J. Bangladesh Acad. Sci. 48(2); 147-160: December 2024 DOI:https://doi.org/10.3329/jbas.v48i2.69400

Journal of Bangladesh Academy of Sciences

Journal homepage: http://www.bas.org.bd/publications/jbas.html

Research Article

Modified Cramer's rule for solving a fully fuzzy system of linear equations with a non-square coefficient matrix

Fatema Khatun*, Md. Sahadat Hossain[1] and Mst. Monjuara Akter[1]

Department of Computer Science and Engineering, Northern University Bangladesh, Dhaka, Bangladesh

ARTICLE INFO

Article History

Received: 17 October 2023
Revised: 04 June 2024
Accepted: 07 July 2024

Keywords: Triangular LR fuzzy number, Trapezoidal LR fuzzy number, Determinant of square and non-square matrix, The fuzzy system of equations.

ABSTRACT

In this modern technological world, several problems arise daily, whose maximum problems have been unsolved in a classical way because of their ambiguity. To solve those problems, the applications of fuzzy mathematics are increasing rapidly, whereas traditional classical methods have failed. Here, we have modified Cramer's rules method for solving the non-square coefficient matrix of the fully fuzzy system of linear equations (FFSLE) using triangular and trapezoidal LR fuzzy numbers. This modified method is compared with another method that can only solve the square coefficient matrix using fuzzy triangular and trapezoidal numbers. At last, we observe that our modified method is more general than others. This process is illustrated with numerical examples.

Introduction

A fuzzy system of linear equations [FSLE} plays a vital role in designing the model in vast real-world problems that ensue in an extensive range of disciplines such as optimization, economics, statistics business (Khatun and Hossain, 2022), and engineering (Buckley and Qu, 1990, 1991; Islam and Hossain, 2022). That's why various researchers are paying attention to the solution methods for the FSLE.

In the previous decades, many researchers have studied and investigated fuzzy system of linear equations first proposed by Friedman et al. (1998) and Ma et al. (2000). For finding a result in the embedding method (Allahviranloo and Hashemi, 2014) they decorated the main SFLE into a crisp system of linear equations (SLE) (Friedman et al., 1998), after then studied dual fuzzy system of linear equations (DFSL) by Allahviranloo and Ghanbari (2012a, 2012b), Allahviranloo and Salahshour (2011), Allahviranloo (2004, 2005), and Abbasi and Jalali (2019); fully fuzzy system of linear equations (FFSLE) by Abbasbandy et al. (2008), Abbasbandy and Alavi (2005), Abbasbandy et al. (2005, 2006) and Asady et al. (2005), and general dual fuzzy linear system (GDFLS) by Zheng and Wang (2006). Later, the FFSLE of the form AX=b was introduced by Dehghan and Hashemi (2006), who analyzed some direct methods in fuzzy to solve FSLE and FFSLE based on classical methods such as matrix inverse method, Cramer's rule and LU-decomposition (Dehghan et al., 2007). FFLS with triangular fuzzy numbers is discussed by Muruganandam et al. (2019), and a method is proposed using Gauss-Jordan Elimination by Abidin et al. (2019). A modified associated linear system is proposed by Malkawi et al. (2018) for solving FFLS where the fuzzy numbers are hexagonal and positive. After that, Ziqan et al. (2022) extended it by using trapezoidal

*Corresponding author:<fatema.rumath@gmail.com>

[1]*Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh*

fuzzy numbers. However, all literature methods can solve only this system's square coefficient matrix. The problem of solving a system of fuzzy linear equations faces serious difficulties when the coefficient matrix of the system is rectangular (non-square coefficient matrix).

In this paper, first, we define the determinate of a non-square matrix (Arunkumar et al., 2011; Radić, 2005; Joshi, 1980)with some basic idea of fuzzy numbers. Furthermore, we modify Cramer's rule method for solving FFSLE of the form AX=b with a square and non-square coefficient matrix of the system, in which all parameters are LR fuzzy numbers (Soylu and Aslan, 2022). For this reason, it is called a fully fuzzy system of linear equations. Here, we discuss this modified method using two fuzzy number systems: a triangular LR fuzzy number and a trapezoidal LR fuzzy number. Here, we emphasize solving the non-square coefficient matrix of the FFSLE, which was not done in the previous literature. The fuzzy number was first defined by Chang and Zadeh (1972). After that, Dubois and Prade (1978) became acquainted with the LR fuzzy number. To extend this method, we use the determinant of the non-square coefficient matrix (Arunkumar et al., 2011; Radić, 2005; Joshi, 1980) of the system and define the rules separately for square and non-square matrix in triangular and trapezoidal LR fuzzy numbers. Finally, we add some numerical examples to illustrate this process.

This paper is decorated by:

Section 2 reviews the basic operations and concept of LR fuzzy numbers with a fully fuzzy system of linear equations and determinants of a non-square matrix. In section 3, the methodology of the solution system of the fully fuzzy system of linear equations for both square and non-square coefficient matrix is modified by Cramer's rule in both number systems. In section 4, we illustrate our methods with some examples. Section 5 presents results, and a discussion with future opinions is provided.

Preliminaries

2.1 Fuzzy number (Muruganandam and Razak, 2013)**:** A fuzzy set U is termed as a fuzzy number on the universe R if

i. U is a normal fuzzy set,

ii. U is a convex fuzzy set and

iii. U is piecewise continuous.

The set of all fuzzy numbers is denoted by $F(E^1)$. A parametric form of this definition is given by Kaleva (Abbasbandy et al., 2005).

2.2 Triangular LR fuzzy number (Soylu and Aslan, 2022; Nikuie and Ahmad, 2014; Guo and Shang, 2013)**:** A triangular fuzzy number $M\ (\mathfrak{m}, \alpha, \beta)$ is said to be a triangular LR fuzzy number if

$$\mu_M(y) = \begin{cases} L\left(\frac{\mathfrak{m} - y}{\alpha}\right) for y \leq \mathfrak{m}, & \alpha > 0 \\ R\left(\frac{y - \mathfrak{m}}{\beta}\right) for y \geq \mathfrak{m}, & \beta > 0 \end{cases}$$

where '$\mathfrak{m}$' mean value, 'α' left spread and 'β' right spread of M, and the strictly decreasing functions L and R defined on [0 1] satisfy the conditions:

$$L(0) = R(0) = 1, \quad L(1) = R(1) = 0$$

$$0 < L(y) < 1, \quad 0 < R(y) < 1, \ \ y \neq 0{,}1$$

2.3 Trapezoidal *LR* fuzzy number (Karthik and Chandrasekaran, 2014)**:** A trapezoidal fuzzy number $M\ (\mathfrak{m}, \mathfrak{n}, \alpha, \beta)$ is said to be a Trapezoidal LR fuzzy number if

$$\mu_M(y) = \begin{cases} L\left(\frac{\mathfrak{m} - y}{\alpha}\right) for y \leq \mathfrak{m}, & \alpha > 0 \\ R\left(\frac{y - \mathfrak{n}}{\beta}\right) for y \geq \ \mathfrak{n}, & \beta > 0 \end{cases}$$

where '$\mathfrak{m}$' and '$\mathfrak{n}$' are the left and right points of mean value, 'α' left spread and 'β' right spread of M, and the strictly decreasing functions L and R defined on [0 1] satisfy the conditions:

$$L(0) = R(0) = 1, \ \ L(1) = R(1) = 0$$

$$0 < L(y) < 1, \quad 0 < R(y) < 1, \ \ y \neq 0{,}1$$

2.4 Equal fuzzy numbers (Elsayed et al., 2020; Guo and Shang, 2013)**:**

i. Two triangular LR fuzzy numbers, $M\ (p,q,r)\ and\ N\ (u,v,w)$, are said to be equal (Guo and Shang, 2013) if $p=u, q=v, r=w$.

ii. Two trapezoidal LR fuzzy numbers, $M(p,q,r,s)$, and $N(u,v,w,x)$, are said to be equal (Elsayed et al., 2020) if $p=u, q=v, r=w, s=x$.

2.5 Fuzzy Arithmetic Operations: In this sector, we discussed the actions of two fuzzy numbers, such as addition, subtraction, multiplication, and scalar multiplication.

a) Triangular LR fuzzy number operations (Guo and Shang, 2013)**:** let $P(p,q,r)\ and\ Q(\ u,v,w)$ be two triangular LR fuzzy numbers then their arithmetic operations are:

i. Addition: $P+Q\ =\ (p+u, q+v,\ r+w)$

ii. Opposite: $-P=-(p,q,r)=(-p,r,q)$ since A is a LR fuzzy number and 'p' is a mean value of P.

iii. Subtraction: $P-Q\ =\ (p-u, q+w, r+v)$

iv. Scalar Multiplication: Let λ be any scalar, then

$$\lambda \otimes P=\lambda \bigotimes (p,q,r) = \begin{cases} (\lambda p, \lambda q, \lambda r), & \lambda \geq 0 \\ (\lambda p, -\lambda r, -\lambda q), & \lambda < 0 \end{cases}$$

v. Multiplication:

- $If\ P>0\ and\ Q>0, then$

$$P \otimes Q\ =\ (p,q,r) \otimes (u,v,w) =\ (pu, pv+qu, pw+ru)$$

- $If\ P<0\ and\ Q>0, then$

$$P \otimes Q\ =\ (p,q,r) \otimes (u,v,w) =\ (pu, qu-pw, ru-pv)$$

- $If\ P<0\ and\ Q<0, then$

$$P \otimes Q\ =\ (p,q,r) \otimes (u,v,w) =\ (pu, -pw-ru, -pv-qu)$$

vi. The cross product of two triangular LR fuzzy numbers is:

$$P \otimes Q\ =\ (p,q,r) \otimes (u,v,w)\ =\ (pu, pv+qu-pu, pw+ru-pu)$$

a) Trapezoidal LR fuzzy number operations: (Karthik and Chandrasekaran, 2014): The Trapezoidal LR fuzzy number operations are the same as Triangular LR fuzzy number operations. Let $P(p,q,r,s)\ and\ Q(\ u,v,w,x)$ be two trapezoidal LR fuzzy numbers, then their arithmetic operations are:

i. Addition: $P+Q\ =\ (p+u, q+v, r+w, s+x)$

ii. Opposite: $-P\ =\ -(p,q,r,s)\ = (-q,-p,s,r)$, since P is a LR fuzzy number and 'p' 'q' are mean values end point of P.

iii. Subtraction: $P-Q\ =\ (p-u, q-v, r+x, s+w)$

iv. Scalar Multiplication: Let λ be any scalar, then

$$\lambda \otimes p =\ \lambda \otimes\ (p,q,r,s) = \begin{cases} (\lambda p, \lambda q, \lambda r, \lambda s), & \lambda \geq 0 \\ (\ \lambda q, \lambda p, -\lambda s, -\lambda r), & \lambda < 0 \end{cases}$$

v. Multiplication:

- $If\ P>0\ and\ Q>0, then$

$$P \otimes Q\ =\ (p,q,r,s) \otimes (\ u,v,w,x) =\ (pu, qv, pw+ru, qx+sv)$$

- $If\ P<0\ and\ Q>0, then$

$$P \otimes Q\ =\ (p,q,r,s) \otimes (\ u,v,w,x) =\ (qu, pv, ru-qx, sv-pw)$$

$If\ P<0\ and\ Q<0, then$

$$P \otimes Q\ =\ (p,q,r,s) \otimes (\ u,v,w,x) =\ (\ qv, pu, -qx-sv, -pw-ru)$$

vi. The cross product of two trapezoidal LR fuzzy numbers is:

$$P \otimes Q = (p,q,r,s) \otimes (\ u,v,w,x) = (pu, qv, pw+ru-pu, qx+sv-qv)$$

2.6 The determinate of a non-square matrix (Arunkumar et al., 2011; Radić, 2005; Joshi, 1980)**:** Let A be a non-square matrix, then the determinant of A is

$$\begin{aligned}\det(A) &= \det(A_1, A_2, \dots, A_k)\\ &= det(A_1, A_2, \dots, A_{k-1})\\ &\quad +(-1)^k \det(A_1 - A_2\\ &\quad + \dots +(-1)^k A_{k-1}, A_k),\end{aligned}$$

Where k is the number of rows or columns of the non-square matrix.

2.1 Fully fuzzy system of linear equations (Dehghan et al., 2007)**:** Consider The n$\times n$ fully fuzzy linear equations:

$$\begin{cases}(\breve{a}_{11}\otimes\hat{y}_1)\oplus(\breve{a}_{12}\otimes\bar{y}_2)\oplus \dots\dots\dots\oplus(\breve{a}_{1n}\otimes\hat{y}_n) = \underset{\sim}{b}_1\\ (\ddot{a}_{21}\otimes\bar{x}_1)\oplus(\ddot{a}_{22}\otimes\hat{y}_2)\oplus \dots\dots\dots\oplus(\breve{a}_{2n}\otimes\hat{y}_n) = \underset{\sim}{b}_2\\ \vdots\\ (\breve{a}_{n1}\otimes\hat{y}_1)\oplus(\breve{a}_{n2}\otimes\hat{y}_2)\oplus \dots\dots\dots\oplus(\breve{a}_{nn}\otimes\hat{y}_n) = \underset{\sim}{b}_n\end{cases} \quad (2.1)$$

The matrix arrangement of the upstairs equations is:

$$\breve{A}\otimes\hat{Y} = \underset{\sim}{b} \qquad (2.2)$$

Or simply $\breve{A}\otimes\hat{Y} = \underset{\sim}{b}$, where the coefficient matrix $\breve{A} = (\breve{a}_{ij}), 1 \le i, j \le n$ is a $n \times n$ fuzzy matrix and $\hat{y}_i, \underset{\sim}{b}_i \in F(E^1)$, $1 \le i \le n$. This arrangement is a fully fuzzy linear Equations (FFSLE) system.

2. Methodology

Analytical methods are sometimes called direct methods. Here, we will discuss the fully fuzzy system of linear equations in Cramer's rule using two types of fuzzy numbers: triangular LR fuzzy numbers and Trapezoidal LR fuzzy numbers. In this method, we will use the cross product of fuzzy numbers and the determinant of the square and non-square matrix.

3.1 Modified Cramer’s rule for triangular LR fuzzy numbers

Using Cramer's rule, we can easily solve a fully fuzzy system of linear equations that states that each entry wi in the solution is a quotient of two determinants. From Equation (2.2) where $\breve{A} = (\breve{a}_{ij})$that $\breve{a}_{ij} = (a_{ij}, b_{ij}, c_{ij})$triangular LR fuzzy numbers with the new notation ($\breve{A}$)= (A, L, M)>0, whose A, L and M are three crisp matrices with the same size of $\breve{A}$ such that $A = (a_{ij}), L = (b_{ij})$ and M = (c_{ij}) are called the center matrix and the left and right spread matrices, respectively. $\underset{\sim}{b} = (b, g, h) > 0$ and $\hat{Y} = (w, x, y) > 0$. Then we have

$$\breve{A}\otimes\hat{Y} = \underset{\sim}{b}$$

$$(A, L, M)\otimes(w, x, y) = (b, g, h) \qquad (3.1)$$

Using the Cross product of two triangular LR fuzzy numbers. It can be written as $(Aw, Ax + Lw - Aw, Ay + Mw - Aw)$ (b, g, h)

Now, using section (2.4), the equality of two fuzzy numbers, we get

$$\begin{aligned}Aw &= b\\ Ax + Lw - Aw &= g \qquad (3.2)\\ Ay + Mw - Aw &= h\end{aligned}$$

i.e., $Aw = b$

$$\begin{aligned}Ax &= g - Lw + Aw \qquad (3.3)\\ Ay &= h - Mw + Aw\end{aligned}$$

By the Cramer's rule, we may write,

i. If A is a square matrix, then

$$w_i = \frac{\det(A^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n$$

ii. If A is a non-square matrix, then

$$\begin{aligned}\det(A) &= \det(A_1, A_2, \dots, A_k)\\ &= det(A_1, A_2, \dots, A_{k-1})\\ &\quad +(-1)^k \det(A_1 - A_2\\ &\quad + \dots +(-1)^k A_{k-1}, A_k)\end{aligned}$$

where k is the number of rows or columns of the non-square matrix.

$$w_i = \frac{\det(A^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n.$$

Where$A^{(i)}$denotes the matrix, we get it by replacing thei^{th} column with b of matrix A. Then, using a solution of w, we will get

$x_i = \frac{\det(A'^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n.$ $y_i = \frac{\det(A''^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n.$

where $A'^{(i)}$ and $A''^{(i)}$ denote the matrix obtained from A by replacing its i^{th} column with $g - Lw + Aw$ and $h - Mw + Aw$, respectively.

3.2 Modified Cramer's rule for trapezoidal LR fuzzy numbers

In this sector, we will use the analytical method to obtain the solution of the fully fuzzy linear system of equations of trapezoidal LR fuzzy numbers (2.2) where $\breve{A} = (\breve{a}_{ij})$ that $\breve{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ LR fuzzy number with the new notation $\breve{A} = (A, L, M, N) > 0$, whose A, L, M, and N are four crisp matrices with the same size of $\breve{A}$ such that $A = (a_{ij}), L = (b_{ij}), M = (c_{ij})$ *and* $N = (d_{ij})$ are called the center-left and center-right matrices and the left and right spread matrices, respectively. $\underset{\sim}{b} = (b, g, h, p) > 0$ and $\hat{Y} = (w, x, y, z) > 0$. Then we have $\breve{A} \otimes \hat{Y} = \underset{\sim}{b}$

$(A, L, M, N) \otimes (w, x, y, z) = (b, g, h, p)$ ………(4.1)

the Cross product of two trapezoidal LR fuzzy numbers. It can be written as

$$(Aw, Lx, Ay + Mw - Aw, Lz + Nx - Lx) = (b, g, h, p)$$

Now, using section (2.4), the equality of two fuzzy numbers, we get

$$Aw = b$$
$$Lx = g$$
$$Ay + Mw - Aw = h \quad …..(4.2)$$
$$Lz + Nx - Lx = p$$

i.e.,

$$Aw = b$$
$$Lx = g$$
$$Ay = h - Mw + Aw \quad …..(4.3)$$
$$Lz = p - Nx + Lx$$

By the Cramer's rule, we may write,

i. If A and L are square matrices, then

$$w_i = \frac{\det(A^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n.$$
$$x_i = \frac{\det(L^{(i)})}{\det(L)}, \quad i = 1,2, \dots, n.$$

ii. If A and L are non-square matrices, then

$$\det(A) = \det(A_1, A_2, \dots, A_k) = det(A_1, A_2, \dots, A_{k-1}) + (-1)^k \det(A_1 - A_2 + \dots + (-1)^k A_{k-1}, A_k)$$

$$\det(L) = \det(L_1, L_2, \dots, L_k) = det(L_1, L_2, \dots, L_{k-1}) + (-1)^k \det(L_1 - L_2 + \dots + (-1)^k L_{k-1}, L_k)$$

where k is the number of rows or columns of the non-square matrix.

$$w_i = \frac{\det(A^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n$$
$$x_i = \frac{\det(L^{(i)})}{\det(L)}, \quad i = 1,2, \dots, n$$

Where $A^{(i)}$ and $L^{(i)}$ denote the matrices we get by replacing the i^{th} column with b and g of matrices A and L, respectively. Then, using the solution of w and x, we will get

$$y_i = \frac{\det(A'^{(i)})}{\det(A)}, \quad i = 1,2, \dots, n$$
$$z_i = \frac{\det(L'^{(i)})}{\det(L)}, \quad i = 1,2, \dots, n$$

where $A'^{(i)}$ and $L'^{(i)}$ denote matrices that are obtained from A and L by replacing their i^{th} column with $h - Mw + Aw$ and $p - Nx + Lx$, respectively.

4. Numerical illustrations

4.1 Examples of triangular LR fuzzy numbers

Example 1: Suppose the fully fuzzy linear system of equations:

$$(6,1,4) \otimes (w_1, x_1, y_1) \oplus (5,2,2) \otimes (w_2, x_2, y_2) \oplus (3,2,1) \otimes (w_3, x_3, y_3) = (58, 30, 60)$$

$$(12,8,20) \otimes (w_1, x_1, y_1) \oplus (14,12,15) \otimes (w_2, x_2, y_2) \oplus (8,8,10) \otimes (w_3, x_3, y_3) = (142,139,257)$$

$$(24,10,34) \otimes (w_1, x_1, y_1) \oplus (32,30,30) \otimes (w_2, x_2, y_2) \oplus (20,19,24) \otimes (w_3, x_3, y_3) = (316,297,514)$$

where $A = \begin{bmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{bmatrix}$,

$L = \begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix}$,

$$M = \begin{bmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{bmatrix}$$

$b = \begin{bmatrix} 58 \\ 142 \\ 316 \end{bmatrix}$, $g = \begin{bmatrix} 30 \\ 139 \\ 297 \end{bmatrix}$, $h = \begin{bmatrix} 60 \\ 257 \\ 514 \end{bmatrix}$,

$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Here, we see that matrix A is square, so we follow the rule (i) in this method. Then we have det(A)= 48 Now we calculate A^1, A^2, and A^3, which are obtained from A by replacing its i^{th} column with b.

$$A^1 = \begin{bmatrix} 58 & 5 & 3 \\ 142 & 14 & 8 \\ 316 & 32 & 20 \end{bmatrix} \Rightarrow det(A^1) = 192$$

$$A^2 = \begin{bmatrix} 6 & 58 & 3 \\ 12 & 142 & 8 \\ 24 & 316 & 20 \end{bmatrix} \Rightarrow det(A^2) = 240$$

$$A^3 = \begin{bmatrix} 6 & 5 & 58 \\ 12 & 14 & 142 \\ 24 & 32 & 316 \end{bmatrix} \Rightarrow det(A^3) = 144$$

Therefore, we get

$w_1 = \frac{192}{48} = 4$, $w_2 = \frac{240}{48} = 5$, $w_3 = \frac{144}{48} = 3$

i.e., $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$

Now we calculate x and y, so first, we calculate $A'^{(i)}$ and $A''^{(i)}$, which denote matrices obtained from A by replacing its i^{th} column with $g - Lw + Aw$ and $h - Mw + Aw$, respectively.

$$g - Lw + Aw$$
$$= \begin{bmatrix} 30 \\ 139 \\ 297 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 68 \\ 165 \\ 366 \end{bmatrix}$$

$$h - Mw + Aw = \begin{bmatrix} 60 \\ 257 \\ 514 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$
$$+ \begin{bmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 89 \\ 214 \\ 472 \end{bmatrix}$$

Now, $A'^1 = \begin{bmatrix} 68 & 5 & 3 \\ 165 & 14 & 8 \\ 366 & 32 & 20 \end{bmatrix} \Rightarrow det(A'^1) = 240$

$$A'^2 = \begin{bmatrix} 6 & 68 & 3 \\ 12 & 165 & 8 \\ 24 & 366 & 20 \end{bmatrix} \Rightarrow det(A'^2) = 264$$

$$A'^3 = \begin{bmatrix} 6 & 5 & 68 \\ 12 & 14165 \\ 24 & 32366 \end{bmatrix} \Rightarrow det(A'^3) = 168$$

Therefore, we get

$x_1 = \frac{240}{48} = 5$, $x_2 = \frac{264}{48} = 5.5$, $x_3 = \frac{168}{48} = 3.5$

i.e., $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5.5 \\ 3.5 \end{bmatrix}$

Now, $A''^1 = \begin{bmatrix} 89 & 5 & 3 \\ 21414 & 8 \\ 4723220 \end{bmatrix} \Rightarrow det(A''^1) = 336$

$$A''^2 = \begin{bmatrix} 6 & 68 & 3 \\ 12 & 165 & 8 \\ 24 & 366 & 20 \end{bmatrix} \Rightarrow det(A''^2) = 336$$

$$A''^3 = \begin{bmatrix} 6 & 5 & 89 \\ 12 & 14 & 214 \\ 24 & 32 & 472 \end{bmatrix} \Rightarrow det(A''^3) = 192$$

Therefore, we get

$y_1 = \frac{336}{48} = 7$, $y_2 = \frac{336}{48} = 7$,

$y_3 = \frac{192}{48} = 4$

i.e., $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix}$

Hence the solution set is

$$\begin{bmatrix} (w_1, x_1, y_1) \\ (w_2, x_2, y_2) \\ (w_3, x_3, y_3) \end{bmatrix} = \begin{bmatrix} (4, & 5, & 7) \\ (5, & 5.5, & 7) \\ (3, & 3.5, & 4) \end{bmatrix}$$

This solution set's figure (Fig. 1) is given below:

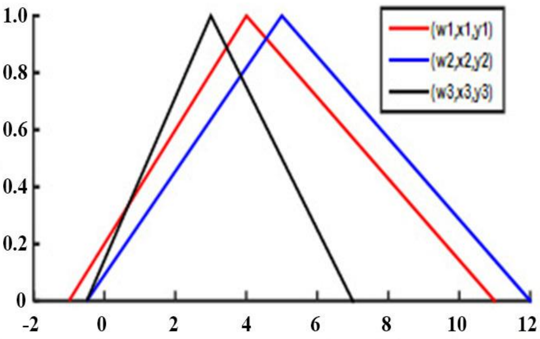

Fig. 1. Solution set of the FFSLE of triangular LR fuzzy numbers in Cramer's rule (i) for square coefficient matrix.

Example 2: Suppose the fully fuzzy system of linear equations:

$$(4,1,1)\otimes(w_1,x_1,y_1) \oplus (6,1,2)\otimes(w_2,x_2,y_2) \oplus (7,2,3)\otimes(w_3,x_3,y_3) = (45,50,5)$$
$$(7,1,2)\otimes(w_1,x_1,y_1) \oplus (5,2,1)\otimes(w_2,x_2,y_2) \oplus (8,0,2)\otimes(w_3,x_3,y_3) = (48,52,5)$$

where $A = \begin{bmatrix}467\\758\end{bmatrix}$, $L = \begin{bmatrix}1 & 12\\1 & 20\end{bmatrix}$, $M = \begin{bmatrix}1 & 23\\2 & 12\end{bmatrix}$

$b = \begin{bmatrix}45\\48\end{bmatrix}$, $g = \begin{bmatrix}50\\52\end{bmatrix}$, $h = \begin{bmatrix}5\\5\end{bmatrix}$,

$w = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$, $x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $y = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

Here, we see that matrix A is a non-square matrix, so we follow the rule (ii) in this method. Then we have

det(A)=$\begin{vmatrix}467\\758\end{vmatrix} = \begin{vmatrix}46\\75\end{vmatrix} + (-1)^3\begin{vmatrix}4-67\\7-58\end{vmatrix} = \begin{vmatrix}46\\75\end{vmatrix} - \begin{vmatrix}-27\\2\ 8\end{vmatrix} = \quad -22+30 = 8$

Now we calculate A^1, A^2, and A^3, which are obtained from A by replacing its i^{th} column with b.

$A^1 = \begin{bmatrix}4567\\4858\end{bmatrix}$,

$\Rightarrow det(A^1) = \begin{vmatrix}4567\\4858\end{vmatrix} = \begin{vmatrix}456\\485\end{vmatrix} + (-1)^3\begin{vmatrix}45-67\\48-58\end{vmatrix} = \begin{vmatrix}456\\485\end{vmatrix} - \begin{vmatrix}397\\438\end{vmatrix} = -63-11 = -74$

$A^2 = \begin{bmatrix}4457\\7488\end{bmatrix}$,

$\Rightarrow det(A^2) = \begin{vmatrix}4457\\7488\end{vmatrix} = \begin{vmatrix}445\\748\end{vmatrix} + (-1)^3\begin{vmatrix}4-457\\7-488\end{vmatrix} = \begin{vmatrix}445\\748\end{vmatrix} - \begin{vmatrix}-417\\-418\end{vmatrix} = -123 + 41 = -82$

$A^3 = \begin{bmatrix}4645\\7548\end{bmatrix}$,

$\Rightarrow det(A^3) = \begin{vmatrix}4645\\7548\end{vmatrix} = \begin{vmatrix}46\\75\end{vmatrix} + (-1)^3\begin{vmatrix}4-645\\7-548\end{vmatrix} = \begin{vmatrix}46\\75\end{vmatrix} - \begin{vmatrix}-245\\2\ 48\end{vmatrix} = -22+186 = 164$

Therefore, we get

$w_1 = \frac{-74}{8} = -9.25, \qquad w_2 = \frac{-82}{8} = -10.25,$

$w_3 = \frac{164}{8} = 20.5$

i.e., $\qquad w = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}-9.25\\-10.25\\20.5\end{bmatrix}$

Now we calculate x and y, so first we calculate $A'^{(i)}$ and $A''^{(i)}$ denote matrices obtained from A by replacing its i^{th} column with $g - Lw + Aw$ and $h - Mw + Aw$, respectively.

$$g - Lw + A = \begin{bmatrix}50\\52\end{bmatrix} - \begin{bmatrix}1 & 12\\1 & 20\end{bmatrix}\begin{bmatrix}-9.25\\-10.25\\20.5\end{bmatrix} + \begin{bmatrix}467\\758\end{bmatrix}\begin{bmatrix}-9.25\\-10.25\\20.5\end{bmatrix} = \begin{bmatrix}73.50\\129.75\end{bmatrix}$$

$$h - Mw + Aw = \begin{bmatrix}5\\5\end{bmatrix} - \begin{bmatrix}1 & 23\\2 & 12\end{bmatrix}\begin{bmatrix}-9.25\\-10.25\\20.5\end{bmatrix} + \begin{bmatrix}467\\758\end{bmatrix}\begin{bmatrix}-9.25\\-10.25\\20.5\end{bmatrix} = \begin{bmatrix}18.25\\40.75\end{bmatrix}$$

Now,

$A'^1 = \begin{bmatrix}73.50\ 67\\129.7558\end{bmatrix}$,

$$\Rightarrow det(A'^1) = \begin{vmatrix}73.50\ 67\\129.7558\end{vmatrix}$$
$$= \begin{vmatrix}73.50\ 6\\129.755\end{vmatrix} + (-1)^3\begin{vmatrix}73.50-6\ 7\\129.75-58\end{vmatrix}$$
$$= \begin{vmatrix}73.50\ 6\\129.755\end{vmatrix} - \begin{vmatrix}67.5\ \ 7\\124.758\end{vmatrix} = -411 + 333.25$$
$$= -77.75$$

$A'^2 = \begin{bmatrix}4\ 73.50\ 7\\7129.758\end{bmatrix}$,

$$\Rightarrow det(A'^2) = \begin{vmatrix}4\ 73.50\ 7\\7129.758\end{vmatrix}$$
$$= \begin{vmatrix}4\ 73.50\\7129.75\end{vmatrix} + (-1)^3\begin{vmatrix}4-73.50\ 7\\7-129.758\end{vmatrix}$$

$$= \begin{vmatrix} 4\ 73.50 \\ 7129.75 \end{vmatrix} - \begin{vmatrix} -69.5\ \ 7 \\ -122.758 \end{vmatrix} = 4.5 - 303.25$$
$$= -298.75$$

$A'^3 = \begin{bmatrix} 46\ 73.50 \\ 75129.75 \end{bmatrix}$,

$\Rightarrow det(A'^3) = \begin{vmatrix} 46\ 73.50 \\ 75129.75 \end{vmatrix} = \begin{vmatrix} 46 \\ 75 \end{vmatrix} +$
$(-1)^3 \begin{vmatrix} 4-6\ 73.50 \\ 7-5129.75 \end{vmatrix} = \begin{vmatrix} 46 \\ 75 \end{vmatrix} - \begin{vmatrix} -2\ 73.50 \\ 2\ 129.75 \end{vmatrix} =$
$-22 + 406.5 = 384.5$

Therefore, we get

$$x_1 = \frac{-77.75}{8} = -9.72,$$
$$x_2 = \frac{-298.75}{8} = -37.34,\ \ x_3 = \frac{384.5}{8} = 48.06$$

i.e., $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9.72 \\ -37.34 \\ 48.06 \end{bmatrix}$$

In a similar way, we get

$A''^1 = \begin{bmatrix} 18.2567 \\ 40.7558 \end{bmatrix}$,

$\Rightarrow det(A''^1) = \begin{vmatrix} 18.2567 \\ 40.7558 \end{vmatrix}$
$= \begin{vmatrix} 18.256 \\ 40.755 \end{vmatrix} + (-1)^3 \begin{vmatrix} 18.25-67 \\ 40.75-58 \end{vmatrix}$
$= \begin{vmatrix} 18.256 \\ 40.755 \end{vmatrix} - \begin{vmatrix} 12.257 \\ 35.758 \end{vmatrix} = -153.25 + 152.25 = -1$

$A''^2 = \begin{bmatrix} 418.257 \\ 740.758 \end{bmatrix}$,

$\Rightarrow det(A''^2) = \begin{vmatrix} 418.257 \\ 740.758 \end{vmatrix}$
$= \begin{vmatrix} 418.25 \\ 740.75 \end{vmatrix} + (-1)^3 \begin{vmatrix} 4-18.257 \\ 7-40.758 \end{vmatrix}$
$= \begin{vmatrix} 418.25 \\ 740.75 \end{vmatrix} - \begin{vmatrix} -14.257 \\ -33.758 \end{vmatrix} = 35.25 - 122.25 = -87$

$A''^3 = \begin{bmatrix} 4618.25 \\ 7540.75 \end{bmatrix}$,

$\Rightarrow det(A''^3) = \begin{vmatrix} 4618.25 \\ 7540.75 \end{vmatrix} = \begin{vmatrix} 46 \\ 75 \end{vmatrix} +$
$(-1)^3 \begin{vmatrix} 4-618.25 \\ 7-540.75 \end{vmatrix}$
$= \begin{vmatrix} 46 \\ 75 \end{vmatrix} - \begin{vmatrix} -218.25 \\ 2\ 40.75 \end{vmatrix} = -22 + 118 = 96$

Therefore, we get

$y_1 = \frac{-1}{8} = -0.125,\qquad y_2 = \frac{-87}{8} = -10.875,$
$y_3 = \frac{96}{8} = 12$

i.e., $$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -0.125 \\ -10.875 \\ 12 \end{bmatrix}$$

Hence the solution set is

$$\begin{bmatrix} (w_1, x_1, y_1) \\ (w_2, x_2, y_2) \\ (w_3, x_3, y_3) \end{bmatrix} = \begin{bmatrix} (-9.25,\ -9.72,\ -0.125) \\ (-10.25, -37.34, -10.875) \\ (20.5,\ \ 48.06,\ \ \ 12) \end{bmatrix}$$

By observing examples 1 and 2, we see that there is no need always to be the matrix square to solve the fully fuzzy system of linear equations in the above process. Square and rectangular, both types of fully fuzzy systems of linear equations, can be solved by this method using rule (i) and rule (ii), respectively. It is more generalized than the initial Cramer's rule method.

This solution set's figure (Fig. 2) is given below:

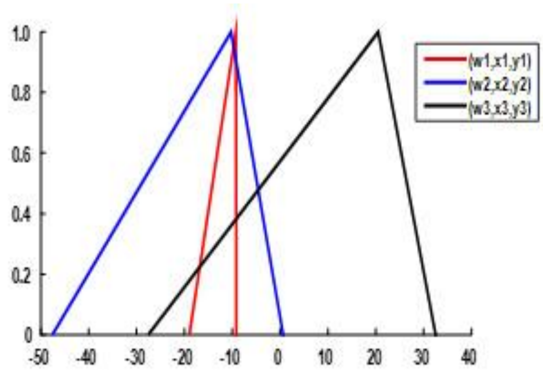

Fig. 2. Solution set of the FFSLE of triangular LR fuzzy numbers in Cramer's rule (ii) for non-square coefficient matrix.

4.2 Examples of fuzzy trapezoidal LR fuzzy numbers

Example 3: Consider the fully fuzzy linear system of equations of trapezoidal LR fuzzy numbers as follows:

$(4,5,1,2)(w_1, x_1, y_1, z_1)$
$\oplus\ (3,4,1,1)(w_2, x_2, y_2, z_2) = (25, 70, 26, 58)$

$(2,3,1,1)(w_1, x_1, y_1, z_1)$
$\oplus\ (5,8,1,2)(w_2, x_2, y_2, z_2) = (35, 90, 25, 55)$

where $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, $L = \begin{bmatrix} 5 & 4 \\ 3 & 8 \end{bmatrix}$,

$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$b = \begin{bmatrix} 25 \\ 35 \end{bmatrix}, g = \begin{bmatrix} 70 \\ 90 \end{bmatrix}$, $h = \begin{bmatrix} 26 \\ 25 \end{bmatrix}$, $p = \begin{bmatrix} 58 \\ 55 \end{bmatrix}$,

$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

Here, the matrices A and L are square, so we follow this method's rule (i). Then we have

det(A)= 14

det(L)= 28

Now we calculate (A^1, A^2) and (L^1, L^2), which are obtained from A and L by replacing their i^{th} column with b and g, respectively. Then

$A^1 = \begin{bmatrix} 25 & 3 \\ 35 & 5 \end{bmatrix} \Rightarrow det(A^1) = 20$

$A^2 = \begin{bmatrix} 4 & 25 \\ 2 & 35 \end{bmatrix} \Rightarrow det(A^2) = 90$

$L^1 = \begin{bmatrix} 70 & 4 \\ 90 & 8 \end{bmatrix} \Rightarrow det(L^1) = 200$

$L^2 = \begin{bmatrix} 5 & 70 \\ 3 & 90 \end{bmatrix} \Rightarrow det(L^1) = 240$

Therefore, we get

$w_1 = \frac{20}{14} = 1.43$, $w_2 = \frac{90}{14} = 6.43$, $x_1 = \frac{200}{28} = 7.14$, $x_2 = \frac{240}{28} = 8.57$

i.e., $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1.43 \\ 6.43 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7.14 \\ 6.07 \end{bmatrix}$

Now we calculate y and z, so first, we calculate$A'^{(i)}$ and $L'^{(i)}$, which denote matrices obtained from A and L by replacing their i^{th} column by $h - Mw + $ Aw and $p - Nx + Lx$, respectively.

$$h - Mw + Aw = \begin{bmatrix} 26 \\ 25 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1.43 \\ 6.43 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1.43 \\ 6.43 \end{bmatrix} = \begin{bmatrix} 43.14 \\ 52.14 \end{bmatrix}$$

$$p - Nx + Lx = \begin{bmatrix} 58 \\ 55 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7.14 \\ 6.07 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 7.14 \\ 6.07 \end{bmatrix} = \begin{bmatrix} 97.64 \\ 105.71 \end{bmatrix}$$

Now, $A'^1 = \begin{bmatrix} 43.14 & 3 \\ 52.14 & 5 \end{bmatrix}$

$\Rightarrow det(A'^1) = 59.28$

$A'^2 = \begin{bmatrix} 4 & 43.14 \\ 2 & 52.14 \end{bmatrix}$

$\Rightarrow det(A'^2) = 122.28$

Therefore, we get

$y_1 = \frac{59.28}{14} = 4.23$, $y_2 = \frac{122.28}{14} = 8.73$,

i.e., $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4.23 \\ 8.73 \end{bmatrix}$

Now, $L'^1 = \begin{bmatrix} 97.64 & 4 \\ 105.71 & 8 \end{bmatrix}$

$\Rightarrow det(A''^1) = 358.28$

$L'^2 = \begin{bmatrix} 5 & 97.64 \\ 3 & 105.71 \end{bmatrix}$

$\Rightarrow det(A''^2) = 235.63$

Therefore, we get

$z_1 = \frac{358.28}{28} = 12.80$, $z_2 = \frac{235.63}{28} = 8.41$,

i.e., $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 12.80 \\ 8.41 \end{bmatrix}$

Hence, the solution set is

$$\begin{bmatrix} (w_1, x_1, y_1, z_1) \\ (w_2, x_2, y_2, z_2) \end{bmatrix} = \begin{bmatrix} (1.43, 7.14, 4.23, 12.80) \\ (6.43, 8.57, 8.73, 8.41) \end{bmatrix}$$

This solution set's figure (Fig. 3) is given below:

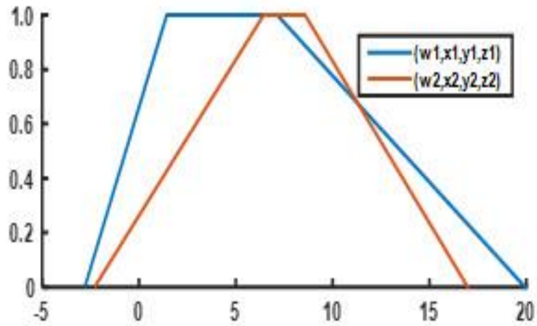

Fig. 3. Solution set of the FFSLE of trapezoidal LR fuzzy numbers in Cramer's rule (i) for square coefficient matrix.

Example 4: Consider the fully fuzzy linear system of equations of trapezoidal LR fuzzy numbers as follows:

$$(1,4,4,4) \otimes (w_1, x_1, y_1, z_1) \oplus (3,6,1,2) \otimes (w_2, x_2, y_2, z_2) \oplus (2,4,3,6) \otimes (w_3, x_3, y_3, z_3) = (10,36,172,160)$$

$$(2,6,6,6)\otimes(w_1,x_1,y_1,z_1)\oplus(8,12,4,3)$$
$$\otimes(w_2,x_2,y_2,z_2)\oplus(6,9,4,6)\otimes(w_3,x_3,y_3,z_3)$$
$$=(20,42,216,186)$$

where $A=\begin{bmatrix}132\\286\end{bmatrix}, L=\begin{bmatrix}4\ 6\ 4\\6129\end{bmatrix} M=\begin{bmatrix}4 & 13\\6 & 44\end{bmatrix},$

$$N=\begin{bmatrix}4 & 26\\6 & 36\end{bmatrix}$$

$b=\begin{bmatrix}10\\20\end{bmatrix},\ g=\begin{bmatrix}36\\42\end{bmatrix}, h=\begin{bmatrix}172\\216\end{bmatrix}, p=\begin{bmatrix}160\\186\end{bmatrix}$

$w=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}, x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix},\quad z=\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}$

Here, we see that the matrices A and L are non-square matrices, so we follow rule (ii) (Joshi, 1980) in this method. Then we have

det(A)= $\begin{vmatrix}132\\286\end{vmatrix}=\begin{vmatrix}13\\28\end{vmatrix}+(-1)^3\begin{vmatrix}1-32\\2-86\end{vmatrix}=\begin{vmatrix}13\\28\end{vmatrix}-\begin{vmatrix}-22\\-66\end{vmatrix}=2-0=2$

det(L)= $\begin{vmatrix}4 & 6\ 4\\6 & 129\end{vmatrix}=\begin{vmatrix}4 & 6\\6 & 12\end{vmatrix}+(-1)^3\begin{vmatrix}4-6\ 4\\6-129\end{vmatrix}=\begin{vmatrix}4 & 6\\6 & 12\end{vmatrix}-\begin{vmatrix}-24\\-69\end{vmatrix}=4$

Now we calculate (A^1, A^2, A^3) and (L^1, L^2, L^3), which are obtained from A and L by replacing their i^{th} column with b and g, respectively. Then

$A^1=\begin{bmatrix}1032\\2086\end{bmatrix},$

$\Rightarrow det(A^1)=\begin{vmatrix}1032\\2086\end{vmatrix}=\begin{vmatrix}103\\208\end{vmatrix}+(-1)^3\begin{vmatrix}10-32\\20-86\end{vmatrix}=\begin{vmatrix}103\\208\end{vmatrix}-\begin{vmatrix}7\ 2\\126\end{vmatrix}=20-18=2$

$A^2=\begin{bmatrix}1102\\2206\end{bmatrix},$

$\Rightarrow det(A^2)=\begin{vmatrix}1102\\2206\end{vmatrix}=\begin{vmatrix}110\\220\end{vmatrix}+(-1)^3\begin{vmatrix}1-102\\2-206\end{vmatrix}=\begin{vmatrix}110\\220\end{vmatrix}-\begin{vmatrix}-9\ 2\\-186\end{vmatrix}=0+18=18$

$A^3=\begin{bmatrix}1 & 3 & 10\\2 & 8 & 20\end{bmatrix},$

$\Rightarrow det(A^3)=\begin{vmatrix}1310\\2820\end{vmatrix}=\begin{vmatrix}13\\28\end{vmatrix}+(-1)^3\begin{vmatrix}1-310\\2-820\end{vmatrix}=\begin{vmatrix}13\\28\end{vmatrix}-\begin{vmatrix}-210\\-620\end{vmatrix}=2-20=-18$

$L^1=\begin{bmatrix}36\ 6\ 4\\42129\end{bmatrix},$

$\Rightarrow det(L^1)=\begin{vmatrix}36\ 6\ 4\\42129\end{vmatrix}=\begin{vmatrix}36\ 6\\4212\end{vmatrix}+(-1)^3\begin{vmatrix}36-6\ 4\\42-129\end{vmatrix}=\begin{vmatrix}36\ 6\\4212\end{vmatrix}-\begin{vmatrix}304\\309\end{vmatrix}=180-150=30$

$$L^2=\begin{bmatrix}4364\\6429\end{bmatrix},$$

$\Rightarrow det(L^2)=\begin{vmatrix}4364\\6429\end{vmatrix}=\begin{vmatrix}436\\642\end{vmatrix}+(-1)^3\begin{vmatrix}4-364\\6-429\end{vmatrix}=\begin{vmatrix}436\\642\end{vmatrix}-\begin{vmatrix}-324\\-369\end{vmatrix}=-48+144=96$

$L^3=\begin{bmatrix}4\ 6\ 36\\61242\end{bmatrix},$

$\Rightarrow det(L^3)=\begin{vmatrix}4\ 6\ 36\\61242\end{vmatrix}=\begin{vmatrix}4\ 6\\612\end{vmatrix}+(-1)^3\begin{vmatrix}4-6\ 36\\6-1242\end{vmatrix}=\begin{vmatrix}4\ 6\\612\end{vmatrix}-\begin{vmatrix}-236\\-642\end{vmatrix}=12-132=-120$

Therefore, we get

$w_1=\frac{2}{2}=1, w_2=\frac{18}{2}=9, w_3=\frac{-18}{2}=-9$

$x_1=\frac{30}{4}=7.5,\quad x_2=\frac{96}{4}=24,$

$$x_3=\frac{-120}{4}=-30$$

i.e., $w=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=\begin{bmatrix}1\\9\\-9\end{bmatrix},\ x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7.5\\24\\-30\end{bmatrix}$

Now we calculate y and z, so first, we calculate $A'^{(i)}$ and $L'^{(i)}$, which denote matrices obtained from A and L by replacing their i^{th} column by $h-Mw+$ Aw and $p-Nx+Lx$, respectively.

$$h-Mw+Aw=\begin{bmatrix}172\\216\end{bmatrix}-\begin{bmatrix}4 & 13\\6 & 44\end{bmatrix}\begin{bmatrix}1\\9\\-9\end{bmatrix}$$
$$+\begin{bmatrix}132\\286\end{bmatrix}\begin{bmatrix}1\\9\\-9\end{bmatrix}=\begin{bmatrix}196\\230\end{bmatrix}$$

$$p - Nx + Lx = \begin{bmatrix}160\\186\end{bmatrix} - \begin{bmatrix}4 & 26\\6 & 36\end{bmatrix}\begin{bmatrix}7.5\\24\\-30\end{bmatrix}$$

$$+ \begin{bmatrix}4\ 6\ 4\\6129\end{bmatrix}\begin{bmatrix}7.5\\24\\-30\end{bmatrix} = \begin{bmatrix}316\\312\end{bmatrix}$$

Now, $A'^{1} = \begin{bmatrix}19632\\23086\end{bmatrix}$

$\Rightarrow det(A'^{1}) = \begin{vmatrix}19632\\23086\end{vmatrix} = \begin{vmatrix}1963\\2308\end{vmatrix} +$
$(-1)^3 \begin{vmatrix}196-32\\230-86\end{vmatrix} = \begin{vmatrix}1963\\2308\end{vmatrix} - \begin{vmatrix}1932\\2226\end{vmatrix} = 878 -$
$714 = 164$

$A'^{2} = \begin{bmatrix}11962\\22306\end{bmatrix}$,

$\Rightarrow det(A'^{2}) = \begin{vmatrix}11962\\22306\end{vmatrix}$

$= \begin{vmatrix}1196\\2230\end{vmatrix} + (-1)^3 \begin{vmatrix}1-1962\\2-2306\end{vmatrix}$

$= \begin{vmatrix}1196\\2230\end{vmatrix} - \begin{vmatrix}-1952\\-2286\end{vmatrix} = -162 + 714 = 552$

$A'^{3} = \begin{bmatrix}13196\\28230\end{bmatrix}$,

$\Rightarrow det(A'^{3}) = \begin{vmatrix}13196\\28230\end{vmatrix} = \begin{vmatrix}13\\28\end{vmatrix} +$
$(-1)^3 \begin{vmatrix}1-3196\\2-8230\end{vmatrix} = \begin{vmatrix}13\\28\end{vmatrix} - \begin{vmatrix}-2196\\-6230\end{vmatrix} = 2 - 716 =$
-714

Therefore, we get

$y_1 = \frac{164}{2} = 82,\ \ y_2 = \frac{552}{2} = 276, \quad y_3 = \frac{-714}{2} =$
-357

i.e., $y = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}82\\276\\-357\end{bmatrix}$

Now, $L'^{1} = \begin{bmatrix}316\ 6\ 4\\312129\end{bmatrix}$,

$\Rightarrow det(L'^{1}) = \begin{vmatrix}316\ 6\ 4\\312129\end{vmatrix} = \begin{vmatrix}316\ 6\\31212\end{vmatrix} +$
$(-1)^3 \begin{vmatrix}316-6\ 4\\312-129\end{vmatrix} = \begin{vmatrix}316\ 6\\31212\end{vmatrix} - \begin{vmatrix}3104\\3009\end{vmatrix} = 1920 -$
$1590 = 330$

$L'^{2} = \begin{bmatrix}43164\\63129\end{bmatrix}$,

$\Rightarrow det(\mathrm{L}'^{2}) = \begin{vmatrix}43164\\63129\end{vmatrix} = \begin{vmatrix}4316\\6312\end{vmatrix} +$
$(-1)^3 \begin{vmatrix}4-3164\\6-3129\end{vmatrix} = \begin{vmatrix}4316\\6312\end{vmatrix} - \begin{vmatrix}-3124\\-3069\end{vmatrix} = -648 +$
$1584 = 936$

$L'^{3} = \begin{bmatrix}4\ 6\ 316\\612312\end{bmatrix}$,

$\Rightarrow det(L'^{3}) = \begin{vmatrix}4\ 6\ 316\\612312\end{vmatrix} = \begin{vmatrix}4\ 6\\612\end{vmatrix} +$
$(-1)^3 \begin{vmatrix}4-6\ 316\\6-12312\end{vmatrix} = \begin{vmatrix}4\ 6\\612\end{vmatrix} - \begin{vmatrix}-2316\\-6312\end{vmatrix} = 12 -$
$1272 = -1260$

Therefore, we get

$z_1 = \frac{330}{4} = 82.5, z_2 = \frac{936}{4} = 234, \quad z_3 = \frac{-1260}{4} =$
-315

i.e., $z = \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix}82.5\\234\\-315\end{bmatrix}$

Hence the solution set is

$$\begin{bmatrix}(w_1, x_1, y_1, z_1)\\(w_2, x_2, y_2, z_2)\\(w_3, x_3, y_3, z_3)\end{bmatrix} = \begin{bmatrix}(1, & 7.5, & 82, & 82.5)\\(9, & 24, & 276, & 234)\\(-9, & -30, & -357, & -315)\end{bmatrix}$$

This solution set's figure (Fig.4) is given below:

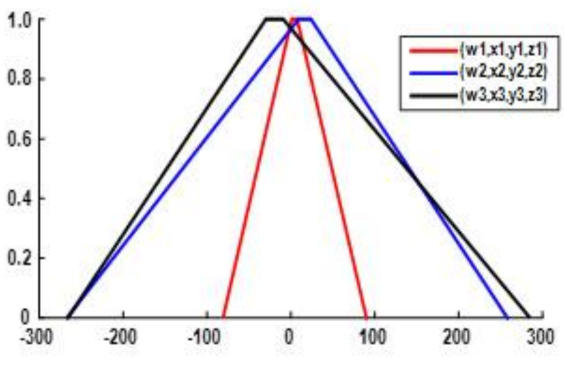

Fig. 4. Solution set of the FFSLE of trapezoidal LR fuzzy numbers in Cramer's rule (ii) for non-square coefficient matrix.

Results and Discussion

In this work, we consider two cases for solving FFSLE: the triangular LR fuzzy numbers and the trapezoidal LR fuzzy numbers. We have used the determinate of square and non-square coefficient matrix, the cross product of two fuzzy numbers, and the equality of fuzzy numbers. We have proposed a modified Cramer's rule method to solve the FFSLE. We have applied this proposed method using four numerical examples in two cases: according to the

first case (triangular LR fuzzy number), figure 1 and figure 2 are a graphical representation of the solution set of square and non-square coefficient matrices of FFSLE, respectively and figure 3 and figure 4 are graphical representation of the solution set of square and non-square coefficient matrices of FFSLE, respectively according to the second case (trapezoidal LR fuzzy number).

Compared with the previous work, we observe that the provided Modified Cramer's rules method can solve FFSLE with both square and non-square coefficient matrix, whereas the previous work could only solve the square coefficient matrix of FFSLE. So, the above-described modified methods are more generalized than the initial methods provided by the previous researchers.

We observed that all the above methods for describing trapezoidal numbers can also work for those methods in triangular numbers, but the converse is not always true.

In conclusion, the Modified Cramer's rules method for trapezoidal numbers is more generalized than those for triangular numbers.

All calculations were made using MATLAB tools.

Future opinion

There are several avenues for further investigation:

- To investigate the formation of a fuzzy system of linear equations (FSLE)and a fully fuzzy system of linear equations (FFSLE) form for bell-shaped fuzzy numbers and Gaussian fuzzy numbers.
- To seek exact solutions to bell-shaped fuzzy numbers and Gaussian fuzzy numbers of FSLE and FFSLE through analytical methods.

Author contribution

Fatema Khatun: Writing-review & editing, writing–original draft, conceptualization, lead, Methodology, Formal analysis, Software, resources, Validation, Investigation, Funding acquisition. **Md. Sahadat Hossain:** Writing-review & editing, Visualization, Supervision, conceptualization, Methodology, Formal analysis, Software, resources, Validation, Data curation. **Mst. Monjuara Akter:** Writing-review & editing, writing–original draft, conceptualization , Methodology, Formal analysis, Validation, Investigation.

Conflict of Interest

The authors declare that they have no conflict of interest.

Acknowledgements

The authors are grateful to the Editor-in-chief, the handling editor, and the anonymous referees for a number of helpful and perceptive comments for improving the manuscript. This work was partially supported by a grant from the National Science and Technology (NST), Government of Bangladesh, by providing NST fellowship. The first and third authors acknowledge this support, respectively.

References

Abbasbandy S and Alavi M. A method for solving fuzzy linear systems. *Iran. J. Fuzzy Syst*. 2005; 2(2): 37-43.

Abbasbandy S, Ezzati Rand Jafarian A. LU decomposition method for solving fuzzy system of linear equations. *Appl. Math.Comput*. 2006; 172(1): 633-643.

Abbasbandy S, Jafarian Aand Ezzati R. Conjugate gradient method for fuzzy symmetric positive definite system of linear equations. *Appl. Math. Comput*. 2005; 171(2): 1184-1191.

Abbasbandy S, Otadi M and Mosleh M. Minimal solution of general dual fuzzy linear systems. *Chaos Solitons Fractals*, 2008; 37(4): 1113-1124.

Abbasi SMM and Jalali A. A novel approach for solving fully fuzzy linear systems and their duality. *J. Intell. Fuzzy Syst.* 2019; 37(2): 2609-2619.

Abidin AS, Mashadi M and Sri G. Algebraic modification of trapezoidal fuzzy numbers to complete fully fuzzy linear equations system using gauss-jacobi method. *Int. J. Manag. Fuzzy Syst.* 2019; 5(2): 40-46.

Allahviranloo T. Numerical methods for fuzzy system of linear equations. *Appl. Math.Comput.* 2004; 155(2): 493-502.

Allahviranloo, T. Successive over relaxation iterative method for fuzzy system of linear equations. *Appl. Math. Comput*. 2005; 162(1): 189-196.

Allahviranloo T and Ghanbari M. A new approach to obtain algebraic solution of interval linear systems. *Soft Comput*. 2012a; 16(1): 121-133.

Allahviranloo T and Ghanbari M. On the algebraic solution of fuzzy linear systems based on interval theory. *Appl. Math. Model*. 2012b; 36(11): 5360-5379.

Allahviranloo T and Hashemi A. The embedding method to obtain the solution of fuzzy linear systems. *Int.J.Ind.Math*. 2014; 6(3): 229-233.

Allahviranloo T and Salahshour S. Fuzzy symmetric solutions of fuzzy linear systems. *J.Comput. Appl. Math*. 2011; 235(16): 4545-4553.

Arunkumar M, Murthy S and Ganapathy G. Determinant for non-square matrices. *Int. J. Math. Sci. Eng. Appl*. 2011; 5(5): 389-401.

Asady B, Abbasbandy S and Alavi M. Fuzzy general linear systems. *Appl. Math. Comput*. 2005; 169(1): 34-40.

Buckley JJ and Qu Y. Solving linear and quadratic fuzzy equations. *Fuzzy Sets Syst*. 1990; 38(1): 43-59.

Buckley JJ and Qu Y. Solving systems of linear fuzzy equations. *Fuzzy Sets Syst*. 1991; 43(1): 33-43.

Chang S and Zadeh L. On fuzzy mapping and control. *IEEE Trans. Syst. Man Cybern*. 1972; 2(1): 30-34.

Dehghan M and Hashemi B. Iterative solution of fuzzy linear systems. *Appl. Math. Comput*. 2006; 175(1): 645-674.

Dehghan M, Hashemi B and Ghatee M. Solution of the fully fuzzy linear systems using iterative techniques. *Chaos, Solitons & Fractals*. 2007; 34(2): 316-336.

Dubois D and Prade H. Operations on fuzzy numbers. *Int. J. Syst. Sci.* 1978; 9(6): 613-626.

Elsayed AAA, Ahmad N and Malkawi G. On the solution of fully fuzzy Sylvester matrix equation with trapezoidal fuzzy numbers. *Comput. Appl. Math*. 2020; 39(4): 1-22.

Friedman M, Ming M and Kandel A. Fuzzy linear systems. *Fuzzy Sets Syst*. 1998; 96(2): 201-209.

Guo X and Shang D. Fuzzy approximate solution of positive fully fuzzy linear matrix equations. *J. Appl. Math.* 2013; 2013: 178209.

Islam MA and Hossain MS. Mathematical comparison of defuzzification of fuzzy logic controller for smart washing machine. *J. Bangladesh Acad.Sci.* 2022; 46(1): 1-8.

Joshi VN. A determinant for rectangular matrices. *Bull. Aust. Math. Soc*. 1980; 21(1): 137-146.

Karthik NJ and Chandrasekaran E. Solving fully fuzzy linear systems with trapezoidal fuzzy number matrices by partitioning the block matrices. *Ann. Pure Appl. Math.* 2014; 8(2): 261-267.

Khatun F and Hossain MS. Mathematically forecasting for generalized business by using fuzzy trapezoidal numbers. *J. Bangladesh Acad.Sci.* 2022; 45(2): 147-154.

Ma M, Friedman M and Kandel A. Duality in fuzzy linear systems. *Fuzzy Sets Syst.* 2000; 109(1): 55-58.

Malkawi G, Rida I and Ahmad N. An associated linear system approach for solving fully fuzzy linear system with hexagonal fuzzy number.

Advances in Science and Engineering Technology International Conferences (ASET). IEEE 2018; 1-7.

Muruganandam S, Razak KA and Rajakumar K. Solving fully fuzzy linear systems by gauss jordan elimination method. In: *Journal of Physics: Conference Series, IOP Publishing.* 2019; 1362: 012087.

Muruganandam S and Razak KA. Matrix inversion method for solving fully fuzzy linear systems with triangular fuzzy numbers. *Int. J. Comput. Appl*. 2013; 65(4): 9-11.

Nikuie M and Ahmad MZ. Minimal solution of singular LR fuzzy linear systems. *Sci. World J.* 2014; 2014: 517218.

Radić M. About a determinant of rectangular matrix and its geometric interpretation. *Beiträge Zur Algebra Und Geometrie*. 2005; 46(2): 321-349.

Soylu G and Aslan ME. LR representation of TA fuzzy arithmetic and its application to solving fuzzy equations. *Iran. J. Fuzzy Syst*. 2022; 19(5): 63-78.

Zheng B and Wang K. General fuzzy linear systems. *App. Math.Comput.* 2006; 181(2): 1276-1286.

Ziqan A, Ibrahim S, Marabeh MAA and Qarariyah A. Fully fuzzy linear systems with trapezoidal and hexagonal fuzzy numbers. *Granul. Comput.* 2022; 7(2): 229-238.