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Short Communication

A note on the Sandor-Smarandache function

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ABSTRACT

The Sandor-Smarandache function, denoted by $SS(n)$, is a recently posed Smarandache-type arithmetic function. This paper concentrates on the function $SS(210m)$, where m (≥ 1) is an integer. At the end of the paper, a table giving values of $SS(210m)$ for $m = 1(1)100$, calculated on a computer, is appended.

Introduction

The Sandor-Smarandache function, due to Sandor (2001), is denoted by $SS(n)$, and is defined as follows: For $n \geq 7$,

$$SS(n) = max\left\{k : 1 \leq k \leq n-2,\ n \text{ divides } \binom{n}{k}\right\}, \quad (1.1)$$

where by convention,

$$SS(1) = 1,\ SS(2) = 1,\ SS(6) = 1. \quad (1.2)$$

Letting $C(n, k) \equiv \binom{n}{k}$, it may be deduced that

$$C(n, k) = \frac{n(n-1)(n-2)\ ...\ (n-k+1)}{k!},\ 0 \leq k \leq n. \quad (1.3)$$

Then, the problem to find SS(n) may be expressed as follows: Given any integer n (≥ 7), find the minimum integer k such that $k!$ divides $(n-1)(n-2)\ldots(n-k+1)$, where$1 \leq k \leq n-2$. With this minimum k, $SS(n)$ is given by $SS(n) = n-k$.

So far as we know, the first extensive study of the Sandor-Smarandache function was made by Majumdar (2018). The problem was later taken up by Majumdar (2019), Islam and Majumdar (2021) and Islam, et al. (2021). The following results have been established by Islam et al. (2021). An alternative simpler proof of Lemma 3 is given here.

Lemma 1**:** Let $m \geq 1$ be an integer. Then,

$$SS(30m) = \begin{cases} 30m-4, & if \quad m = 4s+3,\ s \geq 0 \\ 30m-6, & if \quad m = 2(6t+5),\ t \geq 0 \end{cases}$$

Lemma 2**:** Let m (≥ 1) be an integer, not divisible by 7, such that $m \neq 4s+3$ for any $s \geq 0$, or $m \neq 2(6t+5)$ for any $t \geq 0$. Then, $SS(30m) = 30m-7$.

Lemma 3**:** Let $m \geq 1$ be an integer. Then,

$$SS(210m) = \begin{cases} 210m-4, & if \quad m = 4s+1,\ s \geq 0 \\ 210m-6, & if \quad m = 2(6t+5),\ t \geq 0 \end{cases}$$

Proof: The results may be obtained from Lemma 1 by replacing m by $7m$. Noting that the solutions of the equations $7m = 4x + 3$ and $7m = 2(6y + 5)$ are $m = 4s + 1$ and $m = 2(6t + 5)$ respectively, the lemma follows.

Lemma 4**:** Let $m \geq 1$ be an integer. Then,

$SS(210m) = 210m-8$ if $m = 8s+3,\ s \geq 0$,

or if $m = 2(8t+1),\ t \geq 0$.

Lemma 5**:** Let $m \geq 1$ be an integer. Then, $SS(210m) = 210m-9$ if (exactly) one of the five conditions occur : (1) $m = 4(9u+1),\ u \geq 0$, (2) $m = 4(9v+2),\ v \geq 0$, (3) $m = 36x+31,\ x \geq 0$ is even, (4) $m = 36y+35,\ y \geq 1$ is odd, (5) $m = 2(18z+13),\ z \neq 4t+2,\ t \geq 0$.

Proof: Consider the following simplified expression for $C\ (210m,\ 210m-9)$:

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$$210m\Big[\frac{(210m-1)(105m-1)(70m-1)(105m-2)}{8\times 9}\times$$
$$(42m-1)(35m-1)(30m-1)(105m-4)\Big].$$

To find the condition that the term inside the square bracket is an integer, the following five cases are to be considered, of which the first two cases have been treated in Islam et al. (2021).

Case 1. When 4 divides $105m-4$ and 9 divides $70m-1$.

Case 2. When 4 divides $105m-4$ and 9 divides $35m-1$.

Case 3. When 4 divides $35m-1$ and 9 divides $70m-1$.

In this case,

$35m-1=4\alpha$ for some integer $\alpha \geq 1$,

$70m-1=9\beta$ for some integer $\beta \geq 1$,

with the solutions $m=4a+3$ and $m=9b+4$ $(a, b \geq 0)$ respectively. Now, the combined Diophantine equation is $4a = 9b + 1$, whose solution is $a = 9s + 7$, $s \geq 0$. Whence, $m = 4(9s + 7) + 3 = 36s + 31$. Clearly, such an m violates the condition of Lemma 3, since the equation $36s + 30 = 4c$ has no solution. Also, the solution of the equation $36s+28=8d$ is $s=2t+1$, $t \geq 0$.
Case 4. When 36 divides $35m-1$.
Here, the resulting Diophantine equation is

$35m-1=36\gamma$ (for some integer $\gamma \geq 1$),

with the solution $m=36s+35$, $s \geq 0$. This m does not satisfy the condition of Lemma 3. Also, the solution of the equation $36s+32=8d$ is $s=2t$, $t \geq 0$.
Case 5. When 4 divides $105m-2$ and 9 divides $35m-1$.

Note that, in this case, 2 divides $105m-4$. Now, $105m-2=4\nu$ for some integer $\nu \geq 1$,

$35m-1=9\theta$ for some integer $\theta \geq 1$,

whose solutions are $m=4e+2$ $(e \geq 0)$ and $m=9f+8$ $(f \geq 0)$ respectively. Now, considering the combined Diophantine equation $4e = 9f + 6$, the solution is found to be $e = 9s + 6$, $s \geq 0$. Hence finally, $m = 4(9s + 6) + 2 = 2(18s + 13)$. Obviously, such an m violates the condition of Lemma 3, since the equation $18s + 8 = 6t$ has no solution. Also, the solution of the Diophantine equation $18s + 12 = 8k$ is $s = 4t + 2$, $t \geq 0$.

Lemma 6**:** $SS(210m) = 210m - 10$ if either $m = 4(10s + 7)$, $s \neq 9a + 3$, $s \neq 9b + 4$, $a \geq 0$, $b \geq 0$, or if $m = 2(20t + 9)$, $t > 0$ is odd with $t \neq 3c + 1$, $t \neq 9d + 2$, $c \geq 0$, $d \geq 0$.

Proof: Consider the simplified expression for $C(210m, 210m-10)$:

$$210m\Big[\frac{(210m-1)(105m-1)(70m-1)(105m-2)(42m-1)}{16\times 3\times 5}\times$$
$$(35m-1)(30m-1)(105m-4)(70m-3)\Big].$$

To find the conditions such that the term inside the square bracket is an integer, the following two cases need be considered.

Case 1. When 5 divides $42m - 1$ and 8 divides $105m-4$.

Here, the resulting Diophantine equations are

$42m-1=5\alpha$, for some integer $\alpha \geq 1$,

$105m-4=8\beta$ for some integer $\beta \geq 1$.

The solutions of the above two equations are $m = 5u + 3$ $(u \geq 0)$ and $m = 8v + 4$, $(v \geq 0)$ respectively. Now, the combined Diophantine equation to be considered is $5u = 8v + 1$, whose solution $u = 8s + 5$, $s \geq 0$. Therefore, $m = 5(8s + 5) + 3 = 4(10s + 7)$.

Clearly, such an m does not satisfy the condition of Lemma 3. Also, the solutions of the equations $10s + 6 = 9a$ and $10s + 5 = 9b$, whose solutions are $s = 9x + 3$ and $s = 9y + 4$ respectively.

Case 2. When 8 divides $105m - 2$ and 5 divides $42m-1$.

Note that, in this case, $105m-4$ is also divisibl by 2.

Now, $105m-2=8\gamma$ for some integer $\gamma \geq 1$,

with the solution $m = 8w + 2$, $w \geq 0$. Considering the combined Diophantine equation $5u + 1 = 8w$, the solution is found to be $u = 8t + 3$ $(t \geq 0)$, so that $m = 5(8t + 3) + 3 = 2(20t + 9)$. Obviously, this value of m violates the condition of Lemma 3. Also, the solution of the equation $20t + 4 = 6s$ is $t = 3c + 1$ $(c \geq 0)$, the solution of the equation $20t + 8 = 8s$ is $t = 2e$ $(e \geq 0)$, and the solution of the equation $20t = 18s + 4$ is $t = 9d + 2$ $(d \geq 0)$.

Table 1. Values of $SS(210m)$ for $1 \le m \le 200$

n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)
210	206	8610	8606	17010	17006	25410	25406	33810	33806
420	412	8820	8809	17220	17214	25620	25609	34020	34012
630	622	9030	9022	17430	17422	25830	25822	34230	34222
840	831	9240	9231	17640	17629	26040	26029	34440	34429
1050	1046	9450	9446	17850	17846	26250	26246	34650	34646
1260	1249	9660	9654	18060	18049	26460	26449	34860	34854
1470	1459	9870	9859	18270	18259	26670	26659	35070	35059
1680	1671	10080	10069	18480	18467	26880	26869	35280	35269
1890	1886	10290	10286	18690	18686	27090	27086	35490	35486
2100	2094	10500	10492	18900	18889	27300	27294	35700	35691
2310	2302	10710	10702	19110	19102	27510	27502	35910	35902
2520	2509	10920	10909	19320	19309	27720	27707	36120	36109
2730	2726	11130	11126	19530	19526	27930	27926	36330	36326
2940	2929	11340	11329	19740	19734	28140	28131	36540	36529
3150	3139	11550	11537	19950	19939	28350	28339	36750	36741
3360	3349	11760	11749	20160	20149	28560	28549	36960	36947
3570	3566	11970	11966	20370	20366	28770	28766	37170	37166
3780	3772	12180	12174	20580	20572	28980	28970	37380	37374
3990	3982	12390	12382	20790	20782	29190	29182	37590	37582
4200	4189	12600	12589	21000	20989	29400	29389	37800	37789
4410	4406	12810	12806	21210	21206	29610	29606	38010	38006
4620	4614	13020	13011	21420	21409	29820	29814	38220	38209
4830	4819	13230	13219	21630	21621	30030	30021	38430	38419
5040	5029	13440	13429	21840	21829	30240	30229	38640	38631
5250	5246	13650	13646	22050	22046	30450	30446	38850	38846
5460	5451	13860	13852	22260	22254	30660	30652	39060	39049
5670	5662	14070	14062	22470	22462	30870	30862	39270	39262
5880	5870	14280	14270	22680	22670	31080	31071	39480	39471
6090	6086	14490	14486	22890	22886	31290	31286	39690	39686
6300	6289	14700	14694	23100	23087	31500	31489	39900	39894
6510	6501	14910	14901	23310	23299	31710	31699	40110	40099
6720	6709	15120	15109	23520	23511	31920	31911	40320	40309
6930	6926	15330	15326	23730	23726	32130	32126	40530	40526
7140	7134	15540	15529	23940	23932	32340	32334	40740	40732
7350	7342	15750	15742	24150	24142	32550	32542	40950	40942
7560	7549	15960	15951	24360	24351	32760	32749	41160	41149
7770	7766	16170	16166	24570	24566	32970	32966	41370	41366
7980	7969	16380	16369	24780	24774	33180	33169	41580	41567
8190	8179	16590	16579	24990	24979	33390	33379	41790	41779
8400	8391	16800	16791	25200	25189	33600	33589	42000	41989

***Lemma 7*:** Let m (>) be an integer not divisible by 11; furthermore, let $m \neq 4a + 1$, $m \neq 2\,(6b + 5)$, $m \neq 8c + 3$, $m \neq 2(8d + 1)$, $m \neq 4(9e + 1)$, $m \neq 4\,(9f + 2)$, $m \neq 36g + 31$, $m \neq 36h + 35$, $m \neq 2\,(18i + 13)$, $m \neq 4\,(10j + 7)$, $m \neq 2\,(20k + 9)$ (for any integers $a, b, c, d, e, f, g, h, i, j, k > 0$). Then, $SS(210m) = 210m - 11$.

Proof: The expression $C(210m, 210m-11)$ simplifies as follows :

$$210m\Big[\frac{(210m-1)(105m-1)(70m-1)(105m-2)(42m-1)}{8\times 3\times 11}\times (35m-1)(30m-1)(105m-4)(70m-3)(21m-1)$$

Note that one of $70m - 1$, $35m - 1$, and $70m - 3$ is divisible by 3. Also, it may easily be verified that $(105m-1)\,(21m-1)\,(35m-1)$ is divisible by 8 if m is odd, while for even m, $(105m-2)\,(105m-4)$ is divisible by 8. Hence, the term inside the square bracket is an integer if 11 does not divide m.

Lemmas 3 – 7 show that

$210m - 4 \leq SS\,(210m) \leq 210m - 11$ for all $m \geq 1$ with m not a multiple of 11; furthermore, for any $m \geq 1$,

$$SS(210m) \neq 210m - 5,$$

$$SS(210m) \neq 210m - 7.$$

The accompanying Table 1 gives the values of SS (210 m) for $m = 1$ (1)100, calculated on a computer, using the formula (1.3) for the binomial coefficients.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

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