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Research Article

Interior regularity to the signed solution of the singular doubly nonlinear parabolic equations

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ABSTRACT

We study doubly nonlinear parabolic equation with sign changing solutions. We establish the Hölder regularity of the singular parabolic equations within a parabolic domain.

Introduction

Let $\Omega \subset \mathbb{R}^N$ and for $T > 0$ define the cylindrical doamin $\Omega_T := \Omega \times (0, T]$. Consider the following doubly nonlinear parabolic equation

$$\partial_t(|u|^{p-2}u) - div(|Du|^{p-2}Du) = 0 \quad weakly\ in \quad \Omega_T \tag{1}$$

where $\Delta_p u := div(|Du|^{p-2}Du)$ is the p-Laplacian. For the case $p = 2$ then this operator transforms to well known heat equation. In this manuscript, the weak solution u is unknown and assumed to be locally bounded, real function which depends on both the time and space variables namely x and t in the cylindrical domain.

In our context, the term structural data indicates the parameters p and N. It is also

assumed that the constant $\gamma > 0$, need to be evaluated quantitatively apriori in terms of the structural data. In addition, deno te $\Gamma_T := \partial\Omega_T - \overline{\Omega} \times \{T\}$ to be the parabolic boundary of the cylindrical domain Ω_T. For $\theta > 0$, consider the follo wing backward cylinders of the form $(x_0, t_0) + Q_\varrho(\theta) = (x_0, t_0) + K_\varrho(0) \times (-\theta\varrho^p, 0] = K_\varrho(x_0) \times (t_0 - \theta\varrho^p, t_0]$.

For the case $\theta = 1$, we will call it as Q_ϱ.

The conclusions to be derived from the discoveries presented in this article are summarized as follows.

Theorem 1.1

Let's consider a bounded domain with a smooth boundary, denoted as $\partial\Omega$. Given that u constitutes a local weak solution bounded by (1) in Ω_T, it follows that u exhibits local Hölder continuity within Ω_T. Precisely, there exist constants $\gamma > 1$ and $\beta \in (0,1)$, predetermined based on the data, such that for any compact subset $C \subset \Omega_T$, the inequality

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \gamma \parallel u \parallel_{\infty,\Omega_T} \left(\frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}}}{dist_p(C, \Gamma_T)} \right)^\beta,$$

holds true for any pair of points $(x_1, t_1), (x_2, t_2) \in C$. The following oscillation decay will be demonstrated as part of the proof of the

aforementioned theorem:

$$ess\ osc_{(x_0,t_0)+Q_r} u \leq \gamma ess\ osc_{(x_0,t_0)+Q_\varrho} u \left(\frac{r}{\varrho}\right)^\beta,$$

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for any pair of cylinders $(x_0, t_0) + Q_r \Subset (x_0, t_0) + Q_\varrho \Subset \Omega_T$. A typical covering argument can be used to draw the conclusion of Theorem 1.1 at the end.

A weak solution is defined in Definition 3, and (Kuusi et al., 2021) examines the weak solution's global existence.

1.1 Originality and Importance

The standard equation (1) is referred to as Trudinger's equation. Because of the nonlinear nature of the solution as well as the gradient in its spatial domain, the equation is sometimes known as a doubly nonlinear parabolic equation. Our choice of this particular type of equation for study is particularly intriguing because of its excellent mathematical structure, capable of producing mixed forms of degeneracy and/or singularity in partial differential equations. It also has connections to physical models, such as the dynamics of glaciers in (Mahaffy, 1976), shallow water flows in (Alanso et al., 2008; Feng and Molz, 1997; Hromadka et al., 1985), and friction-dominated flow in a gas network in (Leugering and Mophou, 2018). Another natural connection between the Trudinger equation and the non-linear eigenvalue issue $-\Delta_p u = \lambda |u|^{p-2} u$ (Lindgren and Lindqvist, 2022) is that it is crucial to nonlinear potential theory.V. B. Ogelein, F. Duzzar, and N. Liao examined the Hölder continuity of signed solutions for broader equations under structural constraints in (Bogelion et al., 2021). Through Moser's iteration, Trudinger (Trudinger, 1968b) investigates the Hölder regularity of this equation and finds that, similar to the heat equation, it has a Harnack inequality for non-negative weak solutions. This Harnack inequality is used in (Kussi et al., 2012a; Kuusi et al., 2012b) to prove the Hölder regularity of nonnegative weak solutions. Now, we have the opportunity to discuss our contribution. To ensure Hölder regularity, we remove the constraint of non-negativity from solutions and instead consider sign-changing solutions. The Harnack inequality (Giannnaza and Vespri, 2006, Urbano, 2008) is not applicable in our scenario as it only applies to non-negative solutions. Instead, we employ the positivity expansion to achieve our desired result.

By comparing the oscillation with the supremum/infimum of the solution, our demonstration of interior Hölder regularity unfolds in two primary cases: when the solution approaches zero or when it significantly deviates from zero. Utilizing our equation's scaling invariant property, we can derive the positivity expansion in the first case. Without the use of intrinsic scaling procedures, Proposition 3.1 is analogous to the classical parabolic theory found in (Ladyzenskaja et al., 1968). On the other hand, the behavior of the solution in the latter case resembles that of the parabolic p-Laplacian equation, specifically$u_t = \Delta_p u$. Consequently, success in this second scenario depends on our ability to handle a degenerate case $(p > 2)$ or a singular case $(1 < p < 2)$ equation, for which we utilize the theory that has already been developed in (DiBenedetto, 1993; DiBenedetto et al., 2012). The paper (Nakamura and Misawa, 2018) illustrates the presence of a weak solution to equation (1). Additionally, research into the Hölder regularity of doubly nonlinear equations has been explored in (Ivanov, 1994, 1995; Ivanov and Mkrtychyan, 1994; Kinnunen and Kuusi, 2007; Sarkar, 2022; Vespri, 1992; Vespri and Vestberg, 2020). We want to implement the Theorem 1.1.

Preliminaries

We establish certain notations and tools for technical analysis that will be utilized subsequently. [cf.(Bogelion et al., 2021, DiBenedetto, 2016, 1993, 1986, 1983, Evans, 1998, Ladyzenskaja et al., 1968, Leugering and Mophou, 2018, Trudinger, 1968b)].

2.1 Notation

2.1.1 Concept of Local Weak Solution

Let u be a function belonging to

$$u \in C(0, T; L^p_{loc}(\Omega)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega)) \qquad (2)$$

It is considered a local weak sub(super)-solution to (1) if, for every compact subset C of Ω and each sub-interval $[t_1, t_2] \subset (0, T]$

$$\int_C |u|^{p-2}u\ \zeta\ dx\Big|_{t_1}^{t_2} + \iint_{C\times(t_1,t_2)} [-|u|^{p-2}u\zeta_t + |Du|^{p-2}Du \cdot D\zeta]dx\, dt \leq (\geq) 0 \qquad (3)$$

holds for all non-negative test functions

$$\zeta \in W^{1,p}_{loc}(0,T;L^p(C)) \cap L^p_{loc}(0,T;W^{1,p}_0(C)).$$

ensuring the convergence of all integrals in (3). A function u satisfying both the conditions of being a local weak subsolution and a local weak supersolution to (3) is termed a local weak solution.

2.1.2 Function Spaces on a time-space area

We define several function spaces that operate in space-time domains. For $1 \leq p, q \leq \infty$, $L^q(t_1,t_2\,;\,L^p(\Omega))$ represents a collection of measurable real-valued functions defined on$\Omega \times (t_1, t_2)$, encompassing a finite-region in both space and time and characterized by a norm that may not be bounded:

$$\| v \|_{L^q(t_1,t_2\,;\,L^p(\Omega))}: = \begin{cases} \left(\int_{t_1}^{t_2} \| v(t) \|^q_{L^p(\Omega)}\ dt\right)^{1/q} & \text{if} 1 \leq q < \infty \\ ess\ sup_{t_1\leq t\leq t_2} \| v(t) \|_{L^p(\Omega)} & \text{if} q = \infty \end{cases}$$

where

$$\| v(t) \|_{L^p(\Omega)}: = \begin{cases} \left(\int_\Omega |v(x,t)|^p\ dx\right)^{1/p} & \text{if} 1 \leq p < \infty \\ ess\ sup_{x\in\Omega}|v(x,t)| & \text{if} p = \infty \end{cases}$$

For simplicity, we use $L^p(\Omega \times (t_1,t_2)) = L^p(t_1,t_2\,;\,L^p(\Omega))$ when $p = q$. For $1 \leq p < \infty$, the Sobolev Space $W^{1,p}(\Omega)$ consists of weakly differentiable measurable real-valued functions whose weak derivatives are p-th integrable on Ω, with the norm

$$\| w \|_{W^{1,p}(\Omega)}: = \left(\int_\Omega |w|^p + |\nabla w|^p\ dx\right)^{1/p}$$

where $\nabla w = (w_{x_1}, \dots, w_{x_n})$ indicates, in a distribution sense, the gradient of w, and let $W^{1,p}_0(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ with the norm $\|\cdot\|_{W^{1,p}}$. Additionally, we define $L^q(t_1,t_2\,;\,W^{1,p}_0(\Omega))$ as a function space of measurable real-valued functions on a space-time region with a bounded norm:

$$\| w \|_{L^q(t_1,t_2\,;\,W^{1,p}_0(\Omega))} = \left(\int_{t_1}^{t_2} \| w(t) \|^q_{W^{1,p}(\Omega)}\ dt\right)^{1/q} \qquad (3)$$

Consider $\Omega \subset \mathbb{R}^n$ as a bounded domain. The truncation of a function v for a real number m can be expressed as

$$(v-m)_+ := \max\{(v-m), 0\}; \quad (v-m)_- := -\min\{(v-m), 0\}. \qquad (4)$$

For a measurable function v in $L^1(\Omega)$ and real numbers $m < n$, we introduce the sets

$$\begin{cases} \Omega \cap \{v > n\} := \{x \in \Omega :\ v(x) > n\} \\ \Omega \cap \{v < m\} := \{x \in \Omega :\ v(x) < m\} \\ \Omega \cap \{m < v < n\} := \{x \in \Omega :\ m < v(x) < n\}. \end{cases}$$

2.2 Technical tools

Let's begin by recalling De Giorgi's inequality (refer to DiBenedetto, 1993).

Proposition 2.1 (Inequality of De Giorgi)

Consider $v \in W^{1,1}(B)$ and real numbers

$k, m \in \mathbb{R}$ satisfying $k < m$. Then there exists a positive constant C dependent solely on p as well as n in a way that

$$(k-m)|B \cap \{v > k\}| \leq C\frac{\rho^{n+1}}{|B\cap\{v<m\}|}\int_{B\cap\{k<v<m\}} |\nabla v|\, dx. \qquad (5)$$

Following the approach in (DiBenedetto, 1993), we introduce the auxiliary function

$$\begin{cases} A^+(k,u) := +(p-1)\int_k^u |s|^{p-2}(s-k)_+\, ds \\ A^-(k,u) := -\int_k^u |s|^{p-2}(s-k)_-\, ds \end{cases} \qquad (6)$$

for $u, k \in \mathbb{R}$. In the special case of $k = 0$, we simplify as $A^+(u) = A^+(0,u)$ and $A^-(u) = A^-(0,u)$.

It's evident that $A^\pm \geqslant 0$. We introduce bold notation $\mathbf{b}^\alpha$ to represent the signed α-exponent of b, as defined below

$$b^{\alpha} = \begin{cases} |b|^{\alpha-1}\, b, & b \neq 0, \\ 0, & b = 0. \end{cases}$$

We present a known lemma; cf. (Acerbi and Fusco, 1989, Sarkar, 2023: Lemma 2.2) and (Giaquinta and Modica, 2006) for $\alpha > 1$. This lemma is utilized in the proof of the subsequent lemma:

Lemma 2.2

For each positive value of α, there exists a specific constant β, denoted as β(α), for which the *inequality below holds for any pair of real numbers a, b:*

$$\frac{1}{\beta}|b^{\alpha} - a^{\alpha}| \leqslant (|a| + |b|)^{\alpha-1}|b - a| \leqslant \beta|b^{\alpha} - a^{\alpha}|.$$

Building upon the aforementioned lemma, we establish the following result.

Lemma 2.3

There exists a constant $\beta = \beta(p)$ such that the following inequality holds for all $w, k \in \mathbb{R}$ and $\alpha > 0$:

$$\frac{1}{\beta}(|w| + |k|)^{p-2}(w-k)_{\pm}^{2} \leqslant A^{\pm}(k, w$$
$$\leqslant \beta(|w| + |k|)^{p-2}(w-k)_{\pm}^{2}$$

We introduce a type of time mollification for the solution u to enhance its time regularity: $[u]_h(x,t) \overset{def}{=} \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} u(x,s)ds$ for any $u \in$ $in\ L^1(\Omega_T)$

Lemma 2.4 (Properties of mollification)

(Kinnunen and Lindqvist, 2006)

1. If $u \in L^p(\Omega_T)$, then

$$\| [u]_h(x,t) \|_{L^p(\Omega_T)} \leqslant \| u \|_{L^p(\Omega_T)} \text{ and}$$

$$\frac{\partial [u]_h}{\partial t} = \frac{u - [u]_h}{h} \in L^p(\Omega_T).$$

Moreover, $[u]_h \to u$ in $L^p(\Omega_T)$ as $h \to 0$.

2. If, additionally, $\nabla([u]_h) = [\nabla u]_h$ componentwise,

$$\| \nabla([u]_h) \|_{L^p(\Omega_T)} \leqslant \| \nabla u \|_{L^p(\Omega_T)}$$

and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.

3. Furthermore, if $u_k \to u$ in $L^p(\Omega_T)$, then

$$[u_k]_h \to [u]_h \text{ and } \frac{\partial [u_k]_h}{\partial t} \to \frac{\partial [u]_h}{\partial t}$$

in $L^p(\Omega_T)$. and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.

4. If $\nabla u_k \to \nabla u$ in $L^p(\Omega_T)$, then also $\nabla[u_k]_h \to \nabla[u]_h$ in $L^p(\Omega_T)$.

5. Similar results hold for weak convergence in $L^p(\Omega_T)$.

6. Lastly, if $\varphi \in C(\overline{\Omega}_T)$, then $[\varphi]_h(x,t) + e^{-\frac{t}{h}}\varphi(x,0) \to \varphi(x,t)$ uniformly in Ω_T as $h \to 0$.

Moving forward, we will employ the following energy estimate (as found in Sarkar, 2022).

We briefly outline the estimate before proceeding with the main proof.

Proposition 2.5

Assume that u serves as a subsolution in a local sense for equation (1). In this context, there exists positive constant $\gamma(p)$ in a way that for any cylinders $Q_{R,S} = K_R(x_0) \times (t_0 - S, t_0) \Subset \Omega_T$, the subsequent inequality is satisfied for every non-negative piecewise smooth cutoff function ζ that vanishes along $\partial K(x_0) \times (t_0 - S, t_0)$, as well as for any $k \in \mathbb{R}$:

$$\operatorname{ess\,sup}_{t_0 - S < t < t_0} \int_{K_R(x_0)\times\{t\}} \zeta^p A^{\pm}(k,u)\, dx + \iint_{Q_{R,S}} \zeta^p |D(u-k)_{\pm}|^p\, dx\, dt$$

$$(7) \leqslant \gamma \iint_{Q_{R,S}} [|D\zeta|^p (u-k)_{\pm} + A^{\pm}(k,u)|\partial_t \zeta^p|]\, dx\, dt + \int_{K_R(x_0)\times\{t_0 - S\}} \zeta^p A^{\pm}(k,u)\, dx$$

Positivity expansion

Consider $K \subset \mathbb{R}^n$ and a cylinder $\mathcal{Q} \overset{def}{=} K \times (t_1, t_2) \subset \Omega_T$. Throughout this section, we will utilize the following notations:

$$\mu^+ \geq \operatorname{ess\,sup}_{\mathcal{Q}} u, \quad \mu^- \leq \operatorname{ess\,inf}_{\mathcal{Q}} u, \quad \omega = \mu^+ - \mu^-.$$

We also assume that $(x_0, t_0) \in \mathcal{Q}$ for defining the forward cylinder

$$K_{8\varrho}(x_0) \times (t_0, t_0 + (8\varrho)^p) \subset \mathcal{Q}. \tag{9}$$

In this context, we present the proposition regarding the extension of positivity. The

complete proof can be found in (Sarkar, 2023).

Proposition 3.1

Given that u is locally limited and acts as a sub(super)solution on a local scale for equation (1) within the domain Ω_T, and for a specific point $(x_0, t_0) \in \Omega_T$, as well as for constants M, α, and ϱ, where $M > 0$, and α belongs to the interval (0,1), while $\varrho > 0$ the ensuing conditions are met: (9) and $|\{\pm(\mu^\pm - u(.,t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha|K_\varrho|$.

Subsequently, constants ξ, δ, and η all falling within the range of (0,1), can be identified based solely on the provided information and the value of α. This leads to either

$$|\mu^\pm| > \xi M$$

or

$$\pm(\mu^\pm - u) \geq \eta M$$

almost every where in $K_{2\varrho}(x_0) \times (t_0 + \delta(\frac{1}{2}\varrho)^p, t_0 + \delta\varrho^p)$,

where

$$\xi = \begin{cases} 2\eta, & \text{if } p > 2, \\ 8, & \text{if } 1 < p \leq 2. \end{cases}$$

The proof of Proposition 3.1 follows directly from three lemmas presented in subsequent sections. Here, we provide the statements of these lemmas, which collectively form the foundation for proving the expansion of positivity. For detailed proofs, please refer to (Sarkar, 2022).

3.1 Extension of Positivity in Measure

Lemma 3.2

Take any positive M and $\alpha \in (0,1)$ into account. Consequently, there are δ and ε within the range of (0,1), and their values are exclusively *determined by the provided information and the value of α. In cases where u functions as a locally restricted sub(super)-solution to equation (1) within Ω_T, adhering to the condition*

$$|\{\pm(\mu^\pm - u(\cdot, t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha|K_\varrho|,$$

we have either

$$|\mu^\pm| > 8M$$

or

$$\left|\left\{\pm\left(\mu^\pm - u(\cdot,t)\right) \geq \varepsilon M\right\} \cap K_\varrho(x_0)\right| \geq \frac{\alpha}{2}\left|K_\varrho\right| \quad \text{for all } t \in (t_0, t_0 + \delta\varrho^p). \tag{10}$$

3.2 Lemma of shrinking

Lemma 3.3

Given the assumptions in Lemma 3.2, the second option (10) is true. Let $Q = K_\varepsilon(x_0) \times (t_0, t_0 + \delta\varepsilon^p]$ denote the corresponding cylindrical domain, and let $\tilde{Q} = K_{4\varepsilon}(x_0) \times (t_0, t_0 + \delta\varrho^p] \subset \Omega_T$. A positive constant γ, which is exclusively dependent on the given data and α, exists. This constant is such that for any positive integer j_*, when $1 < p < 2$, the following inequality is legitimate:

$$\left|\left\{\pm(\mu^\pm - u) \leq \frac{\varepsilon M}{2^{j_*}}\right\} \cap \tilde{Q}\right| \leq \frac{\gamma}{j_*^{\frac{p-1}{p}}}|\tilde{Q}|,$$

Similarly, if $p > 2$, the same result holds when $|\mu^\pm| < \varepsilon M 2^{-j_*}$.

3.3 Lemma of the DeGiorgi type

Within this section, we introduce a Lemma resembling DeGiorgi's lemma, but it pertains to cylinders in the format of $Q_\varrho(\theta)$. In the scope of its application, the value of the parameter θ will be a constant universally determined by the provided data. Remarkably, this constant θ remains unaffected by changes in the solution and remains consistent.

Lemma 3.4

Examine a locally bounded function u, which serves as a local sub(super)-solution to equation (1) within Ω_T. Consider the set $(x_0, t_0) + Q_\varrho(\theta) = K_\varrho(x_0) \times (t_0 - \theta\varrho^p, t_0] \Subset \Omega_T$. A constant $\nu \in (0,1)$, relying solely on the given data and θ, is present. If the condition holds that

$$\left|\{\pm(\mu^\pm - u) \leq M\} \cap (x_0, t_0) + Q_\varrho(\theta)\right| \leq \nu|Q_\varrho(\theta)|,$$

then either $|\mu^\pm| > 8M$, or

$$\pm(\mu^\pm - u) \geq \frac{1}{2}M \quad \text{a.e. in} \quad (x_0, t_0) + Q_{\frac{1}{2}\varrho}(\theta).$$

Main Theorem Proof

Proof. Let's introduce the cylinder $Q_0 = K_\varrho(x_0) \times (t_0 - \varrho^p, t_0] \Subset \Omega_T$. For simplicity, we'll assume that the origin and (x_0, t_0) coincide. We begin by noting that

$$\mu^+ = ess\ sup_{Q_0} u, \quad \mu^- = ess\ inf_{Q_0} u, \quad \omega = \mu^+ - \mu^-.$$

Our argument proceeds through two main scenarios, specifically,

$$\begin{cases} \text{when } u \text{ is near zero: } \mu^- \leq \omega \text{ and } \mu^+ \geq -\omega, \\ \text{when } u \text{ is away from zero: } \mu^- > \omega \text{ or } \mu^+ < -\omega. \end{cases} \quad (11)$$

Notice that $(11)_1$ is equivalent to the condition $-2\omega \leq \mu^- \leq \mu^+ \leq 2\omega$. Consequently,

$$|\mu^\pm| \leq 2\omega.$$

4.1 A decrease in oscillation around zero

Within this section, we consider the scenario where the requirement specified in (11) is valid for the initial scenario. It's crucial to emphasize that one of the subsequent possibilities must hold:

$$|\{u(\cdot, -\tfrac{1}{2}\varrho^p) - \mu^- > \tfrac{1}{4}\omega\} \cap K_\varrho| \geq \tfrac{1}{2}|K_\varrho|, \qquad (12)$$

or

$$|\mu^+ - \{u(\cdot, -\frac{1}{2}\varrho^p) > \frac{1}{4}\omega\} \cap K_\varrho| \geq \frac{1}{2}|K_\varrho|,$$

We confine ourselves to the case (12) as both cases can be handled similarly. Employing Proposition 3.1 provides η within the range of (0,1) determined solely by the provided information,in a way that

$$u \geq \mu^- + \eta\omega \ \ a.e.in \ \ Q_1 = K_{\varrho_1} \times (-\varrho_1^p, 0], \quad with \ \ \varrho_1 = \frac{1}{2}\varrho.$$

This results in a decrease in oscillation,i,e. we have

$$ess\ osc_{Q_1} u \leq (1-\eta)\omega := \omega_1.$$

We can now proceed with the induction. Assume that up to $i = 1,2,\cdots,j-1$, we have constructed

$$\begin{cases} \varrho_i = \dfrac{1}{2}\varrho_{i-1}, \omega_i = (1-\eta\omega_{i-1}, Q_i = K_{\varrho_i} \times (-\varrho_i^p, 0], \\ \mu_i^+ = ess\ sup_{Q_i} u, \mu_i^- = ess\ inf_{Q_i} u, ess\ osc_{Q_i} u = \leq \omega_i. \end{cases}$$

We consider the situation where the initial condition in (11) is valid for all indices

$i = 1,2,\cdots,j-1$, i.e.,

$$\mu_i^- \leq \omega_i \ \ and \ \ \mu_i^+ \geq -\omega_i.$$

This allows us to reiterate the initial argument, which we have done for all $i = 1,2,\cdots,j$,

$$ess\ osc_{Q_i} u \leq (1-\eta)\omega_{i-1} := \omega_i.$$

Thus, by repeating the aforementioned iterative inequality, we obtain the following for all values of i ranging from 1 to j.

$$ess\ osc_{Q_i} u \leq (1-\eta)^i \omega := \omega(\tfrac{\varrho_i}{\varrho})^{\beta_0} \ \ where \ \ \beta_0 = \frac{-ln(1-\eta)}{ln2}. \qquad (13)$$

4.2 A decrease in oscillation away from zero

Assuming that j is the smallest index in this section that satisfies the second condition in (11), meaning either $either\ \mu_j^- > \omega_j\ or\ \mu_j^+ < -\omega_j$, we will consider the case where $\mu_j^- > \omega_j$, as the opposite case follows similarly. Given that j marks the initial index for this scenario, it implies that $\mu_{j-1} < \omega_{j-1}$. Additionally, an estimation is made:

$$\mu_j^- \leq \mu_{j-1}^- + \omega_{j-1} + \Omega_{j-1} - \omega_j \leq 2\omega_{j-1} - \omega_j = \frac{1+\eta}{1-\eta}\omega_j.$$

Consequently, we have:

$$\omega_j < \mu_j^- \leq \tfrac{1+\eta}{1-\eta}\omega_j. \qquad (14)$$

Starting from index j, equation (1) exhibits resemblance to a parabolic p-Laplacian equation within Q_j, as indicated by condition (14). Hence, the ability to solve the parabolic p-Laplacian equation plays a role in determining the possibility of reducing oscillation. For this purpose, we temporarily omit the subscript j from our symbol and define $v = \frac{u}{\mu^-}$ across the region $Q = K_\varrho \times (-\varrho^p, 0)$. It can be readily confirmed that v fulfills:

$$\partial_t v^{p-1} - \text{div}(|Dv|^{p-2}Dv) = 0 \ \ weakly\ in\ \ Q,$$

where $(x,t) \in Q$, $v \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$. Additionally,

$$1 \leq v \leq 2 \ \ a.e., \ \ Q. \qquad (15)$$

By leveraging the established regularity theory for the parabolic p-Laplacian (see DiBenedetto, 1993, DiBenedetto et al., 2012), the equation satisfied by $w := v^{p-1}$ turns out to be more amenable, namely,

$$\partial_t w - \text{div}A(x,t,w,Dw) = 0 \ \ weakly\ in\ \ Q, \qquad (16)$$

where for $(x,t) \in Q$, $w \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, we use

$$A(x,t,w,\xi) = \left(\frac{1}{p-1}\right)^{p-1} |w|^{\frac{(2-p)(p-2)}{p-1}} w|\xi|^{p-2}\xi.$$

It is evident that w occupies the identical functional space (2) as u and v because of (15), leading to $1 \leq w \leq 2^{p-1}$ in Q. Reapplying (15), it can be confirmed that there exist positive constants $\gamma_0(p)$ and $\gamma_1(p)$ such that $A(x,t,w,\xi).\xi \geq \gamma_0(p)|\xi|^p$ *and* $|A(x,t,w,\xi)| \leq \gamma_1(p)|\xi|^{p-1}$, (17)

for almost every $(x,t) \in Q$, and any $w \in \mathbb{R}$, and any $\xi \in \mathbb{R}^N$. This implies that w serves as a regional weak solution to the equation resembling the parabolic p-Laplacian.

Proposition 4.1

For $\mathrm{p} > 1$, let w be a bounded, local, weak solution to (1) in $\mathrm{Q} := \mathrm{Q}_\varrho$, and define

$$\widetilde{\omega} = \text{ess osc}_{\mathrm{Q}} \mathrm{w}.$$

If for some constants σ in (0,1), the condition

$$ess\ osc_{Q_{\sigma\varrho}(\theta)} w \leq \widetilde{\omega}\ \ holds,\ where\ \ \theta = \widetilde{\omega}^{2-p}, \tag{18}$$

then there exist constants $\beta_1 \in (0,1)$ and $\gamma > 1$ dependent solely on N, p, $\tilde{C}_0$, $\tilde{C}_1$, and σ, such that for all $0 < r < \varrho$,

$$ess\ osc_{Q_r(\theta)} w \leq \gamma \widetilde{\omega} \left(\frac{r}{\varrho}\right)^{\beta_1}.$$

To apply this proposition suitably for cases where $1 < p < 2$, we initially verify the fulfillment of the condition (18). In fact, recollect that $v = \frac{u}{\mu^-}$, $w = v^{p-1}$, and

$\omega = ess\ osc_Q u$, using (15) and invoking the mean value theorem results in

$$(p-1)2^{p-2} ess\ osc_Q v \leq \widetilde{\omega} = ess\ osc_Q w \leq (p-1) ess\ osc_Q v.$$

As $ess\ osc_Q v = \frac{\omega}{\mu^-}$, this becomes

$$(p-1)2^{p-2}\frac{\omega}{\mu^-} \leq \widetilde{\omega} \leq (p-1)\frac{\omega}{\mu^-}.$$

Considering (14), we have

$$c \overset{\text{def}}{=} \frac{1-\eta}{1+\eta}(p-1)2^{p-2} \leq \widetilde{\omega} \leq (p-1) \leq 1.$$

Since $\widetilde{\omega} \leq 1$, we have $Q_\varrho(\theta) \subset Q_\varrho$, so the condition (18) in proposition 4.1 is fulfilled for $\sigma = 1$. This leads to the conclusion of Proposition 4.1. Furthermore, the set inclusion is actually obtained by the earlier bound on $\widetilde{\omega}$:

$$Q_r(\theta_0) \subset Q_r(\theta)\ \ where\ \ \theta_0 = c^{2-p}.$$

Utilizing this set inclusion and rewriting the oscillation decay in Proposition 4.1 in terms of u, we deduce that for any $0 < r < \varrho$,

$$ess\ osc_{Q_r(\theta_0)} u \leq \gamma\omega(\frac{r}{\varrho})^{\beta}\ \ with\ \ \beta = \min\{\beta_0, \beta_1\}.$$

For the case of $1 < p < 2$, appropriate rescaling produces the desired oscillation decay and finalizes the demonstration of Theorem 1.1.

References

Alonso R, Santillana M and Dawson C. on the diffusive wave approximation of the shallow water equations. *European J. Appl. Math.* 2008; 19(5): 575-606.

Acerbi E. and Fusco N, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$, *J. Math. Appl*. 1989; 140(1): 115-135.

Bogelion V, Duzaar F, and Liao N. On the Hölder Regularity of Signed Solutions to a Doubly Nonlinear Equation. *J. Funct. Anal.*, 2021; 281(9):

DiBenedetto E, Gianazza U and Vespri V. The Harnack's Inequality for Degenerate and Singular Parabolic Equations. Springer Monographs in Mathematics; 2012.

DiBenedetto E. Continuity of weak solutions to a general porous medium equation. *J. Indiana Univ. Math.* 1983; 32:83-118.

DiBenedetto E. Degenerate Parabolic Equations (Universititext). 1st edition, Springer-Verlag; 1993.

DiBenedetto E. On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients. *Ann. Sc. Norm. Super. Pisa-CI. Sci.* 1986; 13(3): 487-535.

DiBenedetto E. Real Analysis. Birkhäuser Springer; 2016.

Evans LC. Partial Differential Equations. *Graduate Studies in Mathematics,* Vol. 19, American Mathematical Society; 1998. pp. 1-695.

Feng F and Molz F. J. A 2-d diffusion based, wetland flow model. *J. Hydrol.* 1997; 196: 230-250.

Giannaza U and Vespri V. A Harnack inequality for solutions of doubly nonlinear parabolic equations. *J. Appl. Funct. Anal.* 2006; 1(3): 271-284.

Giaquinta M and Modica G. Remarks on the regularity of the minimizers of certain degenerate functionals. *J. Appl. Funct. Anal.* 2006; 1(3): 271-284.

Hromadka TV Berenbrock CE, Freckleton JR and Guymon GL. A two dimensional dam- break flood plain model. *Adv. Water Resour*. 1985; 8: 7-14.

Ivanov AV, and Mkrtychyan PZ. Regularity up to the boundary for generalized solutions to the first boundary-value problem for doubly degenerate quasilinear parabolic equations. *J. Math. Sci.* 1994; 70(6): 2112-2122.

Ivanov AV. Hölder estimates for equations of fast diffusion type. *Algebra I Analiz*. 1994; 6(4): 101-142.

Ivanov AV. The classes $B_{m,l}$ and Hölder estimates of weak solutions for quasilinear parabolic equations admitting doubly degeneracy. *Zap. Nauchn. S. Peterburg. Otdel. Mat. Inst. Steklov.* (POMI) 1992; 197: 42-70.

Kinnunen J and Kuusi T. Local behaviuor of solutions to doubly nonlinear parabolic equations. *Math. Ann.* 2007; 337(3): 705-728.

Kinnunen J and Lindqvist P. Pontwise behaviuor of semicontinuous supersolutions to a parabolic solutions to doubly nonlinear quasilinear parabolic equation. *Ann. Mat. Pura Appl.* 2006; 185(3): 411-435.

Kuusi T, Misawa M and Nakamura K. Global existence for the p-Sobolev flow. *J. Differ. Equ.* 2021; 279: 245-281.

Kuusi T, Siljander J and Urbano JM. Local Hölder continuity for doubly nonlinear parabolic equations. *J. Indiana Univ. Math.* 2012a; 61(1): 399-430.

Kuusi 1 T, Siljander J, Laleoglu R and Urbano JM. Hölder continuity for Trudinger's equations in mrasure spaces. *Calc. Var. Partial Differ. Equ.* 2012b; 45(1-2): 193-229.

Ladyzenskaja OA, Solonnikov VA and Ural'ceva N. Linear and quasilinear equations of parabolic type. *Math. Mono*.1968: 23; pp648.

Leugering G and Mophou G. Instantaneous optimal control of friction dominated flow in a gas-network, in "Shape Optimization, Homogenization and Optimal Control". International series of Numerical Mathematics. 2018;169, Birkhaüser, Cham.

Lindgren E and Lindqvist P. On a Comparison Principle for Trudingers equation. *Adv. Calc. Var.* 2022; 15(3): 401-415.

Mahaffy MW. A three dimensional numerical model of ice sheets: Tests on the Barnes ice cap, northwest territories. *J. Geophys. Res.* 1976; 81(6):1059-1066.

Nakamura K and Misawa M. Existence of weak solution to the p-Sobolev flow. *Nonlinear Anal.* 2018; 175: 157-172.

Sarkar A.H. Positivity expansion to signed solution for doubly nonlinear parabolic equations. Jagannath Univ. J. Sci. 2023 (in press).

Sarkar AH. Energy estimates of signed solution to the doubly nonlinear parabolic equations. *Jagannath. Univ. J. Sci*. 2022; 9(1): 43-49.

Trudinger NS. Pointwise estimates and quasilinear parabolic equations. Comm. *Pure Appl. Math.* 1968b; 21(7): 205-226.

Trudinger NS. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. *Schoula. Sup. Pisa.* 1968a; 22: 265-274.

Urbano JM. The Method of Intrinsic Scaling: A Systematic Approach to Regularity for Degenerate and Singular PDEs (Lecture Notes in Mathematics, 1930), Springer-Verlag; 2008.

Vespri V, and Vestberg M. An extensive study of the regularity properties of solution to doubly singular equations. *arXiv:2001.0414.1v1, 2020.*

Vespri V. On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations. *Manuscripta Math.* 1992; 75(1): 65-80.