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Research Article

Triangle group associated with generalized modular equation

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ABSTRACT

In this study, we investigate the triangle group $G=\left(\frac{1}{1-2s},\infty,\infty\right)$ associated with the generalized modular equation $\frac{{}_2F_1(s,1-s;1;1-\beta)}{{}_2F_1(s,1-s;1;\beta)}=p\,\frac{{}_2F_1(s,1-s;1;1-\alpha)}{{}_2F_1(s,1-s;1;\alpha)}$, where $p\in\mathbb{N}\setminus\{1\}$ and $s\in\left(0,\frac{1}{2}\right]$. We determine the generators of group G and prove that the group G is a subgroup of the Hecke group H_k. Also, we show that G is an even-type subgroup of H_k. We provide examples in the cases of Ramanujan's theories of signatures 2, 3, and 4.

2020 Mathematics Subject Classification: 11F06; 33C05.

Introduction

The great mathematician Srinivasa Ramanujan investigated the generalized modular equation

$$\frac{{}_2F_1(s,1-s;1;1-\beta)}{{}_2F_1(s,1-s;1;\beta)}=p\,\frac{{}_2F_1(s,1-s;1;1-\alpha)}{{}_2F_1(s,1-s;1;\alpha)},\qquad(1.1)$$

where

$$\alpha,\beta\in(0,1),s\in\left(0,\frac{1}{2}\right],p\in\mathbb{N}\setminus\{1\},$$

and provided many remarkable formulas and identities (Berndt, 1991; 1998). Here, ${}_2F_1$ is the Gaussian hypergeometric function defined as

$${}_2F_1(a,b;\,c;\alpha)=\sum_{j=0}^{\infty}\frac{(a,j)(b,j)}{(c,j)\,j!}\alpha^j,$$

where $a,b,c\in\mathbb{C}$ with $c\neq 0,-1,-2,\ldots$, $|\alpha|<1$, and (a,j) denotes the Pochhammer symbol given by

$$(\alpha,j)=\begin{cases}1 & \text{if } j=0,\\ \alpha(\alpha+1)\cdots(\alpha+j-1) & \text{if } j\geq 1.\end{cases}$$

The integer p is called the degree or order, and $\frac{1}{s}$ is called the signature of the equation (1.1).

The function ${}_2F_1$ can be extended to the slit plane $\mathbb{C}\setminus[1,+\infty)$ by Euler's integral representation formula (Bateman, 1953; Whittaker and Watson, 1927). The identities given by Ramanujan were published in his unpublished notebooks without original proofs (Ramanujan, 1957; 1988).

Ramanujan mainly investigated the generalized modular equation in the theories of signatures 2, 3, 4, and 6. Before the 1980s, there were no organized and developed theories associated with the generalized modular equation in the theories of signatures 2, 3, 4, and 6. Later, many mathematicians studied Ramanujan's theories and tried to prove the results provided by Ramanujan. For example, Borwein and Borwein (1987), Berndt (1985; 1989; 1991; 2006), and Berndt et al. (1995) proved many results given by Ramanujan and organized the theories related to Ramanujan's modular equation for $\frac{1}{s}=2,3,4,$ and 6. In their proofs, they used hypergeometric functions and the nontrivial identities for Jacobi's theta functions in addition to several new ideas. Also, Anderson et al. (1997) and Anderson et al. (2000) have studied Ramanujan's theories of modular equations from other perspectives. Alam and Sugawa (2022) provided a geometric method to prove Ramanujan's modular equations arising from the

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generalized modular equation. Alam (2024) studied the Hecke groups associated with the generalized modular equation in the theories of signatures $\frac{1}{s} = 2$, 3, and 4.

This paper studies the triangle groups associated with the generalized modular equation. We show that the triangle group $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$, where $s \in \left((0,\frac{1}{2}\right]$, is generated by

$$A_1 = \begin{pmatrix} 1 & 2\sin \pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 \\ 2\sin \pi s & 1 \end{pmatrix},$$

and is related to the generalized modular equation. The generators of the triangle group G can also be expressed by

$$A_1 = \begin{pmatrix} 1 & 2\sin \pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A'_2 = \begin{pmatrix} 4\sin^2 \pi s - 1 & -2\sin \pi s \\ 2\sin \pi s & -1 \end{pmatrix}$$

where $s \in \left(0,\frac{1}{2}\right]$. For $k \geq 3$, the Hecke group H_k is generated by

$$U = \begin{pmatrix} 1 & 2\cos\frac{\pi}{k} \\ 0 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We prove that the triangle group G is a subgroup of H_k and show that G is an even-type subgroup of H_k. Finally, we give examples in the cases of Ramanujan's theories of signatures $\frac{1}{s} = 2$, 3, and 4.

Preliminaries

Let us consider the following hypergeometric differential equation

$$\alpha(1-\alpha)\frac{d^2w}{d\alpha^2} + \{c - (a+b+1)\alpha\}\frac{dw}{d\alpha} - abw = 0. \tag{2.1}$$

The equation (2.1) has two linearly independent solutions

$$w_1 = {}_2F_1(a, b;\ c; \alpha)$$

and

$$w_2 = {}_2F_1(a, b;\ a+b+1-c; 1-\alpha).$$

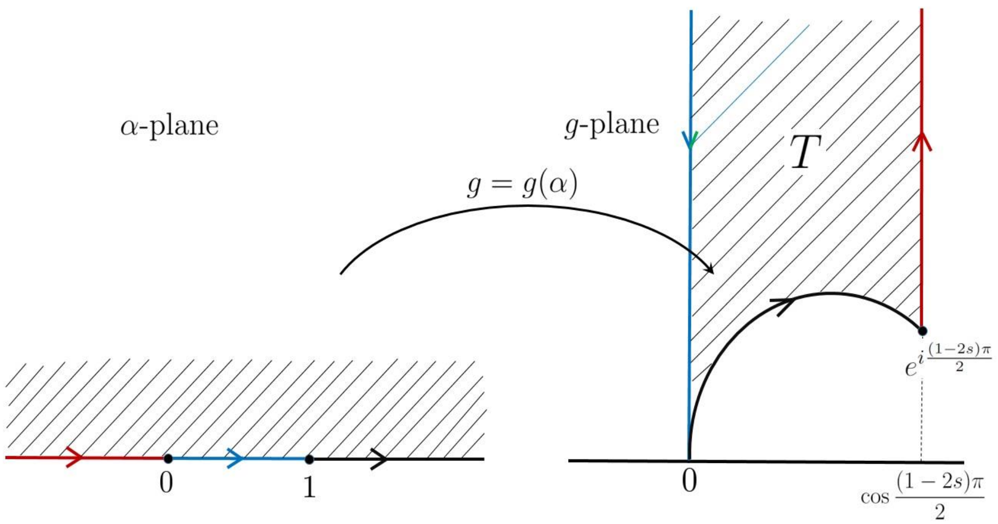

Fig. 1. The map g transforms the upper half of the α-plane to T on the g-plane.

We denote the upper half-plane $\{\alpha \in \mathbb{C} : \mathrm{Im}\, \alpha > 0\}$ by $\mathcal{H}$. If

$$g(\alpha) = i\,\frac{{}_2F_1(a, b;\ \mathrm{a + b + 1 - c}; 1 - \alpha)}{{}_2F_1(a, b;\ \mathrm{c}; \alpha)},$$

then g maps $\mathcal{H}$ conformally onto a hyperbolic triangle T (Fig. 1). At the vertices $g(0)$, $g(1)$, and $g(\infty)$, the interior angles of T are $(1 - c)\pi, (c - a - b)\pi$ and $(b - a)\pi$, respectively (Nehari, 1952).

For $s \in \left(0, \frac{1}{2}\right]$, let

$$a = s,\ \ b = 1 - s\ \ \text{and}\ \ c = 1.$$

Then

$$g(\alpha) = i\,\frac{{}_2F_1(s, 1 - s;\ 1; 1 - \alpha)}{{}_2F_1(s, 1 - \mathrm{s};\ 1; \alpha)}. \quad (2.2)$$

The following lemma describes the above facts.

Lemma 2.1 *(Lemma 4.1 of (Anderson et al., 2010)). Consider the map*

$$g(\alpha) = i\,\frac{{}_2F_1(s, 1 - s;\ 1; 1 - \alpha)}{{}_2F_1(s, 1 - s;\ 1; \alpha)},$$

where $s \in \left(0, \frac{1}{2}\right]$, *then* g *maps*

$$\mathcal{H} = \{\alpha \in \mathbb{C} : Im\ \alpha > 0\}$$

onto the following hyperbolic triangle

$$T = \left\{ g \in \mathcal{H}: 0 < Re\ g < cos\,\frac{(1 - 2s)\pi}{2},\ \left|2g\ cos\,\frac{(1 - 2s)\pi}{2} - 1\right| > 1 \right\}$$

in the g*-plane. At the vertices* $g(\infty) = e^{i\frac{(1-2s)}{2}}$, $g(0) = \infty$ *and* $g(1) = 0$, *the interior angles of* T *are* $(1 - 2s)\pi$, 0, *and* 0, *respectively.*

The following set of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d\ \in \mathbb{R},\ \ ad - bc = 1 \right\}$$

constructs a group known as the unimodular group and is denoted by $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{PSL}_2(\mathbb{R})$ is defined as

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\},$$

where I_2 is the 2×2 identity matrix (Serre, 1973; Katok, 1992). The action of the group P $\mathrm{L}_2(\mathbb{R})$ on $\mathcal{H}$ is as follows:

$$\alpha \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \frac{a\alpha + b}{c\alpha + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in\ \mathrm{PSL}_2(\mathbb{R})$ and $\alpha \in \mathcal{H}$.

Consider the boundary $\partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$ of $\mathcal{H}$. Vertical lines and semicircles orthogonal to the real axis, known as geodesics (Katok, 1992). The group $\mathrm{PSL}_2(\mathbb{R})$, together ith the transformation

$$\gamma(\alpha) = -\overline{\alpha}$$

construct the group $\mathrm{Isom}(\mathcal{H})$ of isometries of $\mathcal{H}$ (Gannon, 2007), i.e.,

$$\mathrm{Isom}(\mathcal{H}) \cong \mathrm{PSL}_2(\mathbb{R}) \cup \gamma\mathrm{PSL}_2(\mathbb{R})$$

and the group of analytic automorphisms of $\mathcal{H}$, denoted by $\mathrm{Aut}(\mathcal{H})$, is the group $\mathrm{PSL}_2(\mathbb{R})$.

Let the internal angles of a triangle be $\frac{\pi}{m_1}$, $\frac{\pi}{m_2}$, $\frac{\pi}{m_3}$, then the triangle is

(i) Euclidean if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1$,

(ii) spherical if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1$,

(iii) hyperbolic if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$.

In our study, we are interested in the hyperbolic triangle. We will denote the hyperbolic triangle by T. If the angles of two hyperbolic triangles are the same, then they are congruent. The area of T depends on the angles $\frac{\pi}{m_1}$, $\frac{\pi}{m_2}$, $\frac{\pi}{m_3}$ and is given by the following theorem known as Gauss-Bonnet theorem (Katok, 1992)).

Theorem 2.2 *(Gauss-Bonnet). Let* T *be a hyperbolic triangle with angles* $\frac{\pi}{m_1}$, $\frac{\pi}{m_2}$, *and* $\frac{\pi}{m_3}$, *then*

$$Area(T) = \pi\left(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right).$$

The upper half-plane $\mathcal{H}$ can be tessellated by the successive reflections of the hyperbolic triangle T about its sides. Let K be the group generated by the three reflections of T about its sides; then, it is the discrete subgroup of Isom($\mathcal{H}$). Let G be the subgroup of K such that G has only orientation-preserving isometries. Then,

$$G = K \cap \mathrm{PSL}_2(\mathbb{R}).$$

Group G is the triangle group with signature (m_1, m_2, m_3). One can also represent the triangle group G as

$$\langle A, B \,|\, A^{m_1} = B^{m_2} = (AB)^{m_3} = 1 \rangle,$$

where A, B and AB represent the rotations, respectively, by $\frac{2\pi}{m_1}$, $\frac{2\pi}{m_2}$ and $\frac{2\pi}{m_3}$ about the vertices of T.

Let D be a subset of $\mathcal{H}$, and let G be a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. If the following conditions are satisfied, then D is called a fundamental domain for G (Shimura, 1971):

(i) all points of D are G-inequivalent,

(ii) the subset D is open and connected,

(iii) if $x \in \mathcal{H}$ and y is a point of the closure of D, then x is G-equivalent to y.

When the subgroup G is a triangle group, the fundamental domain for G is given by the hyperbolic triangle T and its reflection about one of its sides. Note that one can construct a fundamental domain for a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ in different ways.

For $k \geq 3$, the Hecke group H_k is generated by

$$U = \begin{pmatrix} 1 & \delta_k \\ 0 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\delta_k = 2\cos\frac{\pi}{k}$. The element V has a fixed point at $\alpha = i$ of order 2. The Hecke group H_k is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. For $l \geq 2$, let $k = 2l$. Then H_l is isomorphic to $\mathbb{Z} * \mathbb{Z}/l\mathbb{Z}$ and $G = \langle U_{2l}, V_{2l} \rangle$ is a normal subgroup of H_{2l} of index 2, where

$$U_{2l} = \begin{pmatrix} 1 & \delta_{2l} \\ 0 & 1 \end{pmatrix}$$

and

$$V_{2l} = V^{-1} U_{2l}^{-1} V = \begin{pmatrix} 1 & 0 \\ \delta_{2l} & 1 \end{pmatrix}.$$

If $l = 2$ and $l = 3$, then the Hecke subgroups H_4 and H_6 are important and interesting as the elements of H_4 and H_6 are completely known (Parson, 1977). Note that the Hecke subgroup H_3 is the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$ generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The elements of Hecke group H_k are of the following two types:

(1) $\begin{pmatrix} a\delta_k & b \\ c & d\delta_k \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $ad\delta_k^2 - bc = 1$,

(2) $\begin{pmatrix} a & b\delta_k \\ c\delta_k & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc\delta_k^2 = 1$.

Type (1) is known as the odd type Hecke subgroup, and type (2) is known as the even type Hecke subgroup (Cangul, 1997; Cangul and Singerman, 1998).

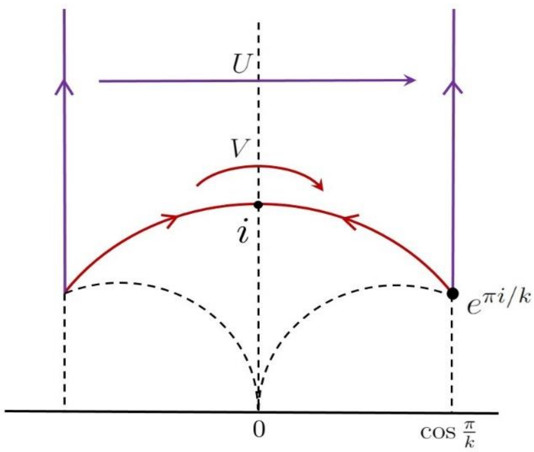

Fig. 2. Fundamental domain for H_k.

Let $W_k = UV = \begin{pmatrix} \delta_k & -1 \\ 1 & 0 \end{pmatrix}$. Then, W_k has a fixed point at $e^{i\pi/k}$ of order k. The following set of points

$$D_k = \left\{\alpha \in \mathcal{H} : |\alpha| \geq 1, |Re\ \alpha| \leq \cos\frac{\pi}{k}\right\}$$

is a fundamental domain for the Hecke group H_k (Fig. 2). One can easily see that the group H_k is a triangle group with signature $(2, k, \infty)$.

Main Results

Theorem 3.1. *The triangle group associated with the generalized modular equation*

$$\frac{{}_2F_1(s,1-s;1;1-\beta)}{{}_2F_1(s,1-s;1;\beta)} = p\,\frac{{}_2F_1(s,1-s;1;1-\alpha)}{{}_2F_1(s,1-s;1;\alpha)}$$

is $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$ *generated by*

$$A_1 = \begin{pmatrix} 1 & 2\sin \pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 \\ 2\sin \pi s & 1 \end{pmatrix},$$

where $s \in \left(0, \frac{1}{2}\right]$.

Proof. From Lemma 2.1, we have the function

$$g(\alpha) = i\,\frac{{}_2F_1(s,1-s;\ 1;1-\alpha)}{{}_2F_1(s,1-s;\ 1;\alpha)}$$

maps the upper half α-plane to the hyperbolic triangle T with angles $(1-2s)\pi$, 0 and 0 at $g(\infty) = e^{i\frac{(1-2s)\pi}{2}}$, $g(0) = \infty$ and $g(1) = 0$, respectively, in the upper half g-plane.

Let

$$\theta_1 = \frac{\pi}{m_1},\quad \theta_2 = \frac{\pi}{m_2} \text{ and } \theta_3 = \frac{\pi}{m_3}$$

be the internal angles of a hyperbolic triangle T, then T can be continued across its sides as a single-valued function if and only if $m_j > 1$ and $m_j \in \mathbb{N} \cup \{\infty\}$ for $j = 1, 2, 3$ (Sansone and Gerretsen, 1969). Therefore,

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$$

and

$$m_1 = \frac{1}{1-2s},\quad m_2 = \infty \text{ and } m_3 = \infty.$$

It follows that we can tessellate $\mathcal{H}$ by the triangle T. Since a triangle group preserves a tessellation by a triangle, the triangle group associated with the generalized modular equation is

$$G = (m_1, m_2, m_3) = \left(\frac{1}{1-2s}, \infty, \infty\right).$$

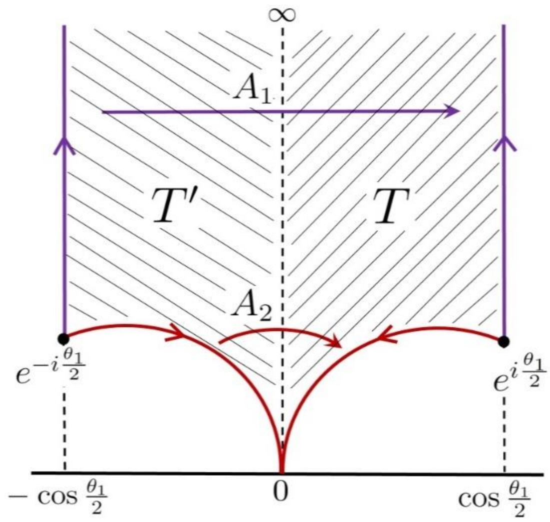

Fig. 3. Fundamental domain for the triangle group $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$.

If we reflect the hyperbolic triangle T about the geodesic side joining 0 and ∞, then we obtain the hyperbolic triangle T' with vertices at ∞, 0 and $e^{-i\frac{\theta_1}{2}}$ (see Fig. 3). The triangle T represents $\mathcal{H}$, and the triangle T' represents the lower half-plane. Note that one can reflect T about any side of T. If the geodesic side between $e^{-i\frac{\theta_1}{2}}$ and ∞ is identified with the geodesic side between $e^{i\frac{\theta_1}{2}}$ and ∞, then the side-pairing transformation is

$$A_1 = \begin{pmatrix} 1 & 2\cos\frac{\theta_1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\cos\frac{(1-2s)\pi}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\sin \pi s \\ 0 & 1 \end{pmatrix}.$$

The transformation A_1 divides the upper half of the α-plane into infinite strips parallel to the y-axis and width $2\cos\frac{\theta_1}{2}$. If the geodesic side between 0 and $e^{-i\frac{\theta_1}{2}}$ is identified with the geodesic side between 0 and $e^{i\frac{\theta_1}{2}}$, then the side-pairing transformation is

$$A_2 = \begin{pmatrix} 1 & 0 \\ 2\cos\frac{\theta_1}{2} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 2\sin\pi s & 1 \end{pmatrix}.$$

Therefore, the triangle group $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$ is generated by

$$A_1 = \begin{pmatrix} 1 & 2\sin\pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 \\ 2\sin\pi s & 1 \end{pmatrix}.$$

Remark 1. The triangle group G acts properly discontinuously on $\mathcal{H}$, and we obtain the quotient surface $G\backslash\mathcal{H}$, which is the thrice punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

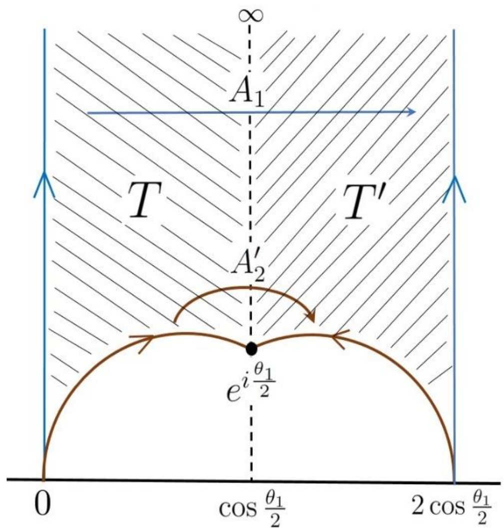

Fig. 4. The modified fundamental domain for $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$.

Lemma 3.2. The generators of the triangle group $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$ can be expressed by

$$A_1 = \begin{pmatrix} 1 & 2\,sin\,\pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A'_2 = \begin{pmatrix} 4\,sin^2\pi s - 1 & -2\,sin\,\pi s \\ 2\,sin\,\pi s & -1 \end{pmatrix},$$

where $s \in \left(0, \frac{1}{2}\right]$.

Proof. By Theorem 3.1, the generators of the triangle group $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$ are

$$A_1 = \begin{pmatrix} 1 & 2\sin\pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 \\ 2\sin\pi s & 1 \end{pmatrix},$$

where $s \in \left(0, \frac{1}{2}\right]$. The fundamental domain for the triangle group G is modified as follows. If we reflect the hyperbolic triangle T about the geodesic side joining $e^{i\frac{\theta_1}{2}}$ and ∞, then we obtain the hyperbolic triangle T' whose vertices are at $2\cos\frac{\theta_1}{2} = 2\sin\pi s$, $e^{i\frac{\theta_1}{2}} = e^{i\frac{(1-2s)\pi}{2}}$, and ∞ (Fig. 4).

The triangle T represents $\mathcal{H}$, and the triangle T' represents the lower half-plane. If the geodesic side between 0 and ∞ is identified with the geodesic side between $2\sin\pi s$ and ∞, then the side-pairing transformation is

$$A_1 = \begin{pmatrix} 1 & 2\sin\pi s \\ 0 & 1 \end{pmatrix}$$

and if the geodesic side between 0 and $e^{i\frac{(1-2s)\pi}{2}}$ is identified with the geodesic side between $2\sin\pi s$ and $e^{i\frac{(1-2s)\pi}{2}}$, then the side-pairing transformation is

$$A'_2 = -A_1 A_2^{-1}$$
$$= -\begin{pmatrix} 1 & 2\sin\pi s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\sin\pi s & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4\sin^2\pi s - 1 & -2\sin\pi s \\ 2\sin\pi s & -1 \end{pmatrix}.$$

Therefore, A_1 and A'_2 are the generators of G.

Remark 2. The generator A'_2 is an elliptic element of order $m_1 = \frac{1}{1-2s}$.

Theorem 3.3. The group associated with the generalized modular equation

$$\frac{{}_2F_1(s,1-s;1;1-\beta)}{{}_2F_1(s,1-s;1;\beta)} = p\,\frac{{}_2F_1(s,1-s;1;1-\alpha)}{{}_2F_1(s,1-s;1;\alpha)}$$

is a subgroup of the Hecke group H_k.

Proof. According to Theorem 3.1, the triangle group $G=\left(\frac{1}{1-2s},\infty,\infty\right)$ is associated with the generalized modular equation. The generators of the group G are

$$A_1=\begin{pmatrix}1 & 2\sin\pi s\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\cos\frac{(1-2s)\pi}{2}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\cos\frac{\pi}{2m_1}\\ 0 & 1\end{pmatrix}$$

and

$$A_2=\begin{pmatrix}1 & 0\\ 2\sin\pi s & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2\cos\frac{\pi}{2m_1} & 1\end{pmatrix}.$$

For $k\geq 3$, the Hecke group H_k is generated by

$$U=\begin{pmatrix}1 & 2\cos\frac{\pi}{k}\\ 0 & 1\end{pmatrix}$$

and

$$V=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Let $k=2m_1$, then

$$A_1=U$$

and

$$A_2=V^{-1}U^{-1}V.$$

Since the generators of the group G can be expressed in terms of the generators of the Hecke group H_k, we conclude that G is a subgroup of H_k.

Lemma 3.4. The triangle group $G=\left(\frac{1}{1-2},\infty,\infty\right)$ is an even type subgroup of H_k.

Proof. It is known that an even type subgroup of H_k is of the following form:

$$\begin{pmatrix}a & b\delta_k\\ c\delta_k & d\end{pmatrix},$$

where $a,b,c,d\in\mathbb{Z}, \delta_k=2\cos\frac{\pi}{k}$ and $ad-bc\delta_k^2=1$. In the proof of Theorem 3.3, we have seen that the generators of the triangle group $G=\left(\frac{1}{1-2s},\infty,\infty\right)$ are

$$A_1=\begin{pmatrix}1 & 2\cos\frac{\pi}{2m_1}\\ 0 & 1\end{pmatrix}$$

and

$$A_2=\begin{pmatrix}1 & 0\\ 2\cos\frac{\pi}{2m_1} & 1\end{pmatrix}.$$

Let $k=2m_1$ and $a=1, b=2, c=0, d=1$ or $a=1, b=0, c=2, d=1$. Then, we conclude that G is an even-type subgroup of H_k.

Remark 3. The triangle group G can be represented by

$$G=\left\{\begin{pmatrix}a & b\delta_k\\ c\delta_k & d\end{pmatrix}: a,b,c,d\in\mathbb{Z}\ and\ ad-bc\delta_k^2=1\right\}.$$

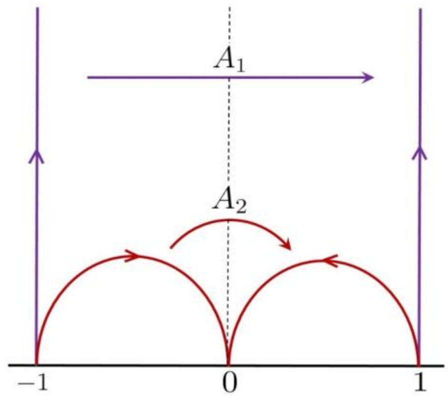

Fig. 5. Fundamental domain for the triangle group $G=(\infty,\infty,\infty)$.

Example 3.1. For the signature $\frac{1}{s}=2$, the generalized modular equation is

$$\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)} = p\,\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}.$$

In this case, the corresponding triangle group is $G=(\infty,\infty,\infty)$ and the generators of G are

$$A_1=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \text{ and } A_2=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}.$$

The fundamental domain for $G = (\infty, \infty, \infty)$ is shown in Fig. 5. The vertices of the triangle T are at 0, 1, and ∞; the angles of T are 0, 0, and 0.

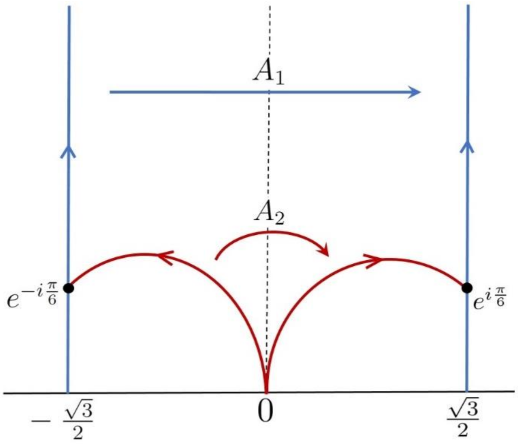

Fig. 6. Fundamental domain for the triangle group $G = (3, \infty, \infty)$.

Example 3.2. If the signature $s = \frac{1}{3}$, then the generalized modular equation is

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right)} = p\,\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right)}$$

and the corresponding triangle group is $G = (3, \infty, \infty)$ and the generators of G are

$$A_1 = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix}.$$

The fundamental domain for $G = (3, \infty, \infty)$ is shown in Fig. 6. In this case, the triangle T has internal angles 0, 0, and $\frac{\pi}{3}$ at the vertices ∞, 0, and $e^{i\frac{\pi}{6}}$, respectively.

Example 3.3. If the signature $\frac{1}{s} = 4$, then the generalized modular equation is

$$\frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \beta\right)} = p\,\frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \alpha\right)}.$$

In this case, the corresponding triangle group is $G = (2, \infty, \infty)$ generated by

$$A_1 = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}.$$

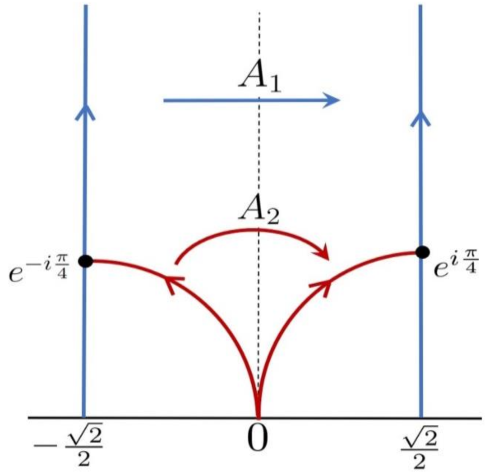

Fig. 7. Fundamental domain for the triangle group $G = (2, \infty, \infty)$.

The fundamental domain for $G = (2, \infty, \infty)$ is shown in Fig. 7. The internal angles of the triangle T are 0, 0, and $\frac{\pi}{2}$ at the vertices ∞, 0, and $e^{i\frac{\pi}{4}}$, respectively.

Conclusion

We have studied the triangle group $G = \left(\frac{1}{1-2s}, \infty, \infty\right)$ associated with the generalized modular equation

$$\frac{{}_2F_1(s, 1-s; 1; 1-\beta)}{{}_2F_1(s, 1-s; 1; \beta)} = p\,\frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}$$

where $s \in \left(0, \frac{1}{2}\right]$ and $p \in \mathbb{N} \setminus \{1\}$. It has been proved that the triangle group G is generated by

$$A_1 = \begin{pmatrix} 1 & 2\sin \pi s \\ 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 \\ 2\sin \pi s & 1 \end{pmatrix}.$$

Also, we have proved that the group G is a subgroup of the Hecke group H_k. In fact, the group G is an even-type subgroup of H_k. Finally, three examples have been given in the cases of signatures $\frac{1}{s} = 2$, 3, and 4.

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Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this article.

Author's Contributions

Md. Shafiul Alam contributed to conceptualization, formal analysis, supervision, and manuscript drafting. Bijan Krishna Saha contributed to validation, analysis, and manuscript editing.

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