between the time versus solution of Eq. (12) is attained by the presented technique (represented by dots) and numerical technique when $\omega = 10,\ \epsilon = 0.5,\ P = 20$.

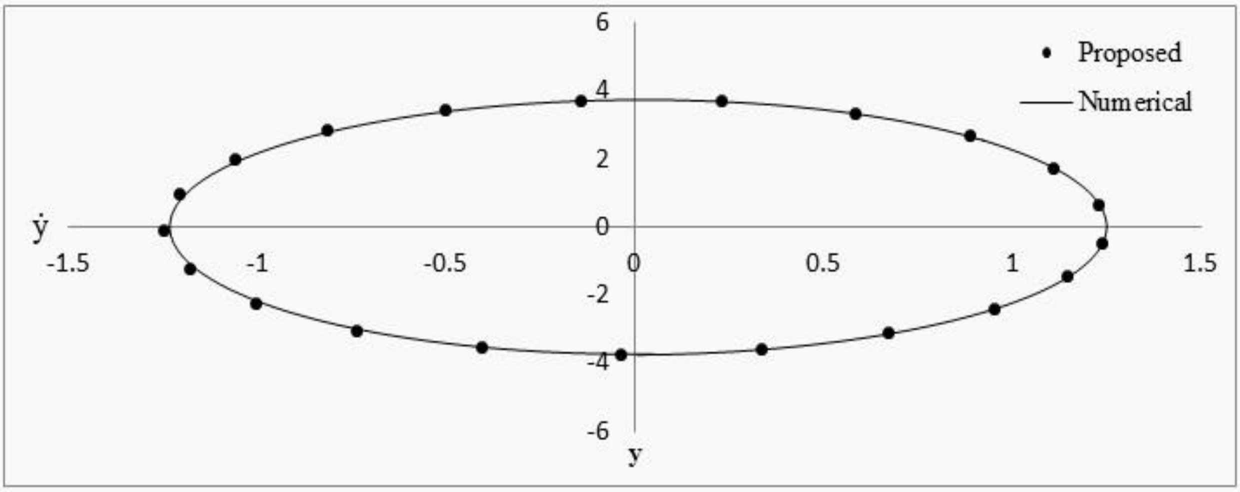

Fig. 2(a). Comparison of analytical and numerical solutions in the phase plane for $\omega = 3,\ \epsilon = 0.1,\ P = 10.$

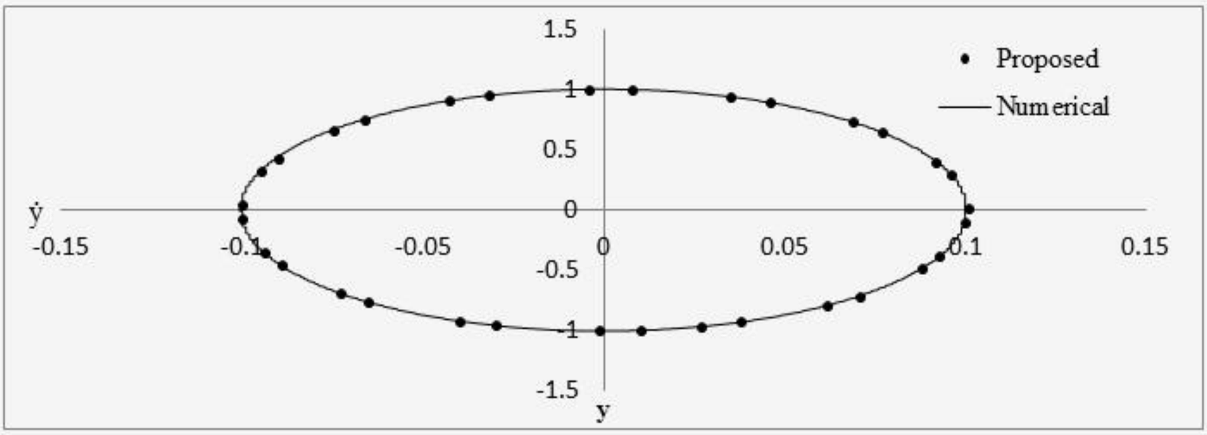

Fig. 2(b). Comparison of analytical and numerical solutions in the phase plane for$\omega = 5,\ \epsilon = 0.1,\ P = 10.$

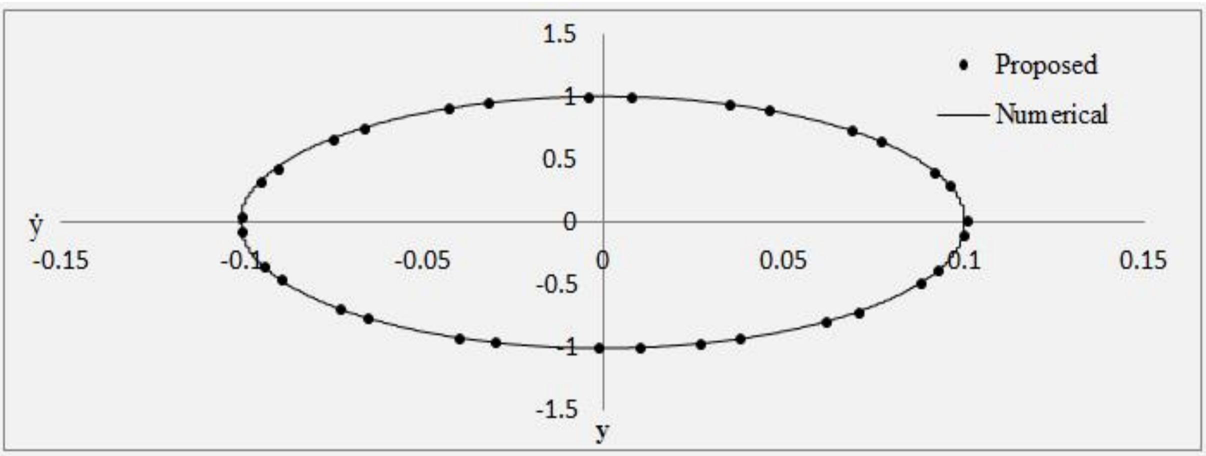

Fig. 2(c). Comparison of analytical and numerical solutions in the phase plane for $\omega = 5,\ \epsilon = 0.1,\ P = 12$

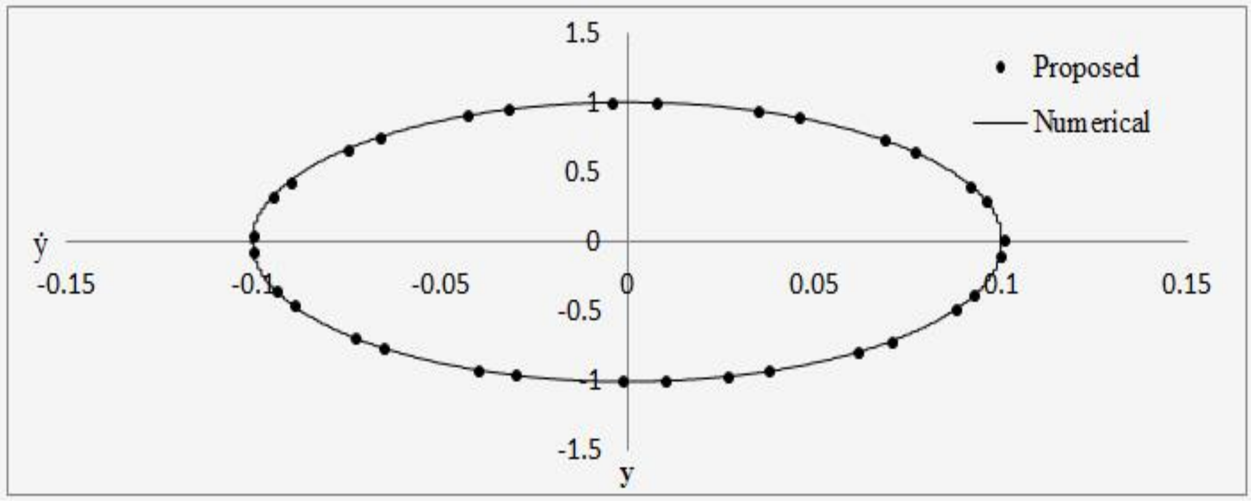

Fig. 2(d). Comparison of analytical and numerical solutions in the phase plane for $\omega = 10,\ \epsilon = 1.0,\ P = 10$

Conclusion

A modified harmonic balance method is developed and justified in this study to handle forced Van der Pol equation. The significant convenience in our technique is that a system of linear algebraic equations is tackled by using a nonlinear one. As a result, the computational attempt is reduced for handling a group of nonlinear algebraic equations. But it needs an enormous computational effort to handle a set nonlinear algebraic equation in the classical harmonic balance method. The graphic representations show good agreement between approximate and numerical solutions. The comparison indicates the accuracy and the exactness of the present technique in solving the forced Van der Pol equation. The study results exhibit adequate understanding with those solutions computed by the fourth-order Runge-Kutta method for several significant damping and several values of the system parameters. Thus, the present approach may consider a suitable technique for handling the forced Van der Pol equation in vibration engineering.

Acknowledgment

The authors are grateful to the unknown reviewers for their valuable comments in preparing this research document.

Conflicts of Interest

The authors declare that they have no conflict of interest. The authors alone are responsible for the content and writing of the paper.

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