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Research Article

T_0 Separation axioms in intuitionistic topological spaces

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ABSTRACT

In this paper, we aim to investigate T_0 separation axioms in intuitionistic topological spaces. After presenting some characterizations of T_0 separation axioms, we provide interrelationships among those and their non-implications in counterexamples. Furthermore, we show that our notions satisfy hereditary and topological properties. Moreover, we establish that some of these notions satisfy productive and projective properties. 2000 Mathematics Subject Classification. 54A99.

Introduction

After the grand introduction of the fuzzy Set by Zadeh (Zadeh, 1965) in 1965, Atanassov (Atanassov, 1984, 1986) proposed the notion of an intuitionistic fuzzy set as the generalization of fuzzy Set considering the degree of membership and non-membership in 1983. Later, Coker (Coker, 1996, 1997) introduced the concept of an intuitionistic set, which is, in one way, the specialization of an intuitionistic fuzzy set and, in another way, the generalization of an ordinary set. Intuitionistic set theory, as a building framework for constructive mathematics, and its logic have influenced many later researchers in developing intuitionistic topology. It has many applications in various areas, particularly computer science, formal verification, and constructive mathematics. It was Coker (2000) who first applied the notion of topology to an intuitionistic set and investigated its various topological consequences. Bayhan and Coker (Bayhan and Coker, 2001) and Prova and Hossain (Prova and Hossain, 2020, 2022) dealt with separation axioms in intuitionistic topological spaces. Selvanayaki and Ilango (Selvanayaki and Ilango, 2016, 2017) studied homeomorphisms and generators in intuitionistic topological spaces.

Besides, Bayhan and Coker (Bayhan and Coker, 1996), Ahmed (Ahmed et al., 2014 a & b), and Prova and Hossain (Prova and Hossain, 2022) studied separation axioms in intuitionistic fuzzy topological spaces. Islam (Islam et al., 2018b) studied intuitionistic $L-T_0$ spaces, and Islam (Islam et al., 2018a) studied level separation on intuitionistic fuzzy T_1 spaces. Mahbub (Mahbub et al., 2019, 2021, 2022) studied a particular type of connectedness and compactness in intuitionistic fuzzy topological spaces.

In the literature on separation axioms and related outcomes in intuitionistic topological spaces, we studied and investigated as far as we didn't get T_0 separation axioms in detail. However, it is offered well for T_1, T_2, and others. In this paper, we present the T_0 separation axioms, following Bayhan and Coker (Bayhan and Coker, 2001) for T_1 separation axioms, in possibly various and modified ways with investing their interrelationships and topological consequences.

We start with listing some basic concepts and results introduced by Coker (Coker, 1996), Bayhan and Coker (Bayhan and Coker, 2001), and

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Selvanayaki and Ilango (Selvanayaki and Ilango, 2016, 2017) to

construct the path for our principal purpose. Afterward, we give some new and modified notions for T_0 separation axioms and find the relationships among those, revealing some counterexamples for non-implications too. Furthermore, we show that our defined notions satisfy hereditary and topological properties. Finally, we observe that two of these notions are productive and projective.

Preliminaries

In this section, we list some basic concepts of intuitionistic Set and intuitionistic topological space.

Definition (Coker, 1996)**:** Let X be a nonempty set. An intuitionistic set (IS for short) A is an object having the form $A = \langle X, A_1, A_2 \rangle$ $A2$, where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. Set A_1 is called the Set of members of A, while $set\, A_2$ is called the Set of non-members of A.

Definition (Coker, 1996)**:** Let X be a nonempty set and let the IS's A and B be $A = \langle X, A_1,\ A_2 \rangle$ and $B = \langle X, B_1,\ B_2 \rangle$ respectively. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of IS's in X, where $A_i = \langle X, A_i^{(1)},\ A_i^{(2)} \rangle$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \langle X, A_2, A_1 \rangle$;
(d) $\cup A_i = \langle X, \cup A_i^{(1)}, \cap A_i^{(2)} \rangle$;
(e) $\cap A_i = \langle X, \cap A_i^{(1)}, \cup A_i^{(2)} \rangle$;
(f) $[\quad] A = \langle X, A_1, A_1^c \rangle$;
(g) $(\quad) A = \langle X, A_2^c, A_2 \rangle$;
(h) $\underset{\sim}{\phi} = \langle X, \emptyset, X \rangle$, $\underset{\sim}{X} = \langle X, X, \emptyset \rangle$.

Definition (Coker, 1996)**:** Let X be a nonempty set and $p \in X$ be a fixed element in X. Then IS's: $\underset{\sim}{p} = \langle X, \{p\}, \{p\}^c \rangle$ and $\underset{\approx}{p} = \langle X, \emptyset, \{p\}^c \rangle$ are called an intuitionistic point (IP in short) and a vanishing intuitionistic point (VIP in short) respectively in X.

Definition (Coker, 1996)**:** Let $\underset{\sim}{p}$ be an IP, $\underset{\approx}{p}$ be a VIP, and $A = \langle X, A_1,\ A_2 \rangle$ be an IS in X. Then

(a) $\underset{\sim}{p} \in A$ if and only if $p \in A_1$;
(b) $\underset{\approx}{p} \in A$ if and only if $p \notin A_2$.

Definition (Coker, 1996)**:** Let $A = \langle X, A_1, A_2 \rangle$ and $B = \langle X, B_1, B_2 \rangle$ are IS's in X and Y respectively, then

(a) the preimage of B under f is the IS in X, defined by $f^{-1}(B) = \langle X, f^{-1}(B_1), f^{-1}(B_2) \rangle$;
(b) the image of A under f, denoted by $f(A)$, is the IS in Y, defined by $f(A) = \langle Y, f(A_1), f_{-}(A_2) \rangle$, where $f_{-}(A_2) = (f(A_2^c))^c$.

Corollary (Coker, 1996)**:** Let A, A_i $(i \in J)$ be IS in X, B, B_j $(j \in K)$ be IS in Y, and $f: X \to Y$ is a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$;
(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
(c) $A \subseteq f^{-1}(f(A))$ and if f is one-one, then $A = f^{-1}(f(A))$;
(d) $f(f^{-1}(B)) \subseteq B$ and if f is onto, then $f(f^{-1}(B)) = B$.

Definition (Coker, 1997)**:** An intuitionistic topology (IT for short) on a nonempty set X is a family τ of IS's in X satisfying the following axioms:

(a) $\underset{\sim}{\emptyset}, \underset{\sim}{X} \in \tau$,
(b) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(c) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called an intuitionistic topological space (ITS for short), and any IS in τ is known as an intuitionistic open set (IOS for short) in X.

Definition (Selvanayaki and Ilango, 2017)**:** Let X be a nonempty set and $A = \langle X, A_1,\ A_2 \rangle$ be an IS in X. Then

the intuitionistic generator of A, denoted as $G(A)$, is defined as the collection of IS's of the form $\langle X, A_1,\ A_2 \rangle$,$\langle X, A_2,\ A_1 \rangle$, $\langle X, \emptyset, A_1 \cup A_2 \rangle$ and $\langle X, A_1 \cup A_2, \emptyset \rangle$;

(a) the intuitionistic prime generator of A, denoted as $G_p(A)$, is the collection of IS's of the form

$\langle X, A_1, \emptyset\rangle, \langle X, \emptyset, A_2\rangle$, $\langle X, A_1, A_2\rangle$, $\langle X, \emptyset, A_1 \cup A_2\rangle$ and $\langle X, A_1 \cup A_2, \emptyset\rangle$.

Definition (Selvanayaki and Ilangom, 2017)**:** Let X be a nonempty set and A be any IS in X. Then

(a) the collection $G(A)$, along with $\underset{\approx}{\emptyset}$, X, forms a topology, and it is called intuitionistic generator topology generated by A and is denoted by (X, τ_G);

(b) the collection $G_p(A)$ along with $\underset{\sim}{\emptyset}, X$ forms a topology, and it is called intuitionistic prime generator topology generated by A and is denoted by (X, τ_{pG}).

Definition (Bayhan and Coker, 2001)**:** Let A and B be two IS's in X and Y, respectively. Then the product intuitionistic Set (PIS for short) of A and B on $X \times Y$ is defined by $U \times V = \langle (X,) Y, A_1 \times B_1, (A_2^c \times B_2^c)^c\rangle$, where $A = \langle X, A_1, A_2\rangle$ and $B = \langle X, B_1, B_2\rangle$.

Definition (Selvanayaki and Ilangom, 2016)**:** A bijection $f: (X, \tau) \to (Y, \sigma)$ is called intuitionistic homeomorphism if f is both intuitionistic continuous and intuitionistic open.

T_0 Separation Axioms in ITS's

In this section, we define six notions for T_0 separation axioms in ITSs and show some of their features and properties: hereditary, topological property, productive, and projective. We form the separation axioms for T_0 from the separation axioms for T_1 in Bayhan and Coker (Bayhan and Coker, 2001) with some necessary modifications.

Definition: Let (X, τ) be an ITS. Then (X, τ) is said to be

(a) $T_0(i)$ if for all $x, y \in X$, with $x \neq y$, there exist $U \in \tau$ such that $\underset{\sim}{x} \in U, \underset{\sim}{y} \notin U$ or $\underset{\sim}{y} \in U, \underset{\sim}{x} \notin U$ (cf. Bayhan and Coker, 2001);

(b) $T_0(ii)$ if for all $x, y \in X$, with $x \neq y$, there exist $U \in \tau$ such that $\underset{\approx}{x} \in U, \underset{\approx}{y} \notin U$ or $\underset{\approx}{y} \in U, \underset{\approx}{x} \notin U$ (cf. Bayhan and Coker, 2001);

(c) $T_0(iii)$ if for all $x, y \in X$, with $x \neq y$, there exist $U \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \bar{\underset{\sim}{y}}$ or $\underset{\sim}{y} \in U \subseteq \bar{\underset{\sim}{x}}$ (cf. Bayhan and Coker, 2001);

(d) $T_0(iv)$ if for all $x, y \in X$, with $x \neq y$, there exist $U \in \tau$ such that $\underset{\approx}{x} \in U \subseteq \bar{\underset{\approx}{y}}$ or $\underset{\approx}{y} \in U \subseteq \bar{\underset{\approx}{x}}$ (cf. Bayhan and Coker, 2001);

(e) $T_0(v)$ if for all $x, y \in X$, with $x \neq y$, there exists nonempty $U \in \tau$ such that $\underset{\sim}{y} \notin U$ or $\underset{\sim}{x} \notin U$ (cf. Bayhan and Coker, 2001).

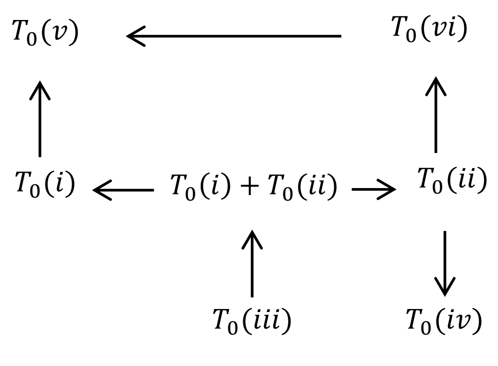
[In this case, we use the non-emptiness of U as an external restriction];1

(f) $T_0(vi)$ if for all $x, y \in X$, with $x \neq y$, there exists nonempty $U \in \tau$ such that $\underset{\approx}{y} \notin U$ or $\underset{\approx}{x} \notin U$ (cf. Bayhan and Coker, 2001).

[In this case, we use the non-emptiness of U as an external restriction].

Remarks: In the first four T_0 separation axioms [$T_0(i)$ to $T_0(iv)$] in the above definition, according to the characterization, $U \in \tau$ is nonempty by default because for any $x, y \in X$, to satisfy $T_0(i)$ to $T_0(iv)$, we have to satisfy either of these four: $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\approx}{x}, \underset{\approx}{y} \in U$, which $\underset{\sim}{\emptyset}$ as a U fails to do. But in the case of the last two T_0 separation axioms ($T_0(v)$ and $T_0(vi)$), we impose non-emptiness of U externally because, otherwise, every ITS become $T_0(v)$ and $T_0(vi)$ automatically for the character of $\underset{\sim}{\emptyset}$ as a U]

Theorem: Let (X, τ) be an ITS, then the following implications are valid:



Proof

(i) $T_0\ (i) \Rightarrow T_0(v)$ and **(ii)** $T_0\ (ii) \Rightarrow T_0(vi)$.

Proofs of (i) and (ii) are easy to obtain and can be done directly from the corresponding definitions.

Conversely, these are untrue [see counterexamples 4 and 5 in the examples section].

(iii) $T_0\ (ii) \Rightarrow T_0(iv)$

Let (X, τ) be an ITS satisfying $T_0(ii)$. We want to show that it is $T_0(iv)$ too, i.e., for all $x, y \in X$, with $x \neq y$, there exists $U \in \tau$ such that $\underset{\approx}{x} \in U \subseteq \bar{\underset{\approx}{y}}$ or $\underset{\approx}{y} \in U \subseteq \bar{\underset{\approx}{x}}$.

Choose arbitrary $x \neq y$ in X, then by $T_0\ (ii)$, there exist $U = \langle X, U_1, U_2\rangle \in \tau$ such that $\underset{\approx}{x} \in U, \underset{\approx}{y} \notin U$ or $\underset{\approx}{y} \in U, \underset{\approx}{x} \notin U$ is true, or, $\underset{\approx}{x} \in U, y \in U_2$ or $\underset{\approx}{y} \in U, x \in U_2$ is true, or, $\underset{\approx}{x} \in U, y \notin U_1$, or $\underset{\approx}{y} \in U, x \notin U_1$ is true, or, $\underset{\approx}{x} \in U, y \notin U_1 \subseteq \{y\}^C$ (and obviously $U_2 \supseteq \emptyset$), or $\underset{\approx}{y} \in U, x \notin U_1 \subseteq \{x\}^C$ (and obviously $U_2 \supseteq \emptyset$) is true, or, $\underset{\approx}{x} \in U \subseteq \langle X, \{y\}^c, \emptyset\rangle = \bar{\underset{\approx}{y}}$ or $\underset{\approx}{y} \in U \subseteq \langle X, \{x\}^c, \emptyset\rangle = \bar{\underset{\approx}{x}}$. Hence, it is $T_0\ (iv)$.

Conversely, this is untrue [see counterexample 1 in the examples section].

(iv) $T_0\ (vi) \Rightarrow T_0(v)$

Let (X, τ) be an ITS satisfying $T_0\ (vi)$. We want to show that it is $T_0\ (v)$ too, i.e., for all $x, y \in X$, with $x \neq y$, there exist nonempty $U \in \tau$ such that $\underset{\sim}{y} \notin U$ or $\underset{\sim}{x} \notin U$.

Choose arbitrary $x \neq y$ in X; then by $T_0\ (vi)$, there exist nonempty $U = \langle X,\ U_1, U_2\rangle \in \tau$ such that $\underset{\approx}{y} \notin U$ or $\underset{\approx}{x} \notin U$ is true, or, $y \in\ U_2$ or $x \in\ U_2$ is true, or, $y \notin\ U_1$ or $x \notin\ U_1$ is true, or, $\underset{\sim}{y} \notin U$ or $\underset{\sim}{x} \notin U$ is true. Hence, it is $T_0(v)$.

Conversely, this is untrue [see counterexample 5 in the examples section].

(v) $T_0\ (iii) \Rightarrow T_0(i) +\ T_0(ii)$.

Let (X, τ) be an ITS satisfying $T_0\ (iii)$. We want to show that it is simultaneously $T_0\ (i)$ and $T_0\ (ii)$.

To show $T_0\ (i)$, choose arbitrary $x \neq y$ in X, then by $T_0(iii)$, there exist $U \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \bar{\underset{\sim}{y}}$ or $\underset{\sim}{y} \in U \subseteq \bar{\underset{\sim}{x}}$ is true, or, $\underset{\sim}{x} \in U = \langle X, U_1, U_2\rangle \subseteq \langle X, \{y\}^c, \{y\}\rangle$ or $\underset{\sim}{y} \in U = \langle X, U_1, U_2\rangle \subseteq \langle X, \{x\}^c, \{x\}\rangle$ is true, or, $\underset{\sim}{x} \in U$, with $U_1 \subseteq \{y\}^c$ and $U_2 \supseteq \{y\}$ or $\underset{\sim}{y} \in U$, with $U_1 \subseteq \{x\}^c$ and $U_2 \supseteq \{x\}$ is true, or, $\underset{\sim}{x} \in U$, with $y \notin U_1$ and $y \in U_2$or $\underset{\sim}{y} \in U$, with $x \notin U_1$ and $x \in U_2$ is true, or, $\underset{\sim}{x} \in U$, with $y \notin U_1$ or $\underset{\sim}{y} \in U$, with $x \notin U_1$ is true , or, $\underset{\sim}{x} \in U$, with $\underset{\sim}{y} \notin U$ or $\underset{\sim}{y} \in U$, with $\underset{\sim}{x} \notin U_1$ is true. Hence, it is $T_0(i)$.

To show $T_0(ii)$, choose arbitrary $x \neq y$ in X, then by $T_0\ (iii)$ there exist $U \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \bar{\underset{\sim}{y}}$ or $\underset{\sim}{y} \in U \subseteq \bar{\underset{\sim}{x}}$ is true, or, $\underset{\sim}{x} \in U = \langle X, U_1, U_2\rangle \subseteq \langle X, \{y\}^c, \{y\}\rangle$ or $\underset{\sim}{y} \in U = \langle X, U_1, U_2\rangle \subseteq \langle X, \{x\}^c, \{x\}\rangle$ is true, or, $x \in U_1$, with $U_1 \subseteq \{y\}^c$ and $U_2 \supseteq \{y\}$, or $y \in U_1$, with $U_1 \subseteq \{x\}^c$ and $U_2 \supseteq \{x\}$ is true, or, $x \notin U_2$, with $y \notin U_1$ and $y \in U_2$ or $y \notin U_2$, with $x \notin U_1$ and $x \in U_2$ is true, or, $\underset{\approx}{x} \in U$, with $y \notin U_1$ and $y \in U_2$ or $\underset{\approx}{y} \in U$, with $x \notin U_1$ and $x \in U_2$ is true, ort, $\underset{\approx}{x} \in U$, with $y \notin U_1$ or $\underset{\approx}{y} \in U$, with $x \notin U_1$ is true, or, $\underset{\approx}{x} \in U$, with $\underset{\approx}{y} \notin U$ or $\underset{\approx}{y} \in U$, with $\underset{\approx}{x} \notin U$ is true. Hence, it is $T_0(ii)$.

Conversely, this is untrue [see counterexamples 1, 2, and 3 in the examples section].

Examples

Counterexample 1:

Let $X = \{p, q\}$, and τ be a topology on X given by $\mathcal{T} = \{\underset{\sim}{X}, \underset{\sim}{\emptyset}, \langle X, \{p\}, \emptyset\rangle, \langle X, \{q\}, \emptyset\rangle, \langle X, \emptyset, \emptyset\rangle\}$. We get IOS's containing $\underset{\sim}{p}$ as $\underset{\sim}{X}$ and $\langle X, \{p\}, \emptyset\rangle$and IOS's containing $\underset{\sim}{q}$ as $\underset{\sim}{X}$ and $\langle X, \{q\}, \emptyset\rangle$. Thus choosing $U = \langle X, \{p\}, \emptyset\rangle \in \tau$, we get $\underset{\sim}{p} \in U, \underset{\sim}{q} \notin U$. Therefore, it is $T_0(i)$. In addition, there does not exist open U to satisfy $\underset{\sim}{q} \in U \subseteq \bar{\underset{\sim}{p}} = \langle X, \{q\}, \{p\}\rangle$ or $\underset{\sim}{p} \in U \subseteq \bar{\underset{\sim}{q}} = \langle X, \{p\}, \{q\}\rangle$. Therefore, it is not $T_0(iii)$.

On the other hand, we get, IOS's containing $\underset{\approx}{p}$ are $\underset{\sim}{X}, \langle X, \{P\}\rangle\langle\{p, \emptyset, X, q\}, \emptyset\rangle$ and $\langle X, \emptyset, \emptyset\rangle$, and IOS's containing $\underset{\approx}{q}$ are $\underset{\sim}{X}, \langle X, \{q\}\rangle\langle\{p, \emptyset, X, q\}, \emptyset\rangle$ and $\langle X, \emptyset, \emptyset\rangle$. Therefore, however, if we choose U, we never get $\underset{\approx}{p} \in U, \underset{\approx}{q} \notin U$or $\underset{\approx}{q} \in U, \underset{\approx}{p} \notin U$. Therefore, it is not $T_0(ii)$. Furthermore, choosing $U = \langle X, \{p\}, \emptyset \rangle$, we get $\underset{\approx}{p} \in U \subseteq \bar{\underset{\approx}{q}} = \langle X, \{p\}, \emptyset \rangle$. Therefore, it is $T_0(iv)$. Moreover, for $p, q \in X$, nonempty open U does not exist to satisfy $\underset{\approx}{p} \notin U$ or $\underset{\approx}{q} \notin U$. Hence, it is not $T_0(vi)$.

Therefore, this is a topological space which is $T_0(i)$ and $T_0(iv)$, but not $T_0(ii)$, $T_0(iii)$ and $T_0(vi)$. Hence, $T_0(i)$ and $T_0(iv)$ can't assert $T_0(ii)$, $T_0(iii)$ or $T_0(vi)$.

Counterexample 2:

Let $X = \{p, q\}$, and τ be a topology on X given by $\tau = \{\underset{\sim}{X}, \underset{\sim}{\emptyset}, \langle X, \emptyset, \{p\} \rangle, \langle X, \emptyset, \{q\} \rangle, \langle X, \emptyset, \emptyset \rangle\}$. We get, IOS's containing $\underset{\sim}{p}$ is $\underset{\sim}{X}$ only, and similarly IOS's containing $\underset{\sim}{q}$ is $\underset{\sim}{X}$ only. Therefore, there does not exist open U to get $\underset{\sim}{p} \in U, \underset{\sim}{q} \notin U$ or $\underset{\sim}{q} \in U, \underset{\sim}{p} \notin U$. Therefore, it is not $T_0(i)$. Analogously, we can conclude that it is not $T_0(iii)$.

Again, we get, IOS's containing $\underset{\approx}{p}$ are $\underset{\sim}{X}, \langle X, \{\emptyset, q\} \rangle$ and $\langle X, \emptyset, \emptyset \rangle$, and IOS's containing $\underset{\approx}{q}$ are $\underset{\sim}{X}, \langle X, \emptyset, \{p\} \rangle$ and $\langle X, \emptyset, \emptyset \rangle$. Therefore, by choosing $U = \langle X, \{\emptyset, q\} \rangle$, we get $\underset{\approx}{p} \in U, \underset{\approx}{q} \notin U$. Therefore, it is $T_0(ii)$. Furthermore, choosing $U = \langle X, \{\emptyset, q\} \rangle$, we get $\underset{\approx}{p} \in U \subseteq \bar{\underset{\approx}{q}} = \langle X, \{p\}, \emptyset \rangle$. Therefore, it is $T_0(iv)$.

Therefore, this is a topological space that is $T_0(ii)$ and $T_0(iv)$, but not $T_0(i)$ and $T_0(iii)$. Hence, $T_0(ii)$ and $T_0(iv)$ can't assert $T_0(i)$ or $T_0(iii)$.

Counterexample 3:

Let $X = \{p, q\}$, and τ be a topology on X given by $\tau = \{\underset{\sim}{X}, \underset{\sim}{\emptyset}, \langle X\{P\} \rangle \langle \{X, p, \emptyset, X, \emptyset, p\} \rangle\}$; an intuitionistic generator topology generated by $A = \langle X, \{\emptyset, p\} \rangle$. We get, IOS's containing $\underset{\sim}{p}$ are $\underset{\sim}{X}$ and $\langle X, \{p\}, \emptyset \rangle$ and IOS's containing $\underset{\sim}{q}$ is $\underset{\sim}{X}$ only. Thus choosing $U = \langle X, \{p\}, \emptyset \rangle \in \tau$, we get $\underset{\sim}{p} \in U, \underset{\sim}{q} \notin U$. Therefore, it is $T_0(i)$. Furthermore, choosing $U = \langle X, \{p\}, \emptyset \rangle \in \tau$ or $U = \underset{\sim}{X} \in \tau$, we fail to get $\underset{\sim}{p} \in U \subseteq \bar{\underset{\sim}{q}} = \langle X, \{p\}, \{q\} \rangle$, and similarly, choosing $U = \underset{\sim}{X} \in \tau$, we fail to get $\underset{\sim}{q} \in U = \underset{\sim}{X} \subseteq \bar{\underset{\sim}{p}} = \langle X, \{q\}, \{p\} \rangle$. Therefore, it is not $T_0(iii)$.

On the other hand, IOS's containing $\underset{\approx}{p}$ are $\underset{\sim}{X}$ and $\langle X, \{p\}, \emptyset \rangle$, and open Set containing $\underset{\approx}{q}$ are $\underset{\sim}{X}$, $\langle X, \{p\}, \emptyset \rangle$, and $\langle X, \{\emptyset, p\} \rangle$. Choosing $U = \langle X, \{\emptyset, p\} \rangle$, we get $\underset{\approx}{q} \in U, \underset{\approx}{p} \notin U$. Therefore, it is $T_0(ii)$. Furthermore, choosing $U = \langle X, \{\emptyset, p\} \rangle$, we get $\underset{\approx}{q} \in U \subseteq \bar{\underset{\approx}{p}} = \langle X, \{q\}, \emptyset \rangle$. Therefore, it is $T_0(iv)$.

Therefore, this is a topological space that is $T_0(i)$, $T_0(ii)$, and $T_0(iv)$, but not $T_0(iii)$. Hence, $T_0(i)$, $T_0(ii)$ and $T_0(iv)$ can't assert $T_0(iii)$.

Counterexample 4:

Let $X = \{p, q, r\}$, and $\tau = \{\underset{\sim}{X}, \underset{\sim}{\emptyset}, \langle \{X, p\}, \{q, r\} \rangle, \langle x\{q, r\}, \{p\} \rangle\}$ is a topology on X; It is an intuitionistic generator topology, generated by $A = \langle X, \{p\}\{q, r\} \rangle$. We get, IOS's containing $\underset{\sim}{p}$ are $\underset{\sim}{X}$ and $\langle X, \{q, r\}\{p\} \rangle$ p, q, r, and IOS's containing $\underset{\sim}{q}$ are $\underset{\sim}{X}$ and $\langle X, \{q, r\}, \{p\} \rangle$, and IOS's containing $\underset{\sim}{r}$ are $\underset{\sim}{X}$ and $\langle X, \{q, r\}, \{p\} \rangle$. Thus for $q, r \in X$, there does not exist open U to get $\underset{\sim}{q} \in U, \underset{\sim}{r} \notin U$ or $\underset{\sim}{r} \in U, \underset{\sim}{q} \notin U$. Therefore, it is not $T_0(i)$. Analogously, it is not $T_0(iii)$. Additionally, for any pair from $p, q, r \in X$, choosing $U = \langle X, \{p\}, \{q, r\} \rangle \in \tau$, we get the necessary $\underset{\sim}{q} \notin U$ or $\underset{\sim}{r} \notin U$ as required. Hence, it is $T_0(v)$.

On the other hand, IOS's containing $\underset{\approx}{p}$ are $\underset{\sim}{X}$ and $\langle X, \{p\}, \{q, r\} \rangle$, containing $\underset{\approx}{q}$ are $\underset{\sim}{X}$ and $\langle X, \{q, r\}, \{p\} \rangle$, and containing $\underset{\approx}{r}$ are $\underset{\sim}{X}$ and $\langle X, \{q, r\}, \{p\} \rangle$. For $q, r \in X$, there does not exist open U to satisfy $\underset{\approx}{q} \in U, \underset{\approx}{r} \notin U$ or $\underset{\approx}{r} \in U, \underset{\approx}{q} \notin U$. Therefore, it is not $T_0(ii)$. Analogously, there is no open U to get $\underset{\approx}{q} \in U \subseteq \bar{\underset{\approx}{r}} = \langle X, \{p, q\}, \{r\} \rangle$ or $\underset{\approx}{r} \in U \subseteq \bar{\underset{\approx}{q}} = \langle X, \{p, r\}, \{q\} \rangle$. Therefore, it is not $T_0(iv)$ too. Additionally, for any pair from $p, q, r \in X$ and choosing $U = \langle X, \{p\}, \{q, r\} \rangle \in \tau$, we get the necessary $\underset{\approx}{q} \notin U$ or $\underset{\approx}{r} \notin U$ as required. Hence, it is $T_0(vi)$.

Therefore, this is a topological space which is $T_0(v)$, $T_0(vi)$, but not $T_0(i)$, $T_0(ii)$, $T_0(iii)$ and $T_0(iv)$.

Hence, $T_0(v)$ and $T_0(vi)$ can'nt assert $T_0(i)$ or $T_0(ii)$ or $T_0(iii)$ or $T_0(iv)$.

Counterexample 5:

Let $X = \{p, q, r\}$, and τ be a topology on X given by $\tau = \{\underset{\sim}{X}, \underset{\sim}{\emptyset}, \langle\{x,\}\rangle\langle\{p\},\rangle\langle X, p, \emptyset, X, \emptyset, p, X, \emptyset, \emptyset\rangle\}$; an intuitionistic prime generator topology generated by $A = \langle X, \{p\}, \emptyset\rangle$. We get IOS's containing $\underset{\sim}{p}$ are $\underset{\sim}{X}$ and $\langle X, \{p\}, \emptyset\rangle$, IOS's containing $\underset{\sim}{q}$ is $\underset{\sim}{X}$ only, and IOS's containing $\underset{\sim}{r}$ is $\underset{\sim}{X}$ only too. Therefore, for $q, r \in X$, there does not exist open U to get $\underset{\sim}{p} \in U, \underset{\sim}{q} \notin U$ or $\underset{\sim}{q} \in U, \underset{\sim}{p} \notin U$. Therefore, it is not $T_0(i)$ and not $T_0(iii)$ as well. In addition, for each pair from $p, q, r \in X$, choosing $U = \langle X, \emptyset, \emptyset\rangle \in \tau$, we get $\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{r} \notin U$. Therefore, it is $T_0(v)$.

Again IOS's containing $\underset{\approx}{p}$ are $\underset{\sim}{X}, \langle\{X, p\}, \emptyset\rangle$ and $\langle X, \emptyset, \emptyset\rangle$, IOS's not containing $\underset{\approx}{p}$ are $\langle X, \{\emptyset\, p\}\rangle$ and $\underset{\sim}{\emptyset}$; IOS's containing $\underset{\approx}{q}$ are $\underset{\sim}{X}$, $\langle X, \{p\}, \emptyset,\rangle, \langle X, \emptyset, \{p\}\rangle$ and $\langle X, \emptyset, \emptyset\rangle$, IOS's not containing $\underset{\approx}{q}$ is $\underset{\sim}{\emptyset}$ only. IOS's containing $\underset{\approx}{r}$ are $\underset{\sim}{X}$, $\langle X, \{p\}, \emptyset\rangle\langle X, \emptyset, \{p\}\rangle$ and $\langle X, \emptyset, \emptyset\rangle$, IOS's not containing $\underset{\approx}{r}$ is $\underset{\sim}{\emptyset}$ only. We see, for $q, r \in X$, there does not exist open U to satisfy $\underset{\approx}{q} \in U, \underset{\approx}{r} \notin U$ or $\underset{\approx}{r} \in U, \underset{\approx}{q} \notin U$. Therefore, it is not $T_0(ii)$ too. However, for $p, q \in X$, choosing $U = \langle X, \{p\}, \emptyset\rangle$, we get $\underset{\approx}{p} \in U \subseteq \underset{\approx}{\bar{q}} = \langle X, \{p, r\}, \emptyset\rangle$, the similar results hold for $p, r \in X$ and for $q, r \in X$ too. Therefore, it is $T_0(iv)$. Similarly, for $q, r \in X$, nonempty open U does not exist to satisfy $\underset{\approx}{q} \notin U$ or $\underset{\approx}{r} \notin U$. Hence, it is not $T_0(vi)$.

Therefore, this is a topological space which is $T_0(iv)$, $T_0(v)$, but is not $T_0(i)$, $T_0(ii)$, $T_0(iii)$ and $T_0(vi)$. Hence, $T_0(iv)$ and $T_0(v)$ can't assert $T_0(i)$ or $T_0(ii)$ or $T_0(iii)$ or $T_0(vi)$.

Properties

This section proves that our defined notions satisfy the hereditary and topological properties. Moreover, we prove that the two of these notions are productive and projective.

Theorem: A homeomorphic image of a $T_0(i)$ space is a $T_0(i)$ space.

Proof.

Let $f: (X, \tau) \to (Y, \sigma)$ be a homeomorphism, i.e., a bijection open and continuous (by definition in the primaries section). Suppose that (X, τ) is a $T_0(i)$ space. We want to show that (Y, σ) is $T_0(i)$ too.

Choose two arbitrary $y_1, y_2 \in Y$, with $y_1 \neq y_2$, then $f^{-1}(y_1), f^{-1}(y_2) \in X$, And since f is a bijection, so is f^{-1}, with its existence. In particular, as f^{-1} is one-one, $f^{-1}(y_1) \neq f^{-1}(y_2)$ in X. Suppose that $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. As (X, τ) is a $T_0(i)$ space, with $x_1 \neq x_2$ in X, then there exists $U \in \tau$ such that $\underset{\sim}{x_1} \in U, \underset{\sim}{x_2} \notin U$ or $\underset{\sim}{x_2} \in U, \underset{\sim}{x_1} \notin U$. Now $\underset{\sim}{x_1} \in U, \underset{\sim}{x_2} \notin U$ or $\underset{\sim}{x_2} \in U, \underset{\sim}{x_1} \notin U$ implies $f(\underset{\sim}{x_1}) \in f(U), f(\underset{\sim}{x_2}) \notin f(U)$ or $f(\underset{\sim}{x_2}) \in f(U), f(\underset{\sim}{x_1}) \notin f(U)$. As f is open, $f(U) = R(say) \in \sigma$. Since f is onto, $f(\underset{\sim}{x_1}) = f(f^{-1}\underset{\sim}{(}y_1)) = \underset{\sim}{y_1}$ and $f(\underset{\sim}{x_2}) = f(f^{-1}\underset{\sim}{(}y_2)) = \underset{\sim}{y_2}$ [by corollary in the preliminaries section]. Therefore, we get $R \in \sigma$ such that $\underset{\sim}{y_1} \in R, \underset{\sim}{y_2} \notin R$ and $\underset{\sim}{y_2} \in R, \underset{\sim}{y_1} \notin R$.

Theorem: Homeomorphic image of a $T_0(r)$ space is a $T_0(r)$ space for $r = i, ii, \dots, vi$.

The proof is the same for $r = ii, iii, \dots, vi$. as for $r = i$ in the above theorem.

Each $T_0(r)$ separation axioms for $r = i, ii, \dots, vi$ is a topological property.

Theorem: Inverse homeomorphic image of a $T_0(i)$ space is a $T_0(i)$ space.

Proof.

Let $f: (X, \tau) \to (Y, \sigma)$ be a homeomorphism, i.e., a bijection open and continuous. Suppose that (Y, σ) is a $T_0(i)$ space. We want to show that (X, τ) is $T_0(i)$ too. For any two arbitrary $x_1, x_2 \in X$, with $x_1 \neq x_2$, since f is a bijection, particularly one-one, therefore, $f(x_1) \neq f(x_2)$ in Y.

Choose two arbitrary $x_1, x_2 \in X$, with $x_1 \neq x_2$, then $f(x_1), f(x_2) \in Y$, And since f is a bijection, and in particular one-one, therefore, $f(x_1)\, f(x_2)$ in Y.

Suppose $f(x_1) = y_1$ and $f(x_2) = y_2$. As (Y, σ) is a $T_0(i)$ space, with this $y_1 \neq y_2$ in Y, there must exist $R \in \sigma$ such that $\underset{\sim}{y_1} \in R, \underset{\sim}{y_2} \notin R$ or $\underset{\sim}{y_2} \in R, \underset{\sim}{y_1} \notin R$. Now $\underset{\sim}{y_1} \in R, \underset{\sim}{y_2} \notin R$ or $\underset{\sim}{y_2} \in R, \underset{\sim}{y_1} \notin R$ implies $f^{-1}(\underset{\sim}{y_1}) \in f^{-1}(R), f^{-1}(\underset{\sim}{y_2}) \notin f^{-1}(R)$ or $f^{-1}(\underset{\sim}{y_2}) \in f(R),\ f^{-1}(\underset{\sim}{y_1}) \notin f^{-1}(R)$. As f is continuous, and $R \in \sigma$, therefore, $f^{-1}(R) = U(say) \in \tau$. Since f is one-one, $f^{-1}(\underset{\sim}{y_1}) = f^{-1}(f(\underset{\sim}{x_1})) = \underset{\sim}{x_1}$ and $f^{-1}(\underset{\sim}{y_2}) = f^{-1}(f(\underset{\sim}{x_2})) = \underset{\sim}{x_2}$ [By corollary in the preliminaries section]. Therefore, we get $U \in \sigma$ such that $\underset{\sim}{x_1} \in U, \underset{\sim}{x_2} \notin U$ and $\underset{\sim}{x_2} \in U, \underset{\sim}{x_1} \notin U$.

Theorem: Inverse homeomorphic image of a $T_0(r)$ space is a $T_0(r)$ space for $r = i, ii, \ldots, vi$.

The proof is the same for $r = ii, \ldots, vi.$ as for $r = i$ in the above theorem.

Theorem: If $(X,)\ \tau$ is $T_0(i)$, then for any subset $A \subseteq X$, the subspace (A, τ_A) is also $T_0(i)$.

Proof.

Let (X, τ) be a $T_0(i)$ space and $A \subseteq X$, with the subspace topology τ_A on A. We want to show that (A, τ_A) is also $T_0(i)$.

Let $x, y \in A$, with $x \neq y$, then $x, y \in X$, with $x \neq y$, hold the same. As (X, τ) is a $T_0(i)$ space, therefore, we must have $U \in \tau$ such that $\underset{\sim}{x} \in U, \underset{\sim}{y} \notin U$ or $\underset{\sim}{y} \in U, \underset{\sim}{x} \notin U$. Now, for this $U \in \tau$, we get $U_A = U \cap A$ in τ_A. This must satisfy $\underset{\sim}{x} \in U_A, \underset{\sim}{y} \notin U_A$ or $\underset{\sim}{y} \in U_A, \underset{\sim}{x} \notin U_A$. Hence, (A, τ_A) is $T_0(i)$.

Theorem: If $(X,)\ \tau$ is $T_0(r)$, then for any subset $A \subseteq X$, the subspace (A, τ_A) is also $T_0(r)$ for $r = ii, iii, \ldots, vi$.

The proof is the same for $r = ii, iii \ldots, vi.$ as for $r = i$ in the above theorem.

This shows that each of $T_0(r)$ for $r = i, ii, \ldots, vi$ is hereditary.

Theorem: If $(X,)\ \tau$, and (Y, σ) be two ITSs. If

(a) $(X,)\ \tau$, and (Y, σ) both are $T_0(i)$, then so is $(X \times Y, \sigma \times \tau)$.

(b) $(X,)\ \tau$, and (Y, σ) both are $T_0(ii)$, then so is $(X \times Y, \sigma \times \tau)$.

Proof

(a) Let (X, τ) and (Y, σ) are both $T_0(i)$. We want to prove that $(X \times Y, \sigma \times \tau)$ is $T_0(i)$ too. Choose arbitrary points (x_1, y_1) and (x_2, y_2) in $X \times Y$, with $(x_1, y_1) \neq (x_2, y_2)$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. For $x_1 \neq x_2$ in X, as $(X, \tau)$is $T_0(i)$, there must exist $U \in \tau$ such that such that $\underset{\sim}{x_1} \in U, \underset{\sim}{x_2} \notin U$ or $\underset{\sim}{x_2} \in U, \underset{\sim}{x_1} \notin U$. Then we have IOS's $U \times \underset{\sim}{Y} = \langle X \times Y, U_1 \times Y, (U_2^c \times \phi^c)^c \rangle$ in $\tau \times \sigma$ such that $(\underset{\sim}{x_1, y_1}) \in U \times \underset{\sim}{Y}$, $(\underset{\sim}{x_2, y_2}) \notin U \times \underset{\sim}{Y}$ or $(\underset{\sim}{x_2, y_2}) \in U \times \underset{\sim}{Y}$, $(\underset{\sim}{x_1, y_1}) \notin U \times \underset{\sim}{Y}$.

Similarly, for $y_1 \neq y_2$ in Y, as (Y, σ) is $T_0(i)$, there must exist $R \in \sigma$ such that such that $\underset{\sim}{y_1} \in R, \underset{\sim}{y_2} \notin R$ or $\underset{\sim}{y_2} \in R, \underset{\sim}{y_1} \notin R$. Then we have IOS's $\underset{\sim}{X} \times R = \langle X \times Y, X \times R_1, (\phi^c \times R_2^c)^c \rangle$ in $\tau \times \sigma$ such that $(\underset{\sim}{x_1, y_1}) \in \underset{\sim}{X} \times R$, $(\underset{\sim}{x_2, y_2}) \notin \underset{\sim}{X} \times R$ or $(\underset{\sim}{x_2, y_2}) \in \underset{\sim}{X} \times R$, $(\underset{\sim}{x_1, y_1}) \notin \underset{\sim}{X} \times R$.

(b) Let (X, τ) and (Y, σ) are both $T_0(ii)$. We want to prove that $(X \times Y, \sigma \times \tau)$ is $T_0(ii)$ too. Choose arbitrary points (x_1, y_1) and (x_2, y_2) in $X \times Y$, with $(x_1, y_1) \neq (x_2, y_2)$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. For $x_1 \neq x_2$ in X as $(X, \tau)$is $T_0(ii)$, there must exist $U \in \tau$ such that $\underset{\approx}{x_1} \in U$, $\underset{\approx}{x_2} \notin U$ or $\underset{\approx}{x_2} \in U$, $\underset{\approx}{x_1} \notin U$. Then we have IOS's $U \times \underset{\sim}{Y} = \langle X \times Y, U_1 \times Y, (U_2^c \times \phi^c)^c \rangle$ in $\tau \times \sigma$ such that $(\underset{\approx}{x_1, y_1}) \in U \times \underset{\approx}{Y}$, $(\underset{\approx}{x_2, y_2}) \notin U \times \underset{\approx}{Y}$ or $(\underset{\approx}{x_2, y_2}) \in U \times \underset{\approx}{Y}$, $(\underset{\approx}{x_1, y_1}) \notin U \times \underset{\approx}{Y}$.

Similarly, for $y_1 \neq y_2$ in Y, as (Y, σ) is $T_0(ii)$, there must exist $R \in \sigma$ such that such that $\underset{\approx}{y_1} \in R$, $\underset{\approx}{y_2} \notin R$ or $\underset{\approx}{y_2} \in R$, $\underset{\approx}{y_1} \notin R$. Then we have IOS's $\underset{\approx}{X} \times R = \langle X \times Y, X \times R_1, (\phi^c \times R_2^c)^c \rangle$ in $\tau \times \sigma$ such that $(\underset{\approx}{x_1, y_1}) \in \underset{\approx}{X} \times R$, $(\underset{\approx}{x_2, y_2}) \notin \underset{\approx}{X} \times R$ or $(\underset{\approx}{x_2, y_2}) \in \underset{\approx}{X} \times R$, $(\underset{\approx}{x_1, y_2}) \notin \underset{\approx}{X} \times R$.

Hence, $T_0(i)$ and $T_0(ii)$ are productive.

Theorem: If $(X,)$ τ, and (Y,σ) be two ITS's. If

(a) $(X\times Y,\sigma\times\tau)$ is $T_0(i)$, then $(X,)$τ, and (Y,σ) both are $T_0(i)$.

(b) $(X\times Y,\sigma\times\tau)$ is $T_0(ii)$, then $(X,)$τ, and (Y,σ) both are $T_0(ii)$.

Proof.

(a) Let $(X\times Y,\sigma\times\tau)$ is $T_0(i)$. We want to show that (X,τ) and (Y,σ) are $T_0(i)$.

To show (X,τ) follows $T_0(i)$, choose arbitrary points $x_1,x_2\in X$, with $x_1\neq x_2$, and fix $y\in Y$, then in $X\times Y$, $(x_1,y)\neq(x_2,y)$, and as $(X\times Y,\sigma\times\tau)$ is $T_0(i)$, we must have $U\times R\in\tau\times\sigma$ such that $(\underset{\sim}{x_1},y)\in U\times R$, $(\underset{\sim}{x_2},y)\notin U\times R$ or $(\underset{\sim}{x_2},y)\in U\times R$, $(\underset{\sim}{x_1},y)\notin U\times R$. This implies that $(x_1,y)\in U_1\times R_1$, $(x_2,y)\notin U_1\times R_1$ or $(x_2,y)\in U_1\times R_1$, $(x_1,y)\notin U_1\times R_1$. More specifically, we get $x_1\in U_1$, $x_2\notin U_1$ or $x_2\in U_1$, $x_1\notin U_1$. Therefore, for $x_1,x_2\in X$, with $x_1\neq x_2$, we get $U\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\notin U$ or $\underset{\sim}{x_2}\in U$, $\underset{\sim}{x_1}\notin U$. This shows that (X,τ) is $T_0(i)$.

Now, to show (Y,σ) is $T_0(i)$, choose arbitrary points $y_1,y_2\in Y$, with $y_1\neq y_2$, and fix $x\in X$. Then in $X\times Y,(x,y_1)\neq(x,y_2)$ and as $(X\times Y,\sigma\times\tau)$ is $T_0(i)$, we must have $U\times R\in\tau\times\sigma$ such that $(x,\underset{\sim}{y_1})\in U\times R$, $(x,\underset{\sim}{y_2})\notin U\times R$ or $(x,\underset{\sim}{y_2})\in U\times R$,$(x,\underset{\sim}{y_1})\notin U\times R$. This implies that $(x,y_1)\in U_1\times R_1$, $(x,y_2)\notin U_1\times R_1$ or $(x,y_2)\in U_1\times R_1$, $(x,y_1)\notin U_1\times R_1$. More specifically, we get $y_1\in R_1$, $y_2\notin R_1$ or $y_2\in R_1$, $y_1\notin R_1$. Therefore, for $y_1,y_2\in Y$, with $y_1\neq y_2$, we get $R\in\sigma$ such that $\underset{\sim}{y_1}\in R$, $\underset{\sim}{y_2}\notin R$ or $\underset{\sim}{y_2}\in R$, $\underset{\sim}{y_1}\notin R$. This shows that (Y,σ) is $T_0(i)$.

(b) Let $(X\times Y,\sigma\times\tau)$ is $T_0(ii)$. We want to show that (X,τ) and (Y,σ) are $T_0(ii)$.

To show (X,τ) follows $T_0(ii)$, choose arbitrary points $x_1,x_2\in X$, with $x_1\neq x_2$, and fix $y\in Y$ then in $X\times Y,(x_1,y)\neq(x_2,y)$ and as $(X\times Y,\sigma\times\tau)$ is $T_0(ii)$, we must have $U\times R\in\tau\times\sigma$ such that $(\underset{\approx}{x_1},y)\in U\times R$, $(\underset{\approx}{x_2},y)\notin U\times R$ or $(\underset{\approx}{x_2},y)\in U\times R$,$(\underset{\approx}{x_1},y)\notin U\times R$. This implies that $(x_1,y)\notin U_2\times R_2$,$(x_2,y)\in U_2\times R_2$ or $(x_2,y)\notin U_2\times R_2$, $(x_1,y)\in U_2\times R_2$. we get $x_1\notin U_2$, $x_2\in U_2$ or $x_2\notin U_2$, $x_1\in U_2$. Therefore, for $x_1,x_2\in X$, with $x_1\neq x_2$, we get $U\in\tau$ such that $\underset{\approx}{x_1}\in U$, $\underset{\approx}{x_2}\notin U$ or $\underset{\approx}{x_2}\in U$, $\underset{\approx}{x_1}\notin U$. This shows that (X,τ) is $T_0(ii)$.

Now, to show (Y,σ) is $T_0(ii)$, choose arbitrary points $y_1,y_2\in Y$, Y, $y_1\neq,y_2$, and Fix $x\in X$. Then in $X\times Y,(x,y_1)\neq(x,y_2)$ and as $(X\times Y,\sigma\times\tau)$ is $T_0(ii)$, we must have $U\times R\in\tau\times\sigma$ such that $(x,\underset{\approx}{y_1})\in U\times R$, $(x,\underset{\approx}{y_2})\notin U\times R$ or $(x,\underset{\approx}{y_2})\in U\times R$, $(x,\underset{\approx}{y_1})\notin U\times R$. This implies that $(x,y_1)\notin U_2\times R_2$, $(x,y_2)\in U_2\times R_2$ or $(x,y_2)\notin U_2\times R_2$, $(x,y_1)\in U_2\times R_2$. More specifically, we get $y_1\notin R_2$, $y_2\in R_2$ or $y_2\notin R_2$, $y_1\in R_2$. Hence, for $y_1,y_2\in Y$, with $y_1\neq y_2$, we get $R\in\sigma$ such that $\underset{\approx}{y_1}\in R$, $\underset{\approx}{y_2}\notin R$ or $\underset{\approx}{y_2}\in R$, $\underset{\approx}{y_1}\notin R$. This shows that (Y,σ) is $T_0(ii)$. Hence, $T_0(i)$ and $T_0(ii)$ are projective.

Conclusions

In this paper, we provide some new and modified notions for T_0 separation axioms, analyze interrelationships among them, and give necessary counter examples for non-implications. We show that our defined notions satisfy hereditary and topological properties, and two of these given notions are productive and projective. These results are very encouraging for further study in this area, especially for other higher separation axioms.

Credit authorship contribution statement

Rajendra Chandra Bhowmik:

Conceptualization, Investigation, Methodology, Formal analysis, Writing-original draft, Writing review editing;

Md. Sahadat Hossain: Validation, Supervision.

Conflict of interest

The authors declare that they have no conflict of interest.

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