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Research Article

The Reve's Puzzle with a single cheat of the Divine rule

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ABSTRACT

A new variant of the Reve's puzzle, which allows (at most) one violation (or, "cheat") of the "divine rule", has been introduced in an earlier paper. Letting $S(n)$ be the minimum number of moves required to solve the new generalization with n discs, this paper finds an explicit expression of $S(n)$, taking into account all the possible schemes. Some results related to the generalization to the general case of c cheats are derived.

Introduction

The Tower of Hanoi problem, posed by the French number theorist Lucas (1883), is as follows: Given are three pegs, S, P and D and n (≥ 1) discs of different radii, d_1, d_2, …, d_n. Initially, the discs rest on the source peg S in a tower, with the smallest disc at the top. The problem is to transfer the tower to the destination peg, D, in minimum number of "moves", where each "move" transfers only one (topmost) disc from one peg to another peg, under the "divine rule" that, no disc can ever be put directly on top of a smaller disc. It is known that $2^n - 1$ moves are necessary to solve the problem.

The 4-peg generalization, commonly known as the Reve`s puzzle, is due to the English puzzlist Dudeney (1958) and is as follows: There are n discs of varying sizes, and four pegs. Initially, the source peg S contains the tower of n discs, as shown in Fig. 1. The objective is to transfer the tower to the destination peg D (using the two auxiliary pegs P_1 and P_2), in minimum number of moves, under the "divine rule" described above.

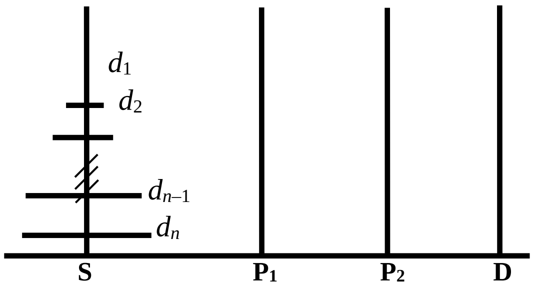

Fig. 1. The initial configuration of the Reve's puzzle.

For details on the Reve's puzzle and its various generalizations, the reader is referred to Majumdar (2012, 2013) and Hinz et al. (2018).

A new version of the Tower of Hanoi problem has been proposed and solved by Chen et al. (2007). In the new variant, the objective is to transfer the tower from the source peg S to the destination peg D in minimum number of moves, such that, for (at most) c moves, the player may put some disc directly on top of a smaller one, thereby violating the "divine rule". Chen et al. (2007) call each such move a "cheat" of the player.

In an earlier paper, Majumdar (2022) proposed a generalization of the problem of Chen et al. (2007) to

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the Reve's puzzle, which permits (at most) one relaxation (or, "cheat") of the "divine rule". Thus, during the transfer process, at most once, a larger disc may be placed directly on top of a smaller disc.

Letting $S(n)$ be the minimum number of moves required to solve the Reve's puzzle with n discs and (at most) one relaxation (or, "cheat") of the "divine rule", this paper derives a closed-form expression of $S(n)$, considering in detail all possible schemes. This is done in the third section. The next section gives the necessary background materials, while the following section gives some observations on multiple optimal strategies. The paper concludes with Conclusion in the final section.

Background materials

Letting $R(n)$ be the minimum number of moves required to solve the Reve's puzzle with n (≥ 1) discs, $R(n)$ satisfies the dynamic programming equation (DPE) below (see, for example, Roth (1974), Wood (1981), Hinz (1989), Chu and Johnsonbaugh (1991), Majumdar (1994, 2012) and Hinz et al. (2018)): For $n \geq 4$,

$$R(n) = \min_{1 \leq \ell \leq n-1} \left\{2R(\ell) + 2^{n-\ell} - 1\right\}, \tag{1a}$$

with

$$R(n) = 2n - 1 \text{ for all } 1 \leq n \leq 3. \tag{1b}$$

Obviously, $R(0) = 0$.

Recall that the DPE (1a) is obtained by following the steps below:

1. the tower of the topmost ℓ (≥ 1) smallest discs is moved from the peg S to the peg P_1, using all the four pegs available, in (minimum) $R(\ell)$ moves,
2. next, the tower of the remaining $n-\ell$ discs (on S) is moved to the peg D, using the three pegs available, in (minimum) $2^{n-\ell} - 1$ moves,
3. finally, the tower of ℓ discs is shifted (from P_1) to D, in (minimum) $R(\ell)$ moves, to complete the tower on D.

Now, ℓ is chosen so as to minimize the total numbers of moves, which results in the DPE (1a).

Recently, Bousche (2014) presented an analytical proof of optimality of the above scheme, which leads to the DPE (1a).

The solution of the optimality equation (1a) is given below. For proofs, the reader is referred to Majumdar (1994, 2012), Hinz et al. (2018) and Majumdar (2021).

***Lemma* 1:** The expression of $R(n)$ is given as follows:

(1) for $s = 1, 2, \ldots$, $R\left(\frac{s(s+1)}{2}\right)$ is attained at the unique point $\ell = \frac{s(s-1)}{2}$, with

$$R\left(\frac{s(s+1)}{2}\right) = 2^s(s-1) + 1,$$

(2) for $\frac{s(s+1)}{2} < n < \frac{(s+1)(s+2)}{2}$, $R(n)$ is attained at $\ell = n-s-1, n-s$, with

$$R(n) = 2^s \left\{n - \frac{s(s-1)}{2} - 1\right\} + 1.$$

Moreover, for $\frac{s(s+1)}{2} \leq n < \frac{(s+1)(s+2)}{2}$,

$$R(n+1) - R(n) = 2^s.$$

In Lemma 1 above, ℓ is the value at which the function $2R(\ell) + 2^{n-\ell} - 1$ in equation (1a) is minimized. The values of $R(n)$ for some small n are tabulated in Table 1.

The result below readily follows from Lemma 1. An alternative proof is given in Majumdar (1994, 2012, 2016).

***Corollary* 1:** For $n \geq 1$,

$$R(n+1) - R(n) \geq R(n) - R(n-1).$$

The following result would be required later.

***Corollary* 2:** For $n \geq 6$, $R(n+1) - R(n) > 4$.

***Proof*:** Since for $n \geq 6$ (see Table 1),

$$R(n+1) - R(n) \geq R(7) - R(6) = 8 > 4 = R(6) - R(5),$$

the result follows (by Corollary 1).

Two other results of importance here are the following ones.

***Corollary* 3:** $R(n+1) - R(n) = 4$ if and only if $n = 3$, 4, 5.

***Proof*:** The proof is evident from the values tabulated in Table 1.

***Corollary* 4:** $R(n+1) - R(n) = 2^n$ if and only if $n = 1$.

***Proof*:** First note that, $R(1)$ and $R(2)$ both are attained at the point $\ell = 0$. To prove the result, it may be assumed, without loss of generality, that $R(n)$ and $R(n+1)$ both are attained at the point $\ell = L$, so that

$$R(n) = 2R(L) + 2^{n-L} - 1,$$

$$R(n+1) = 2R(L) + 2^{n+1-L} - 1,$$

and hence,

$$R(n+1) - R(n) = 2^{n-L}.$$

Then, by the given condition,

$$2^{n-L} = 2^n,$$

so that $L = 0$, and consequently, $n = 1$.

Letting $S_3(n, c)$ be the minimum number of moves required to solve the problem of Chen et al. (2007) with n (≥ 1) discs and (at most) c (≥ 1) relaxations (or, "cheats") of the "divine rule", $S_3(n, c)$ is given in the lemma below. The lemma is due to Chen et al. (2007).

***Lemma* 2:** For any $n \geq 1$, $c \geq 1$,

$$S_3(n,\ c) = \begin{cases} 2n-1, & if\, 1 \leq n \leq r+2 \\ 4n-2c-5, & if\, r+2 \leq n \leq 2r+3 \\ 2^{n-2r}+6r-1, & if\ \ n \geq 2r+3 \end{cases}$$

Recall that, in Lemma 2 above, for $c = 1$, $n \geq 4$, the optimal strategy is as follows:

1. the tower of the topmost $n - 3$ smallest discs is transferred (from the peg S) to the auxiliary peg P, in (minimum) $2^{n-3} - 1$ moves,
2. the disc d_{n-2} is moved (from S) to D,
3. the disc d_{n-1} is shifted (from S) to P, violating the "divine rule",
4. the disc d_{n-2} (on D) is moved to P,
5. the disc d_n is shifted (from S) to D,
6. the disc d_{n-2} is moved (from P) to S,
7. the disc d_{n-1} (on P) is shifted to D,
8. the disc d_{n-2} is moved (from S) to D,
9. finally, the tower of the $n - 3$ smallest discs is transferred (from P) to D, to complete the tower on the destination peg D.

The next section first describes the problem in detail and then finds its solution explicitly.

The problem and its solution

This section considers the Reve`s puzzle with n discs and (at most) one relaxation (or, "cheat") of the "divine rule". More precisely, the problem may be stated as follows: Given is a tower of n (≥ 1) discs (of varying sizes) resting on the source peg S, with the smallest disc at the top. The objective is to transfer the tower (from S) to the destination peg D, using the two auxiliary pegs P_1 and P_2, in minimum number of moves, where each move shifts only one (topmost) disc from one peg to another, under the condition that (at most) once, some disc may be put directly on top of a smaller one, and in any of the remaining moves, no disc can ever be placed directly on a smaller one.

Let $S(n)$ denote the minimum number of moves necessary to solve the problem described above. In an earlier paper, Majumdar (2022, Theorem 2) gives an expression of $S(n)$ by considering three possible schemes. The following theorem supplements the result of Majumdar (2022), and gives an expression of $S(n)$, taking into consideration all the possible schemes.

***Theorem* 1:** For $1 \leq n \leq 8$, $S(n)$ is given by

$$S(n) = \begin{cases} 2n-1, & if\, 1 \leq n \leq 4 \\ R(n-1)+2, & if\, n = 5,\ 6,\ 7 \\ R(n-2)+6, & if\ \ n = 5,\ 6,\ 7,\ 8 \end{cases}$$

and for $n \geq 6$, $S(n)$ is given as follows: Let

$$\frac{(t+2)(t+3)}{2} \leq n < \frac{(t+3)(t+4)}{2}$$

for some integer $t \in \{1, 2, \ldots\}$; then

$$S(n) = R(n-t-2) + 2[R(t) + 2^t + 2].$$

***Proof*:** For $1 \leq n \leq 4$, the proof is immediate.

So, let $n \geq 5$. There are four possible schemes, which are described below.

Scheme 1: follows the steps below:

1. the tower $T = \{d_1, d_2, \ldots, d_k\}$ of the topmost k ($\geq$ 1) smallest discs is shifted from the source peg S to the auxiliary peg P_1 (using all the four available pegs), in (minimum) $R(k)$ moves,
2. the disc d_{k+1} is moved from S to P_1, on top of T, violating the "divine rule",
3. the tower of remaining $n - k - 1$ discs is shifted (from S) to D (using the three available pegs), in (minimum) $2^{n-k-1} - 1$ moves,

4. the disc d_{k+1} is moved from P_1 to D,
5. finally, the tower T (of k discs) on P_1 is transferred to D, (in (minimum) $R(k)$ moves), thereby completing the tower on the destination peg D.

Now, k in Step 1 above is chosen so that the total number of moves is minimized. Thus, Scheme 1 requires minimum

$$\min_{1 \le k \le n-1} 2\{R(k)+1\} + 2^{n-k-1} - 1$$
$$= R(n-1) + 2. \qquad (2)$$

number of moves, where the expression (2) follows by virtue of the equation (1).

Note that, in equation (2), $R(n-1)$ is attained at a point k with $k \le n-2 < n-1$.

Scheme 2: follows the nine steps below:

1. the tower T of the topmost k (≥ 1) smallest discs is transferred from the source peg S to the auxiliary peg P_1, in (minimum) $R(k)$ moves,
2. the disc d_{k+1} is moved from S to P_2,
3. the disc d_{k+2} (on S) is put on top of the tower T, violating the "divine rule",
4. the disc d_{k+1} is placed on d_{k+2} on P_1,
5. the tower of the remaining $n-k-2$ discs (on S) is transferred to D, in (minimum) $2^{n-k-2}-1$ moves,
6. the disc d_{k+1} on P_1 is moved P_2,
7. the disc d_{k+2} is shifted from P_1 to D,
8. the disc d_{k+1} is moved from P_2 to D,
9. finally, the tower T is transferred from P_1 to D, to complete the tower on the destination peg D.

This scheme requires (minimum)

$$2R(k) + 2^{n-k-2} + 5$$

number of moves, and k is to be chosen so as to minimize the total number of moves. Thus, under this scheme, the minimum number of moves required is

$$\min_{1 \le k \le n-1} \{2R(k) + 2^{n-k-2}\} + 5$$
$$= R(n-2) + 6, \qquad (3)$$

where in the last equation (3), equation (1) has been exploited. Recall that, $R(n-2)$ is attained at a point k with $k \le n-3 < n-1$. Thus, the value of $R(n-2)$ is not affected if the range of k is extended (to $n-1$) in equation (3).

Now, for $n \ge 8$ (by Corollary 2),

$$R(n-1) + 2 > R(n-2) + 6.$$

Thus, for $n \ge 8$, Scheme 2 is better than Scheme 1. Again, by Corollary 3,

$$R(n-1) + 2 = R(n-2) + 6.$$

if and only if $n = 5$, 6, 7, so that for these values of n, Scheme 2 is as good as Scheme 1.

Scheme 3: follows the steps below:

1. first, the tower T of the topmost k (≥ 1) smallest discs is transferred from the source peg S to the auxiliary peg P_1, in (minimum) $R(k)$ moves,
2. next, the tower of the remaining $n-k$ largest discs on S is moved to D, using the three available pegs, in (minimum) $S_3(n-k)$ moves, where $S_3(n)$ is given by Lemma 1,
3. finally, the tower T of k discs (on P_1) is shifted to D, in (minimum) $R(k)$ moves, to complete the tower on D.

The above scheme requires (minimum)

$$2R(k) + S_3(n-k) = 2R(k) + 2^{n-k-2} + 5$$

number of moves, which is the same as that for Scheme 2.

Note that, though Scheme 2 and Scheme 3 both give the same number of moves, the (optimal) strategies employed are different in the two cases.

Scheme 4: consists of the steps below:

1. the tower T_1 of the topmost k (≥ 1) smallest discs is shifted from the source peg S to the auxiliary peg P_1, in (minimum) $R(k)$ moves,
2. the tower of the next t consecutive discs, namely, d_{k+1}, d_{k+2}, …, d_{k+t}, denoted by T_2, is moved (from S) to the auxiliary peg P_2, using the three pegs available, in (minimum) $2^t - 1$ moves,
3. the disc d_{k+t+1} is shifted (from S) to the destination peg D,
4. the disc d_{k+t+2} is moved (from S) to P_1, on top of the tower T_1 of k smallest discs, violating the "divine rule",
5. the disc d_{k+t+1} is transferred (from D) to P_1, on the disc d_{k+t+2},
6. the tower of t discs T_2 is shifted (from P_2) to P_1,

on the disc d_{k+t+1}, in (minimum) $R(t)$ moves.

After Step 6, there are two towers on the auxiliary peg P_1, namely, the tower of $t+2$ consecutive discs, $d_{k+1}, d_{k+2}, \ldots, d_{k+t}, d_{k+t+1}, d_{k+t+2}$, on top of the tower T_1 of the smallest k discs. The peg S contains the tower of $n-k-t-2$ largest discs. The situation is depicted in Fig. 2. Note that, the total number of moves necessary in the above six steps is $R(k)+R(t)+2^t+2$.

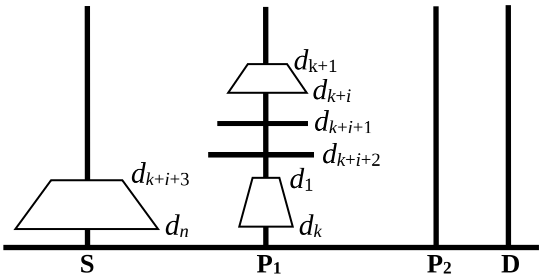

Fig. 2. The configuration after the first six steps in Scheme 4.

Next, follow the steps below to complete the transfer process to the peg D.

7. the tower of the largest $n-k-t-2$ discs, still lying on the peg S, is moved to the peg D, using the three pegs available, in (minimum) $2^{n-k-t-2}-1$ moves,
8. the tower of t discs, T_2, is transferred (from P_1) to P_2, in (minimum) $R(t)$ moves,
9. the disc d_{k+t+1} (on P_1) is shifted to S,
10. the disc d_{k+t+2} (on P_1) is moved to D,
11. the disc d_{k+t+1} is transferred from S to D, on top of the disc d_{k+t+2},
12. the tower T_2 (on P_2) is shifted to D, using the three pegs available, in (minimum) $2^t - 1$ moves,
13. finally, the tower of k smallest discs, T_1 (on P_1), is transferred to D, (in (minimum) $R(k)$ moves) so as to complete the tower on the destination peg D.

To find the minimum number of moves, k is to be chosen accordingly. Hence, the minimum number of moves involved in this scheme is

$$\min_{1 \le k \le n-1} \; 2\{R(k)+R(t)+2^t+2\}+2^{n-k-t-2}-1$$

$$=R(n-t-2)+2[R(t)+2^t+2], \qquad (4)$$

where t is to be determined such that the expression in (4) is minimized. Note that, in (4), $R(n-t-2)$ is attained at a point k with $k \le n-t-2 < n$.

Let

$$S(n, t) \equiv R(n-t-2)+2[R(t)+2^t+2]. \qquad (5)$$

Then, for any $n \ge 1$ fixed,

$$\begin{aligned} S(n, t) &= R(n-t-2)+2[R(t)+2^t+2] \\ &< R(n-t-3)+2[R(t+1)+2^{t+1}+2] \\ &= S(n, t+1) \end{aligned}$$

if and only if

$$R(n-t-2)-R(n-t-3) < 2^{t+1}+2[R(t+1)-R(t)]. \qquad (5a)$$

Note that, for any integer $t \ge 1$,

$$2^{t+1}+2[R(t+1)-R(t)] \le 2^{t+2},$$

with the equality sign if and only if $t = 1$ (by Corollary 4).

Let n be such that

$$\frac{(t+1)(t+2)}{2} \le n-t-3 < \frac{(t+2)(t+3)}{2} \qquad (6)$$

for some integer $t \ge 1$; then, by virtue of part (2) of Lemma 1,

$$R(n-t-2)-R(n-t-3)=2^{t+1}.$$

Note that, the inequality (6) simplifies to

$$\frac{(t+2)(t+3)}{2}+1 \le n < \frac{(t+3)(t+4)}{2}.$$

Now, when $n=\frac{(t+2)(t+3)}{2}$, then

$$R(n-t-2)=R\left(\frac{(t+1)(t+2)}{2}\right)=2^{t+1}t+1,$$

$$\begin{aligned} R(n-t-3) &= R\left(\frac{(t+1)(t+2)}{2}-1\right), \\ &= 2^t(2t-1)+1, \end{aligned}$$

so that

$$R(n-t-2)-R(n-t-3)=2^t.$$

Thus, if $\frac{(t+2)(t+3)}{2} \le n < \frac{(t+3)(t+4)}{2}$, then the inequality (5a) is satisfied.

By Corollary 2,

$$R(n-2)+6 > R(n-3)+10, \text{ if } n \ge 9.$$

Hence, when $n \geq 9$, the fourth scheme (with $t = 1$) is better than the second one; moreover, the fourth scheme is the only optimal scheme when $n \geq 9$. To complete the proof, it is necessary to compare the values of $R(n-1)+2$, $R(n-2)+6$ and $R(n-3)+10$ when $5 \leq n \leq 8$. Since,

$$R(4)+2 = 11 = R(3)+6,$$

it follows that, when $n = 5$, the first and the second schemes both are optimal; again, since

$$R(5)+2 = R(4)+6 = R(3)+10 = 15,$$

$$R(6)+2 = R(5)+6 = R(4)+10 = 19,$$

it follows that, all the three schemes are optimal for $n = 6, 7$; and finally, since

$$R(6)+6 = 23 = R(5)+10,$$

it follows that, for $n = 8$, the second and the third schemes are optimal. Thus, so far as the number of moves is concerned, the second scheme as well as the third one may be disregarded.

Now, using the values in Table 1, it can easily be deduced that, for $4 \leq n \leq 7$,

$$R(n-1) = 4n - 11.$$

Thus, the minimum number of moves under the first scheme is simply $4n - 9$. Finally, note that, this number remains valid when $n = 8, 9$.

Hence, the theorem is established.

For t (≥ 1) fixed, let $k = K(t)$ be the point at which $R(n-t-2)$ is attained, so that

$$R(n-t-2) = 2[R(K) + 2^{n-t-K-3} - 1] + 1.$$

It then follows from equation (4) in Scheme 4 that, $R(K) + R(t) + 2^{n-t-K-3} + 2^t + 1$ is the number of moves required to dismantle the topmost $n-1$ smallest discs on the source peg S and distribute them on the two auxiliary pegs just before transferring the largest disc (from S) to the destination peg D. Therefore, in describing the optimal strategy, it is sufficient to give such a half-way solution.

Lemma 1 may be exploited to find the properties as well as the closed-form expression of $S(n)$. This is done below.

***Theorem* 2:** Let n be such that

$$\frac{(t+2)(t+3)}{2} \leq n < \frac{(t+3)(t+4)}{2}$$

for some integer $t \in \{1, 2, \ldots\}$; then,

(1) the optimal strategy corresponding to $n = \frac{(t+3)(t+4)}{2} - 1$ is unique with

$$S\left(\frac{(t+3)(t+4)}{2} - 1\right) = (2t+3)2^{t+1} + 2R(t) + 5,$$

(2) the optimal strategy corresponding to $n = \frac{(t+2)(t+3)}{2}$ is unique with

$$S\left(\frac{(t+2)(t+3)}{2}\right) = (t+1)2^{t+1} + 2R(t) + 5,$$

(3) if $\frac{(t+2)(t+3)}{2} < n < \frac{(t+3)(t+4)}{2} - 1$, then there are two optimal strategies for $S(n)$ with

$$S(n) = 2^{t+1}\left[n - \frac{t(t+3)}{2} - 2\right] + 2R(t) + 5.$$

***Proof*:** When $n = \frac{(t+3)(t+4)}{2} - 1$, then by part (1) of Lemma 1,

$$R(n-t-2) = R\left(\frac{(t+2)(t+3)}{2}\right) = (t+1)2^{t+2} + 1,$$

and hence, by Theorem 1,

$$S(n) = [(t+1)2^{t+2} + 1] + 2[R(t) + 2^t + 2],$$

which, after simplification, gives part (1) of the theorem. By part (1) of Lemma 1, the integer $K = \frac{(t+1)(t+2)}{2}$ is unique, where K is the number of discs that are to be stored, in a tower, on the auxiliary peg P_1, in Step 1 of Scheme 4, and t is the number of discs to be stored, in a tower, on the peg P_2, in Step 2 of Scheme 4.

Next, let $n = \frac{(t+2)(t+3)}{2}$. Since,

$$R(n-t-2) = R\left(\frac{(t+1)(t+2)}{2}\right) = t2^{t+1} + 1,$$

by Theorem 1,

$$S(n) = (t2^{t+1} + 1) + 2[R(t) + 2^t + 2].$$

After some algebraic manipulation, part (2) of the theorem results. By Lemma 1, the integer $K = \frac{t(t+1)}{2}$ is unique.

Finally, consider the case when

$$\frac{(t+2)(t+3)}{2} < n < \frac{(t+3)(t+4)}{2} - 1.$$

Here, since

$$\frac{(t+1)(t+2)}{2} < n-t-2 < \frac{(t+2)(t+3)}{2},$$

it follows, by part (2) of Lemma 1,

$$R(n-t-2) = 2^{t+1}\left[n - \frac{t(t+3)}{2} - 3\right] + 1.$$

Plugging-in this expression for $R(n-t-2)$ in Theorem 1 and simplifying, the desired expression of $S(n)$ is obtained. Note that, $R(n-t-2)$ is attained at two points, namely, at $K = n-2t-4$, $n-2t-3$.

All these complete the proof.

The above analyses show that, for the new version of the Reve's puzzle with single relaxation (or, "cheat") of the "divine rule", the optimal value function $S(n)$ can be expressed in terms of $R(n)$ only. Note that, of the four schemes, only Scheme 3 makes use of Lemma 2. Also, note that, in Scheme 3, after moving the tower T of k discs (from S) to P_1 and the tower of next $n-k-3$ largest discs (from S) to P_2, there are two possibilities: Form the tower of discs d_{n-1} and d_{n-2} either on the peg P_1 or on the peg P_2 to free the largest disc d_n on S. The analyses show further that, for $n \geq 9$, Scheme 4, which depends on n, is the only optimal scheme. It then follows that, for "sufficiently large" n, the Tower of Hanoi with single relaxation of the "divine rule" does not play any role in the Reve's puzzle variant of the problem, if the concern is in the number of moves only.

Multiplicity of optimal strategies

Recall that, if $n-3$ is a triangular number, $R(n-3)$ is attained at a unique point, otherwise, $R(n-3)$ is attained at exactly two points, so that, in this case, there are two optimal solutions of the problem. It is, therefore, an interesting problem to study the multiple optimal solutions. When $n = 4$, there are three optimal strategies, each requiring 7 moves to transfer the tower from the source peg S to the destination peg D, of which one is mentioned in the proof of Theorem 1. For the second optimal strategy, the half-way solution is as follows: Move the discs d_1 and d_2, in this order, from the peg S to the auxiliary peg P_1 (violating the "divine rule"), and then shift the disc d_3 (from S) to the peg P_2 to free the largest disc d_4 for transfer to D. The third optimal strategy is: Move the disc d_1 from S to P_1, next, shift d_2 (from S) to P_2, finally, place the disc d_3 (on S) on top of d_1 on P_1 (violating the "divine rule") to free d_4. When $n = 5$, the number of moves required is 11, and the number of optimal strategies jumps to 18. The first optimal half-way strategy is: Move (from S) d_1 and d_2, in this order to P_1, violating the "divine rule", next, shift the tower of discs d_3 and d_4 (from S) to P_2 to free the largest disc d_5 for transfer to the destination peg D in the next move. The second optimal strategy is: Move (from S) d_1 to P_1 and d_2 to P_2, next, place d_3 on top of d_1 on P_1, violating the "divine rule", next, move d_4 (from S) to P_2, and finally, move d_2 (from D) to P_2 on top of d_4, so that D is free to receive d_5 in the next move. The third optimal strategy is: After moving (from S) d_1 to P_1 and d_2 to D, d_3 is put on top of d_1 on P_1, violating the "divine rule", next, d_2 is moved (from D) to P_1 on top of d_3, and finally, d_4 is moved (from S) to P_2 to free d_5 on S. The fourth optimal strategy is: Move (from S) d_1 to P_1, d_2 to D and d_3 to P_2, next, d_4 is placed on top of d_1 on P_1, violating the "divine rule", and finally, d_2 is moved (from D) to P_2 on top of d_3, so that d_5 may be shifted to D in the next move. The fifth optimal strategy is: Move (from S) d_1 to P_1, d_2 to D and d_3 to P_2, next, d_4 is placed on top of d_1 on P_1 (violating the "divine rule"), and then d_2 (on D) is moved to P_1 on top of d_4. The sixth optimal strategy is: Move (from S) d_1 to P_1, d_2 to P_2 and d_3 to D, next, put d_4 on top of d_1 on P_1, violating the "divine rule", and then shift d_3 (from D) to P_1 on top of d_4 to free D. The seventh optimal strategy is: After moving d_1 to D, shift the discs d_2 and d_3, in this order, to P_1, violating the "divine rule", next, d_4 is shifted (from S) to P_2, and finally, d_1 is moved (from D) on top of d_4 on P_2. The eighth optimal strategy is : After moving d_1 to D and d_2 and d_3, in this order to P_1 (thereby violating the "divine rule") and shifting d_4 to P_2, the disc d_1 (on D) is put on top of d_3 on P_1. The ninth optimal strategy is: Move (from S) d_1 to D, d_2 to P_1 and d_3 to P_2, next, put d_4 on top of d_2 on P_1, violating the "divine rule", and finally, move d_1 (from D) to P_2 on top of d_3 to free D to receive d_5 in the next move. The tenth optimal strategy is: After moving d_1 to D, d_2 to P_1 and d_3 to P_2, d_4 is placed on top of d_2 on P_1 (violating the "divine rule"), next, d_1 is moved (from D) to P_1 on top of d_4 to free D. The eleventh optimal strategy is: Move d_1 to D, d_2 to P_2 and d_3 to P_1, next, put d_4 on top of d_3 on P_1, violating the "divine rule", and finally, shift d_1 (from D) to P_1 on top of d_4. The twelfth optimal strategy is: Move (from S) the tower of discs d_1 and d_2, next, put d_3 on top of this tower, violating the "divine rule", and finally, move d_4 (from S) to P_2 to free d_5 on S. The thirteenth optimal strategy is: After moving (from S) the tower of discs d_1 and d_2 to P_1, and d_3 to P_2, put d_4 on P_1, on top of the tower of discs d_1 and d_2, violating the "divine rule", to free d_5 on S. The fourteenth optimal strategy is: Move the tower of discs d_1 and d_2 (from S) to P_1, next, shift the discs d_3 and d_4, in this order, from S to P_2, violating the "divine rule", to free d_5. The fifteenth optimal strategy is: First, move (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, next, put d_1 (on D) on top of d_3 on P_1, and finally, move d_4 (from S) to P_1 on top of the tower of discs d_1 and d_3, violating the "divine rule". The sixteenth optimal strategy is: Move (from S) d_1 to P_1, d_2 to P_2 and d_3 to D, next, put d_4 on top of d_2 on P_2, violating the "divine rule", and finally, move d_3 (from D) to P_2 on top of d_4 to free D for d_5. The seventeenth optimal strategy is: Move (from S) d_1 to P_1, next, transfer the tower of discs d_2 and d_3 (from S) to P_2, and finally, put d_4 on top of this tower, violating the "divine rule", to free d_5 on S. The eighteenth optimal strategy is: Move (from S) d_1 to P_1, d_2 to D and d_3 to P_2, next, place d_4 on top of d_3 on P_2 (violating the "divine rule"), and finally, move d_2 (from D) to P_2 on top of d_4 to free the peg D. When $n = 6$, there are as many as 40 optimal strategies, each requiring 15 moves. The first optimal half-way solution is: Move the tower of discs d_1 and d_2 (from S) to P_1, next, put d_3 on top of this tower (violating the "divine rule"), and finally, move the tower of discs d_4 and d_5 (from S) to P_2 to free the largest disc d_6 for transfer to the peg D. The second optimal strategy is: After moving the tower of discs d_1 and d_2 (from S) to P_1, place d_3 on D and d_4 on P_1 on top of the tower of two discs, violating the "divine rule", next, move d_5 (from S) to P_2, and finally, shift d_3 (from D) to P_2 on top of d_5 so that d_6 may be shifted to D in the next move. The third optimal strategy is: After moving (from S) the tower of discs d_1 and d_2 to P_1, and the tower of discs d_3 and d_4 to P_2, d_5 is placed on top of the tower of discs d_1 and d_2 on P_1 (violating the "divine rule) to free d_6 for transfer to D. The fourth optimal strategy is: Move the tower of discs d_1, d_2 and d_3 (from S) to P_1, next, put d_4 on top of this tower (violating the "divine rule"), and finally, shift d_5 (from S) to P_2 to free d_6 on S. If, after moving (from S) the tower of discs d_1, d_2 and d_3 to P_1, and d_4 to P_2, d_5 is put on top of the tower of three discs on P_1 (violating the "divine rule"), the fifth optimal solution is found. Again, if after moving the tower of discs d_1, d_2 and d_3 (from S) to P_1, the discs d_4 and d_5 are placed, in this order, on P_2 (violating the "divine rule"), the sixth optimal strategy is obtained. The seventh optimal strategy is: Move (from S) d_1 to P_1 and d_2 to D, next, put d_3 on top of d_1 on P_1, violating the "divine rule", then, shift d_2 (from D) on top of d_3 on P_1, and finally, move the tower of discs d_4 and d_5 (from S) to P_2, to free d_6 on S. The eighth optimal strategy is: After moving (from S) the tower of discs d_1 and d_2 to P_1 and d_3 to D, shift d_4 to P_1 on top of the tower of two discs (violating the "divine rule"), next, move d_3 (from D) to P_1 on top of d_4, and finally, put d_5 on P_2 to free d_6. The ninth optimal strategy is: Move (from S) d_1 to P_1, and the tower of discs d_2 and d_3 to P_2, next, move (from S) d_4 to D and d_5 to P_2 on top of the tower of two discs (violating the "divine rule"), and finally, move d_4 (from D) to P_2 on top of d_5 to free D. The tenth optimal strategy is: After moving (from S) d_1 from S to P_1, and the tower of discs d_2 and d_3 to P_2, shift (from S) d_4 to D and d_5 to P_1 on top of d_1, violating the "divine rule", and finally, move d_4 from D to P_1 on top of d_5 to free D. The eleventh optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to P_2 and d_4 to D, next, put d_5 on top of d_3 on P_2 (violating the "divine rule"), and finally, shift d_4 (from D) to P_2 on top of d_5 so that, in the next move, d_6 may be shifted to D. The twelfth optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to P_2 and d_4 to D, next, place d_5 on P_1 (on top of the tower of two discs, violating the "divine rule"), and finally, move d_4 (from D) to P_1 on top of d_5 to free D to receive d_6 in the next move. The thirteenth optimal strategy is: Move d_1 to P_1, d_2 to D and d_3 to P_2, next, put d_4 on top of d_1 on P_1 (violating the "divine rule"), then, move d_3 (from P_2) to P_1 on top of d_4, next, move d_2 (from D) to P_1 on top of d_3, and finally, place d_5 on P_2 to free d_6. The fourteenth optimal strategy is: After moving (from S) d_1 to P_1, d_2 to D, d_3 to P_2, place d_4 on top of d_1 on P_1 (violating the "divine rule"), then, shift d_3 (on P_2) to P_1 on top of d_4, next, move d_5 (from S) to P_2, and finally, shift d_2 (on D) to P_2 on top of d_5 to free D. The fifteenth optimal strategy is: Move (from S) d_1 to P_1, d_2 to P_2 and d_3 to D, next, put d_4 on top of d_1 on P_1, violating the "divine rule", then, move d_2 (from P_2) to P_1 on top of d_4, next, place d_5 on P_2, and finally, move d_3 (from D) to P_2 on top of d_5 to free D. The sixteenth optimal strategy is: Move (from S) d_1 to D, d_2 to P_1 and d_3 to P_2, next, put d_4 on top of d_2 on P_1, violating the "divine rule", then, move d_3 (from P_2) to P_1 on top of d_4 and d_1 (from D) to P_1 on top of d_3, and finally, shift d_5 (from S) to P_2 to free d_6. The seventeenth optimal strategy is: After moving (from S) d_1 to D, d_2 to P_1, d_3 to P_2, and d_4 on top of d_2 on P_1 (violating the "divine rule"), put d_3 (on P_2) on top of d_4 on P_1, next, move d_5 (from S) to P_2, and finally, shift d_1 (from D) to P_2 on top of d_5 to free D. The eighteenth optimal strategy is: After moving (from S) d_1 to P_2, d_2 to P_1 and d_3 to D, put d_4 on top of d_2 on P_1 (violating the "divine rule"), next, put d_1 (on P_2) on top of d_4 on P_1, then, move d_5 (from S) to P_2, and finally, shift d_3 (from D) to P_2 on top of d_5 to free D. The nineteenth optimal strategy is: Move (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, next, put d_4 on top of d_3 on P_1, violating the "divine rule", then, move d_2 (from P_2) to P_1 on top of d_4 and d_1 (from D) to P_1 on top of d_2, and finally, shift d_5 (from S) to P_2 to free d_6. The twentieth optimal strategy is: After moving (from S) d_1 to D, d_2 to P_2, d_3 to P_1 and d_4 on top of d_3 on P_1, (violating the "divine rule"), put d_2 (on P_2) on top of d_4 on P_1, and d_5 (on S) to P_2, and finally, shift d_1 (from D) to P_2 on top of d_5 to free d_6. The twenty-first optimal strategy is: Move (from S) d_1 to P_2, d_2 to D and d_3 to P_1, next, put d_4 on top of d_3 on P_1, violating the "divine rule", then, move d_1 (from P_2) to P_1 on top of d_4, after shifting d_5 (from S) to P_2, place d_2 (on D) on top of d_5 on P_2 to free D. The twenty-second optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to D and d_4 to P_2, next, put d_5 on top of the tower of two discs on P_1, violating the "divine rule", and then move d_3 (from D) to P_1, on top of d_5 to free D. The twenty-third optimal strategy is: After moving (from S) the tower of discs d_1 and d_2 to P_1, and the tower of discs d_3 and d_4 to P_2, d_5 is placed on top of the latter tower, violating the "divine rule", to free d_6. The twenty-fourth optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to D and d_4 to P_2, next, place d_5 on top of d_4 on P_2, violating the "divine rule", and finally, put d_3 (on D) on top of d_5 on P_2 to free D. The twenty-fifth optimal strategy is: Move (from S) d_1 to P_2, d_2 to D and d_3 to P_1, next, put d_1 on top of d_3 on P_1, then put d_4 on top of d_1 on P_1, violating the "divine rule", next, move d_5 (from S) to P_2, and finally, put d_2 (on D) on top of d_5 on P_2 to free D. The twenty-sixth optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_1 (on P_2) on top of d_3 on P_1, next, move d_4 to P_1 on top of d_1 (violating the "divine rule"), then, move d_2 (from D) to P_1 on top of d_4, and finally, shift d_5 (from S) to P_2, so that, in the next move, d_6 may be transferred from S to D. The twenty-seventh optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_1 (on P_2) on top of d_3 on P_1, next, move d_4 (from S) to P_2, and d_5 to P_1 on top of d_1, violating the "divine rule", and finally, put d_2 (on D) on top of d_4 on P_2 to free D. The twenty-eighth optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_1 (on P_2) on top of d_3 on P_1, next, move (from S) d_4 to P_2 and d_5 to P_1 on top of d_1, violating the "divine rule", and finally, put d_2 (on D) on top of d_4 on P_2 to free D. The twenty-ninth optimal strategy is: Move (from S) d_1 to P_2, d_2 to D and d_3 to P_1, then put d_1 on top of d_3 on P_1, next, move d_4 and d_5, in this order, to P_2 (violating the "divine rule"), and finally, move d_2 (from D) to P_2 on top of d_5 to free D. If, after moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, and putting d_1 on top of d_3 on P_1, d_4 is moved to P_2, then d_2 is shifted (from D) to P_2 on top of d_4, and finally, d_5 is put on top of d_2 on P_2 (violating the "divine rule"), the thirtieth optimal strategy is obtained. Again, if after moving (from *S*) d_1 to *D*, d_2 to P_2 and d_3 to P_1, and putting d_1 on top of d_3 on P_1, d_4 is first moved to *D*, and after moving d_5 to P_2 on top of d_2 (violating the "divine rule"), d_4 is moved (from *D*) to P_2 on top of d_5, the thirty-first optimal strategy is obtained. The thirty-second optimal strategy is: Move (from *S*) d_1 to *D* and the tower of discs d_2 and d_3 to P_1, next, move (from *S*) d_4 to P_1 on top of this tower (violating the "divine rule") and d_5 to P_2, and finally, move d_1 (from *D*) to P_2 on top of d_5 to free *D*. If, after moving (from *S*) d_1 to *D* and the tower of discs d_2 and d_3 to P_1, d_4 is put on this tower (violating the "divine rule"), and d_5 is shifted to P_2, and finally, d_1 is moved (from *D*) to P_1 on top of d_4, the thirty-third optimal strategy is obtained. The thirty-fourth optimal strategy is: Move (from *S*) d_1 to *D* and the tower of discs d_2 and d_3 to P_1, next, move (from *S*) d_4 to P_2 and d_5 to P_1 on top of the tower (violating the "divine rule"), and finally, move d_1 (from *D*) to P_2 on top of d_4 to free *D*. If, after moving (from *S*) d_1 to *D* and the tower of discs d_2 and d_3 to P_1, d_4 is moved to P_2 and d_5 is shifted to P_1 on top of the tower (violating the "divine rule"), and finally, d_1 is moved (from *D*) to P_1 on top of d_5, the thirty-fifth optimal strategy is obtained. The thirty-sixth optimal strategy is: Move (from *S*) d_1 to *D* and the tower of discs d_2 and d_3 to P_1, next, move (from *S*) d_4 and d_5, in this order, to P_2 (violating the "divine rule"), and finally, move d_1 (from *D*) to P_2 on top of d_5 to free *D* to receive d_6 in the next move. The thirty-seventh optimal strategy is: After moving (from *S*) d_1 to *D* and the tower of discs d_2 and d_3 to P_1, d_4 is put on P_2, and after placing d_1 (on *D*) on top of d_4 on P_2, d_5 is placed on top of d_1 on P_2, violating the "divine rule". The thirty-eighth optimal strategy is: Move (from *S*) d_1 to *D*, d_2 to P_1 and d_3 to P_2, next, shift d_1 (from *D*) to P_2 on top of d_3, then move (from *S*) d_4 to *D* and d_5 to P_2 on top of d_1 (violating the "divine rule"), and finally, move d_4 (from *D*) to P_2 on top of d_5 to free *D*. The thirty-ninth optimal strategy is: Move (from *S*) d_1 to P_2 and the tower of discs d_2 and d_3 to P_1, next, put d_4 on *D*, and after placing d_5 on top of d_1 on P_2 (violating the "divine rule"), put d_4 (on *D*) on top of d_5 on P_2. The fortieth optimal strategy is: Move (from *S*) d_1 to *D*, and d_2 and d_3, in this order, to P_1, violating the "divine rule", next, shift (from *D*) d_1 on top of d_3 on P_1, and finally, move the tower of discs d_4 and d_5 (from *S*) to P_2 to free d_6 on *S*. When $n = 7$, there are as many as 30 optimal strategies, each requiring 19 moves. The first optimal half-way strategy is: Transfer the tower of discs d_1, d_2 and d_3 (from *S*) to P_1, next, put d_4 on top of this tower (violating the "divine rule"), and finally, move the tower of discs d_5 and d_6 (from *S*) to P_2 to free the largest disc d_7. The second optimal strategy is: After transferring the tower of discs d_1, d_2 and d_3 (from *S*) to P_1, move d_4 to *D* and d_5 to P_1 on top of the tower of three discs (violating the "divine rule"), next, move d_6 (from *S*) to P_2 and then shift d_4 (from *D*) to P_2 on top of d_6, so that, in the next move, the largest disc d_7 may be shifted to *D*. If, after shifting the tower of discs d_1, d_2 and d_3 (from *S*) to P_1, the tower of discs d_4 and d_5 is formed on P_2, and finally, d_6 is put on P_1 on top of the tower (violating the "divine rule"), the third optimal strategy is found. Again, if after forming the tower of discs d_1, d_2 and d_3 on P_1, and the tower of discs d_4 and d_5 on P_2, the discs d_6 is put on P_2 on top of the tower of two discs, the fourth optimal strategy is obtained. The fifth optimal strategy is: Move (from *S*) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to *D* and d_5 to P_2, next, put d_6 on top of d_5 on P_2, violating the "divine rule", and finally, shift d_4 (from *D*) to P_2 on top of d_6 to free d_7. The sixth optimal strategy is: Move (from *S*) the tower of discs d_1 and d_2 to P_1 and d_3 to P_2, next, put d_4 on top of the tower of two discs on P_1, violating the "divine rule", then, move d_3 (from P_2) on top of d_4 on P_1, and finally, shift the tower of discs d_5 and d_6 (from *S*) to P_2 to free d_7. The seventh optimal strategy is: Transfer (from *S*) the tower of discs d_1, d_2 and d_3 to P_1 and d_4 to P_2, then, after placing d_5 on P_1 on top of the tower of three discs (violating the "divine rule"), d_4 is put on d_5, and finally, d_6 is moved (from *S*) to P_2 to free d_7. The eighth optimal strategy is: Move (from *S*) the tower of discs d_1 and d_2 to P_1, and the tower of discs d_3 and d_4 to P_2, next, move (from S) d_5 to D and d_6 to P_2 (on top of the tower of two discs, violating the "divine rule"), and finally, put d_5 (on D) on top of d_6 on P_2. If, after moving (from S) the tower of discs d_1 and d_2 to P_1, and the tower of discs d_3 and d_4 to P_2, the tower of discs d_5 and d_6 is formed on P_1 (violating the "divine rule"), the ninth optimal strategy is found. The tenth optimal strategy is as follows: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to P_2 and d_5 to D, then, after placing d_6 on top of d_4 on P_2 (violating the "divine rule"), put d_5 on top of d_6, so that D is ready to accept d_7 in the next move. If, after moving (from S) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to P_2 and d_5 to D, the disc d_6 (on S) is placed on P_1 on top of the tower (violating the "divine rule"), and then, d_5 is put on d_6 on P_1, the eleventh optimal strategy is found. The twelfth optimal strategy is: Move (from S) d_1 to P_1, d_2 to D and d_3 to P_2, next, shift d_4 (from S) to P_1 on top of d_1 (violating the "divine rule"), then put d_3 (on P_2) on top of d_4 on P_1 and d_2 (on D) on top of d_3 on P_1, and finally, move the tower of discs d_5 and d_6 to P_2 to free d_7. The thirteenth optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to D and d_4 to P_2, next, place d_5 (on S) on P_1 on top of the tower of two discs, violating the "divine rule", then, move d_4 (from P_2) on top of d_5 on P_1 and d_3 (on D) on top of d_4 on P_1, finally, move d_6 (from S) to P_2 to free d_7 on S. The fourteenth optimal strategy is: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to D and d_5 to P_2, next, put d_6 on top of the tower on P_1, violating the "divine rule", then, move d_4 (on D) on top of d_6 on P_1 to free D to receive d_7 in the next move. The fifteenth optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to P_2 and d_4 to D, next, place d_5 (on S) on P_1 on top of the tower, violating the "divine rule", then, move d_3 (on P_2) on top of d_5 on P_1, now, shift d_6 (on S) to P_2, and finally, move d_4 (from D) to P_2 on top of d_6 to free D. The sixteenth optimal strategy is: After moving (from S) the tower of discs d_1 and d_2 to P_1, d_3 to D and d_4 to P_2, place d_5 (on S) on P_1 on top of the tower, violating the "divine rule", then, move d_4 (from P_2) on top of d_5 on P_1, next, shift d_6 (on S) on P_2, and finally, move d_3 (from D) to P_2 on top of d_6 to free D. The seventeenth optimal strategy is: Move (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, next, place d_1 (on D) on top of d_3 on P_1, then put d_4 on top of d_1 on P_1, violating the "divine rule", now, place d_2 (on P_2) on top of d_4 on P_1, and finally, move the tower of discs d_5 and d_6 (from S) to P_2 to free d_7. The eighteenth optimal strategy is: After moving (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, put d_1 (on D) on top of d_3 on P_1, next, put d_4 on D and d_5 on top of d_1 on P_1, violating the "divine rule", now, place d_2 (on P_2) on top of d_5 on P_1, and d_6 (on S) on P_2, and finally, move d_4 (on D) to P_2 on top of d_6 to free D. The nineteenth optimal strategy is: Move (from S) d_1 to P_2, d_2 to D and d_3 to P_1, next, put d_1 on top of d_3 on P_1, then shift (from S) d_4 to P_2, and d_5 on top of d_1 on P_1, violating the "divine rule", now, place d_4 (on P_2) on top of d_5 on P_1, next, shift d_6 to P_2, and finally, move d_2 (on D) to P_2 on top of d_6 to free D. The twentieth optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_1 on top of d_3 on P_1, and then shift (from S) d_4 to P_2, and d_5 on top of d_1 on P_1, violating the "divine rule", now, place d_4 (on P_2) on top of d_5 on P_1, and d_2 on top of d_4 on P_1, and finally, move d_6 (on S) to P_2 to free d_7. The twenty-first optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_1 on top of d_3 on P_1, and d_4 on P_2, next, put d_2 (on D) on top of d_4 on P_2, and d_5 (on S) on D, now, place d_6 (on S) on top of d_1 on P_1, violating the "divine rule", and finally, move d_5 (from D) to P_1 on top of d_6 to free D. The twenty-second optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_1 on top of d_3 on P_1, and d_4 on P_2, next, put d_2 (on D) on top of d_4 on P_2, now, shift (from S) d_5 to D and d_6 to P_2 on top of d_2, violating the "divine rule", and finally, move d_5 (on D) to P_2 on top of d_6 to free D. The twenty-third optimal strategy is: Move (from S) d_1 to P_2, d_2 to D and d_3 to P_1, next, place d_2 (on D) on top of d_3 on P_1, then put d_4 on top of d_2 on P_1, violating the "divine rule", now, place d_1 (on P_2) on top of d_4 on P_1, and finally, move the tower of discs d_5 and d_6 (from S) to

P_2 to free d_7. The twenty- fourth optimal strategy is: After moving (from S) d_1 to P_2, d_2 to D and d_3 to P_1, put d_2 (on D) on top of d_3 on P_1, then shift (from S) d_4 to D and d_5 on top of d_2 on P_1, violating the "divine rule", now, place d_1 (on P_2) on top of d_5 on P_1, next, put d_6 on P_2, and finally, move d_4 (on D) on top of d_6 on P_2 to free D. The twenty-fifth optimal strategy is: After moving (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, put d_2 (on P_2) on top of d_3 on P_1, next, shift (from S) d_4 to P_2 and d_5 on top of d_2 on P_1, violating the "divine rule", now, place d_4 (on P_2) on top of d_5 on P_1, then put d_6 on P_2, and finally, move d_1 (on D) on top of d_6 on P_2 to free D. The twenty-sixth optimal strategy is: After moving (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, put d_2 (on P_2) on top of d_3 on P_1, then, shift (from S) d_4 to P_2 and d_5 on top of d_2 on P_1, violating the "divine rule", next, place d_4 (on P_2) on top of d_5 on P_1, and d_1 (on D) on top of d_4 on P_1, and finally, move d_6 (from S) to P_2 to free d_7. The twenty- seventh optimal strategy is: After moving (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, put d_2 (on P_2) on top of d_3 on P_1, next, shift d_4 to P_2 and place d_1 (on D) on top of d_4 on P_2, now, shift (from S) d_5 to D and d_6 on top of d_2 on P_1, violating the "divine rule", now, place d_1 (on P_2) on top of d_5 on P_1, and finally, move d_5 (on D) on top of d_6 on P_1 to free D. The twenty-eighth optimal strategy is: After moving (from S) d_1 to D, d_2 to P_2 and d_3 to P_1, put d_2 (on P_2) on top of d_3 on P_1, next, shift d_4 to P_2 and on d_4 put d_1 (from D), now, shift (from S) d_5 to D and d_6 on top of d_1 on P_2, violating the "divine rule", and finally, move d_5 (on D) on top of d_6 on P_2 to free D. The twenty-ninth optimal strategy is: Move (from S) d_1 to D, d_2 to P_1, d_3 to P_2 and d_4 on top of d_2 on P_1, violating the "divine rule", next, put d_3 (on P_2) on top of d_4 on P_1, and d_1 (on D) on top of d_3 on P_1, and finally, shift (from S) the tower of discs d_5 and d_6 on P_2 to free d_7. The thirtieth optimal strategy is: Move (from S) d_1 to D, d_2 to P_2, d_3 to P_1, and d_4 on top of d_3 on P_1, violating the "divine rule", next, shift d_2 (on P_2) on top of d_4 on P_1, and d_1 (on D) on top of d_2 on P_1, now, shift (from S) the tower of discs d_5 and d_6 to P_2 to free d_7. When $n = 8$, there are nine optimal strategies, each requiring 23 moves. The first optimal half-way solution is: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, and d_4 to D, next, shift d_5 (on S) to P_1 on top of the tower of three discs, violating the "divine rule", then, shift d_4 (on D) to P_1 on top of d_5, and finally, move the tower of discs d_6 and d_7 from S to P_2 to free the largest disc d_8 on S. The second optimal strategy is: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, and the tower of discs d_4 and d_5 to P_2, next, shift (from S) d_6 to D and d_7 on top of the tower of discs d_4 and d_5 on P_2, violating the "divine rule", and finally, move d_6 (on D) on top of d_7 on P_2, so that d_8 may now be shifted to D. The third optimal strategy is: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, and the tower of discs d_4 and d_5 to P_2, next, shift (from S) d_6 to D and d_7 to P_1, on top of the tower of three discs, violating the "divine rule", and finally, put d_6 (on D) on top of d_7 on P_1 to free D. The fourth optimal strategy is: Move (from S) the tower of discs d_1 and d_2 to P_1, d_3 to D and d_4 to P_2, next, put d_5 (on S) on top of the tower of three discs on P_1, violating the "divine rule", then, move d_4 (on P_2) and d_3 (on D), in this order, to P_1 to form the tower of discs d_3, d_4 and d_5, and finally, move the tower of discs d_6 and d_7 to P_2 to free d_8. The fifth optimal strategy is: Transfer (from S) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to D and d_5 to P_2, next, put d_6 (on S) on top of the tower of three discs on P_1, violating the "divine rule", then, move d_5 (on P_2) and d_4 (on D), in this order, to P_1 to form the tower of discs d_4, d_5 and d_6, and finally, move d_7 (from S) to P_2 to free d_8. The sixth optimal strategy is: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to D and d_5 to P_2, next, put d_6 (on S) on top of the tower of three discs on P_1, violating the "divine rule", now, place d_5 (on P_2) on top of d_6 on P_1, then, shift (from S) d_7 to P_2, and finally, move d_4 (on D) on top of d_7 on P_2 to free D. The seventh optimal strategy is: Move (from S) the tower of discs d_1, d_2 and d_3 to P_1, d_4 to P_2 and d_5 to D, next place d_6 (on S) on top of the tower of three discs on P_1, violating the "divine rule", then, move d_4 (from P_2) to P_1 on top of d_6, now, shift (from S) d_7 to P_2, and finally, move d_5 (from D) to P_2 on top of d_7 to free D. The eighth optimal strategy is: Move (from S) d_1 to D, d_2 to P_2, d_3 to P_1, and then put d_2 (on P_2) on top of d_3 on P_1, next, shift (from S) d_4 to P_2 and d_5 to P_1 on top of d_2, violating the "divine rule", now, move d_4 (on P_2) and d_1 (on D), in this order, to P_1 on top of d_5, and finally, move (from S) the tower of discs d_6 and d_7 to P_2 to free d_8 on S. The ninth optimal strategy is: Move (from S) d_1 to P_2, d_2 to D, d_3 to P_1, and then put d_1 (on P_2) on top of d_3 on P_1, next, shift (from S) d_4 to P_2 and d_5 to P_1 on top of d_1, violating the "divine rule", now, move d_4 (on P_2) and d_2 (on D), in this order, to P_1 on top of d_5, and finally, shift (from S) the tower of discs d_6 and d_7 to P_2 to free d_8 on S to move next. There is a unique optimal strategy, namely, Scheme 4 (with $k = 3$), when $n = 9$.

So far, the focus has been on the multiple optimal strategies when $4 \le n \le 8$. In this connection, the following result may be established.

***Lemma* 3:** $S(n, t) = S(n, t + 1)$ if and only if $t = 1$, $10 \le n \le 13$.

***Proof*:** First note that

$$S(n, t) = S(n, t + 1)$$

if and only if

$$R(n - t - 2) - R(n - t - 3) = 2^{t+1} + 2[R(t + 1) - R(t)].$$

Now, $R(n - t - 2) - R(n - t - 3) = 2^{t+2}$ if and only if $t = 1$ (by Corollary 4). In this case, by part (2) of Lemma 1,

$$\frac{(t+2)(t+3)}{2} < n - t - 3 < \frac{(t+3)(t+4)}{2}$$

so that

$$\frac{(t+3)(t+4)}{2} \le n < \frac{(t+3)(t+6)}{2}.$$

Putting $t = 1$, the desired result is obtained.

Applying Lemma 3, three optimal solutions are obtained when $n = 10$, when $n = 11$, there are four optimal solutions, when $n = 12$, there are four optimal solutions, while three optimal solutions are obtained when $n = 13$.

Table 1. *R*(*n*) and *S*(*n*) for 4 ≤ *n* ≤ 8.

n	4	5	6	7	8
***R*(*n*)**	9(3)	13(18)	17(40)	25(30)	33(9)
***S*(*n*)**	7	11	15	19	23

In Table 1 above, the number within parentheses gives the number of optimal strategies. Thus, for example, when $n = 4$, the number of moves required is $R(4) = 9$, and there are three optimal strategies each requiring 9 moves. Table 2 gives the values of $R(n)$ and $S(n)$ for $9 \le n \le 19$.

Table 2. *R*(*n*) and *S*(*n*) for 9 ≤ *n* ≤ 19.

n	*R*(*n*)	*S*(*n*)
9	41	27
10	49	35
11	65	43
12	81	51
13	97	59
14	113	75
15	129	79
16	161	95
17	193	111
18	225	127
19	257	143

Generalizations

An immediate generalization of the present problem is the Reve's puzzle with (at most) c (≥ 2) relaxations (or, "cheat") of the "divine rule". Let $S(n, c)$ be the minimum number of moves required to solve the Reve's puzzle with n (≥ 1) discs when (at most) c relaxations of the "divine rule" are allowed. Then, we can prove the following result.

***Lemma* 4:** For $1 \le n \le c + 3$, $S(n, c) = 2n - 1$.

***Proof*:** It is sufficient to consider the case when $n = c + 3$. In this case, the half-way optimal strategy is to form the inverted tower with the smallest $c + 1$ discs on the peg P_1 (violating the "divine rule" c times), in $c + 1$ moves, followed by the transfer of the disc d_{c+2} to the peg P_2, to free the largest disc d_{c+3} on the peg S. The total number of moves required is

$$2(c + 2) + 1 = 2c + 5 = 2n - 1.$$

Another problem of interest is to extend the concept of the relaxation of the "divine rule" to the p-peg Tower of Hanoi problem with $p \geq 5$.

Conclusion

The primary objective of the paper is to initiate the study on a new generalization of the Reve's puzzle, which permits relaxation of the "divine rule". This paper considers in complete detail the variant with n discs, allowing the possibility of a single relaxation (or, "cheat") of the "divine rule". It is found analytically that the optimal value function, $S(n)$, can be expressed in terms of $R(n)$. It is also found that, for $n \geq 9$, Scheme 4 is the only optimal strategy. When $4 \leq n \leq 8$, there are multiple optimal strategies which have been found. Lemma 4 shows that there are multiple optimal solutions when $10 \leq n \leq 13$.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this article.

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