

A UNIFIED KBM METHOD FOR OBTAINING THE SECOND APPROXIMATE SOLUTION OF A THIRD ORDER WEAKLY NON-LINEAR DIFFERENTIAL SYSTEM WITH STRONG DAMPING AND SLOWLY VARYING COEFFICIENTS

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ABSTRACT

To obtain the second order approximate solution of a third order weakly nonlinear ordinary differential system with strong damping and slowly varying coefficients modeling a damped oscillatory process is considered based on the extension of a unified Krylov-Bogoliubov-Mitropolskii (KBM) method. The asymptotic solution for different initial conditions shows a good coincidence with those obtained by the numerical procedure for obtaining the transient's response. The method is illustrated by an example.

Key words: KBM method, Damped oscillatory process, Strong damping, Slowly varying coefficients

INTRODUCTION

The study of non-linear problems is of crucial importance not only in all areas of physics but also in engineering and in applied mathematics, since most phenomena in the world are essentially non-linear and are described by non-linear equations. It is very difficult to solve non-linear problems and in general, it is often more difficult to get an analytical approximation than a numerical one for a given non-linear problem. There are several methods used to find approximate solutions to non-linear problems, such as the perturbation techniques (Murty 1971, Bojadziev 1983, Alam and Sattar 1997) and harmonic balance based method (Itovich and Moiola 2005). The method has been extended to damped oscillatory and purely non oscillatory systems with slowly varying coefficients by Bojadziev and Edwards (1981). Arya and Bojadziev (1980) studied a system of second order nonlinear hyperbolic differential equation with slowly varying coefficients. They (1981) also studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Feshchenko *et al.* (1967) presented a brief way to determine KBM (Bogoliubov and Yu 1961, Krylov and Bogoliubov 1947) solution (first order) of an n th, $n = 2, 3, \dots$ order differential systems. Alam (2003) investigated a unified KBM method for obtaining the first approximate solution of n th

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order nonlinear systems with slowly varying coefficients. Alam and Sattar (2004) also presented an asymptotic method for obtaining the first approximate solution of a third order non-linear differential system with varying coefficients. Moreover, Alam (2002) investigated a unified KBM method for obtaining the first approximate solution of n th, ($n \geq 3$) order non-linear differential system with constant coefficients. Recently, Roy and Alam (2004) have studied the effect of higher approximation of Krylov-Bogoliubov-Mitropolskii's solution and matched asymptotic solution for second order non-linear differential system with slowly varying coefficients and damping. Sometimes the first approximate solutions obtained in [1-10] give desired results when the linear damping effect is very small. Otherwise, the solutions give incorrect results after a long time $t \gg 1$ where the reduced frequency becomes small. From the present study, it is seen that the most of the authors in references have obtained the first approximate solutions for both constants and varying coefficients. The complicated and no less important case of second order approximate solution of a third order non-linear differential system with strong damping and slowly varying coefficients by a unified KBM method has remained almost untouched. The main goal of this paper is to fill this gap.

METHOD

Let authors consider a third-order weakly non-linear ordinary differential equation with slowly varying coefficients in the following form [1]

$$\ddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon f(x, \dot{x}, \ddot{x}, \tau) \quad (1)$$

where the over dots represent the time derivatives, ε is a small positive parameter which measures the strength of the nonlinearity, $\tau = \varepsilon t$ slowly varying time, $k_j(\tau) \geq 0$, $j = 1, 2, 3$ and f is a given nonlinear function which satisfies $f(-x, -\dot{x}, -\ddot{x}, \tau) = -f(x, \dot{x}, \ddot{x}, \tau)$. The coefficients are varying slowly in the sense that their time derivatives are proportional to ε (Alam and Sallar 1947).

By putting $\varepsilon = 0$, $\tau = \tau_0 = \text{constant}$ in Eq. (1), they obtains the solution of the unperturbed equation. They assume that the unperturbed part of Eq. (1) has three eigenvalues $\lambda_j(\tau_0)$, $j = 1, 2, 3$, where $\lambda_j(\tau_0)$ are constants, but if $\varepsilon \neq 0$ then $\lambda_j(\tau)$ are varying slowly with time t . The solution of the linearized equation of Eq. (1) is obtained in the following form

$$x(t, 0) = \sum_{j=1}^3 a_{j,0} e^{\lambda_j(\tau_0)t}, \quad (2)$$

where $a_{j,0}$, $j = 1, 2, 3$ are arbitrary constants.

Now authors are going to choose a solution of Eq. (1) that reduces to Eq. (2) as a limit $\varepsilon \rightarrow 0$ in the following form according to the KBM method (Bogoliubov *et al.* 1961)

$$x(t, \varepsilon) = \sum_{j=1}^3 a_j(t) + \varepsilon u_1(a_1, a_2, a_3, \tau) + \varepsilon^2 u_2(a_1, a_2, a_3, \tau) + \dots, \quad (3)$$

where each a_j satisfies the following first order differential equation

$$\dot{a}_j = \lambda_j a_j + \varepsilon A_j(a_1, a_2, a_3) + \varepsilon^2 B_j(a_1, a_2, a_3) + \varepsilon^3 \dots. \quad (4)$$

Confining only to the first few terms, 1, 2, 3 in the series expansions of Eq. (3) and Eq. (4), they evaluate the functions u_1, u_2, \dots and $A_j, B_j, \dots, j = 1, 2, 3$ such that each $a_j(t)$ appearing in Eq. (3) and Eq. (4) satisfy the given differential Eq. (1) with an accuracy of ε^{m+1} (Alam 2002). Theoretically, the solution can be obtained up to any order of approximations but, owing to the rapidly growing algebraic complexity for the derivation of the formula, the solution is in general confined to a low order, usually the first order (Alam 2001, 2003, Alam and Sattar 1997, 2004, Arya and Bojadziev 1980, 1981, Bogoliubov and Mitropolskii 1961, Bojadziev and Edwards 1981, Bojadziev 1983, Feshchenko *et al.* 1966). In order to determine these functions it is assumed that the functions u_1, u_2 do not contain the fundamental terms (Alam 2002, 2003, Alam and Sattar 1997, 2004, Bogoliubov and Mitropolskii 1961, Krylov and Bogoliubov 1947) which included in the series expansions (3) at order ε^0 . Now differentiating Eq. (3) three times with respect to time t and using the relations Eq. (4) and substituting the values of $\ddot{x}, \dot{x}, \dot{x}$ together with x into the original Eq. (1) with the slowly varying coefficients $k_1(\tau) = -(\lambda_1(\tau) + \lambda_2(\tau) + \lambda_3(\tau))$, $k_2(\tau) = \lambda_1(\tau)\lambda_2(\tau) + \lambda_2(\tau)\lambda_3(\tau) + \lambda_1(\tau)\lambda_3(\tau)$, $k_3(\tau) = -\lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)$ and expanding the right hand side of Eq. (1) by Taylor's series and equating the coefficients of ε and ε^2 on both sides we obtain the following equations

$$\begin{aligned} & \prod_{j=1}^3 (\Omega - \lambda_j) u_1 + \sum_{j=1}^3 \left(\prod_{k=1, k \neq j}^3 (\Omega - \lambda_k) \right) A_j \\ & + \sum_{j=1}^n \frac{1}{2} \sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_j^{(n-k-2)} \lambda_j' a_j = f^{(0)}(a_1, a_2, a_3, \tau) \end{aligned} \quad (5)$$

$$\prod_{j=1}^3 (\Omega - \lambda_j) u_2 + \sum_{j=1}^3 \left(\prod_{k=1, k \neq j}^3 (\Omega - \lambda_k) \right) B_j + \sum_{j=1}^3 \lambda_j' A_j + \sum_{j=1}^3 \lambda_j'' a_j = f^{(1)}(a_1, a_2, a_3, \tau) \quad (6)$$

where $f(x_0, \dot{x}_0, \ddot{x}_0, \tau) = f^{(0)}(a_1, a_2, a_3, \tau) + \varepsilon f^{(1)}(a_1, a_2, a_3, \tau)$, $x_0 = \sum_{j=1}^3 a_j$, $\Omega = \sum_{j=1}^3 \lambda_j a_j \frac{\partial}{\partial a_j}$,

and $\lambda_j' = \frac{d\lambda_j}{d\tau}$, $n = 3$.

Authors have already assumed that u_1 and u_2 do not contain the fundamental terms and for this reason the solution will be free from secular terms, namely $t \cos t$, $t \sin t$ and $t e^{-t}$. Since the solution will be non-uniform in presence of secular terms. Under these restrictions, we are able to solve Eq. (5) and Eq. (6), by separating this into $n+1$ individual equations for the unknown functions u_1, u_2, A_j and B_j . In general, the functions $f^{(0)}, f^{(1)}, u_1$ and u_2 are expanded in Taylor series in the following forms

$$f^{(0)} = \sum_{m_1=0, m_2=0, \dots, m_n=0}^{\infty, \infty, \dots, \infty} F_{m_1, m_2, m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3}, \quad (7)$$

$$u_1 = \sum_{m_1=0, m_2=0, \dots, m_n=0}^{\infty, \infty, \dots, \infty} U_{m_1, m_2, m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3}, \quad (8)$$

$$f^{(1)} = \sum_{m_1=0, m_2=0, \dots, m_n=0}^{\infty, \infty, \dots, \infty} G_{m_1, m_2, m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3}, \quad (9)$$

and

$$u_2 = \sum_{m_1=0, m_2=0, m_3=0}^{\infty, \infty, \infty} V_{m_1, m_2, m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3}. \quad (10)$$

The eigen values of the unperturbed equation can be written as $\lambda(\tau_0)$ and $-\mu_l(\tau_0) \pm \omega_l(\tau_0)$ where $l=1$. For the above restrictions, it guarantees that u_1 and u_2 must be excluded all terms with $a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}$ of $f^{(0)}$ and $f^{(1)}$ where $m_{2l} - m_{2l+1} = \pm 1$. Since as a linear approximation (*i.e.* $\varepsilon \rightarrow 0$) $a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}$ becomes $e^{\omega_l t}$ when $m_{2l} - m_{2l+1} = 1$ or $e^{-\omega_l t}$ when $m_{2l} - m_{2l+1} = -1$. It is noticed that $e^{\pm \omega_l t}$ are known as the fundamental terms [4, 7, 12]. Usually these are included in equations A_j and B_j . Moreover, it is restricted (by Krylov and Bogoliubov (1947) that the functions A_j and B_j are independent of the fundamental terms. Now to determine the equations for A_1 and B_1 , we followed the assumption of Bojadziev (1983) that u_1 and u_2 do not contain a term te^{-t} (as limit $\mu_l \rightarrow 0$) and obtained the following equations:

$$\begin{aligned} & \left(\prod_{k=2}^n (\Omega - \lambda_k) \right) A_1 + \frac{1}{2} \left(\sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_{2l-1}^{(2l-k-2)} \right) \lambda_1' a_1 \\ & = \sum_{m_1=0, m_{2l}=0, m_{2l+1}=0}^{\infty, \infty, \infty} F_{m_1, m_{2l}, m_{2l+1}} a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} = m_{2l+1}, \end{aligned} \quad (11)$$

$$\begin{aligned} & \left(\prod_{k=2}^n (\Omega - \lambda_k) \right) B_1 + \frac{1}{2} \left(\sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_{2l-1}^{(2l-k-2)} \right) \lambda_1' a_1 \\ & = \sum_{m_1=0, m_{2l}=0, m_{2l+1}=0}^{\infty, \infty, \infty} G_{m_1, m_{2l}, m_{2l+1}} a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} = m_{2l+1}. \end{aligned} \quad (12)$$

Then the equations for u_1, u_2, A_j and $B_j, j=1, 2, \dots, n$ are obtained as

$$\prod_{j=1}^n (\Omega - \lambda_j) u_1 = \sum_{m_1=0, m_{2l}=0, m_{2l+1}=0}^{\infty, \infty, \infty} F_{m_1, m_{2l}, m_{2l+1}}(\tau) a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}} \quad m_{2l} - m_{2l+1} \neq 0, \pm 1, \quad (13)$$

$$\begin{aligned} & \left(\prod_{k=1, k \neq 2l}^n (\Omega - \lambda_k) \right) A_{2l} + \frac{1}{2} \left(\sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_{2l}^{2l-k-2} \right) \lambda_{2l}' a_{2l} \\ & = \sum_{m_{2l}=0, m_{2l+1}=0}^{\infty, \infty} F_{m_{2l}, m_{2l+1}} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} - m_{2l+1} = 1, \end{aligned} \quad (14)$$

$$\begin{aligned} & \left(\prod_{k=1, k \neq 2l+1}^n (\Omega - \lambda_k) \right) A_{2l+1} + \frac{1}{2} \left(\sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_{2l}^{2l-k-2} \right) \lambda'_{2l+1} a_{2l+1} \\ & = \sum_{m_{2l}=0, m_{2l+1}=0}^{\infty, \infty} F_{m_{2l} m_{2l+1} 0} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} - m_{2l+1} = -1 \end{aligned} \quad (15)$$

and

$$\prod_{j=1}^n (\Omega - \lambda_j) u_2 = \sum_{m_1=0, m_{2l}=0, \dots, m_{2l+1}}^{\infty, \infty, \infty} G_{m_1, m_{2l}, m_{2l+1}}(\tau) a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} - m_{2l+1} \neq 0, \pm 1, \quad (16)$$

$$\begin{aligned} & \left(\prod_{k=1, k \neq 2l-1}^n (\Omega - \lambda_k) \right) B_{2l} + \frac{1}{2} \left(\sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_{2l}^{2l-k-2} \right) \lambda'_{2l} a_{2l} \\ & = \sum_{m_{2l}=0, m_{2l+1}=0}^{\infty, \infty} G_{m_{2l}, m_{2l+1}} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} - m_{2l+1} = 1, \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \left(\prod_{k=1, k \neq 2l+1}^n (\Omega - \lambda_k) \right) B_{2l+1} + \frac{1}{2} \left(\sum_{k=0}^{n-2} (n-k)(n-k-1) c_k \lambda_{2l}^{2l-k-2} \right) \lambda'_{2l+1} a_{2l+1} \\ & = \sum_{m_{2l}=0, m_{2l+1}=0}^{\infty, \infty} G_{m_{2l} m_{2l+1}} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}, \quad m_{2l} - m_{2l+1} = -1. \end{aligned} \quad (18)$$

To obtain the particular solutions of Eqs. (11) - (17), they replace the operator Ω by $\sum_{j=1}^3 m_j \lambda_j$, since authors know that $\Omega(a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}) = \sum_{j=1}^n m_j \lambda_j (a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}})$. Hence the determination of second order approximate solution of Eq. (1) is completely determined.

But it is noticed that the solution Eq. (3) is not a standard form of the KBM method. To reduce the standard form of the KBM solution from Eq. (3), authors need to use the following substitutions

$$\begin{aligned} a_1 &= a \\ a_{2l} &= \frac{1}{2} b e^{\varphi t}, \\ a_{2l+1} &= \pm \frac{1}{2} b e^{-\varphi t}, \quad l = (n-1)/2, \end{aligned} \quad (19)$$

where a , b represent the amplitudes and φ represents the phase of the nonlinear physical differential systems.

EXAMPLE

For the practical importance of the above method, authors consider the following third order weakly nonlinear differential equation with strong damping and slowly varying coefficients

$$\ddot{x} + k_1(\tau) \ddot{x} + k_2(\tau) \dot{x} + k_3(\tau) x = \varepsilon x^3 \quad (20)$$

Here $n = 3$, $j = 1, 2, 3$ $f(x, \dot{x}) = x^3$ and $x_0 = a_1 + a_2 + a_3$. So authors have,

$$\begin{aligned} f(x, \dot{x}) &= f(a_1 + a_2 + a_3 + \varepsilon u_1, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \varepsilon(A_1 + A_2 + A_3 + \Omega u_1)) \\ &= f(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) + \varepsilon u_1 f_x(a_1 + a_2 + a_3, \\ &\quad \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) + \varepsilon(A_1 + A_2 + A_3 + \Omega u_1) \\ &\quad \times f_x(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) + \dots \\ &= f^{(0)} + \varepsilon f^{(1)} \end{aligned} \quad (21)$$

where

$$\begin{aligned} f^{(0)} &= f(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) \\ &= a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3 + 3a_1^2 a_3 + 6a_1 a_2 a_3 + 3a_2^2 a_3 + 3a_1 a_3^2 + 3a_2 a_3^2 + a_3^3, \end{aligned} \quad (22)$$

and

$$\begin{aligned} f^{(1)} &= 3u_1 f_x(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) \\ &= 3u_1 (a_1 + a_2 + a_3)^2 \\ &= 3[r_1 a_1^3 a_2^2 + (r_1 + 2r_3) a_1 a_2^4 + (r_1 + r_2) a_1 a_2^2 a_3^2 + (2r_1 + r_2) a_1^2 a_2^3 + 2(r_1 + r_3) a_1 a_2^3 a_3 \\ &\quad + 2r_1 a_1^2 a_2^2 a_3 + r_2 a_1^3 a_3^2 + (r_2 + 2r_4) a_1 a_3^4 + 2r_2 a_1^2 a_2 a_3^2 + 2(r_2 + r_4) a_1 a_2 a_3^3 \\ &\quad + (2r_2 + r_4) a_1^2 a_3^3 + r_3 a_2^5 + r_3 a_2^3 a_3^2 + 2r_3 a_2^4 a_3 + r_4 a_2^2 a_3^3 + r_4 a_3^5 + 2r_4 a_2 a_3^4], \end{aligned} \quad (23)$$

where

$$\begin{aligned} r_1 &= \frac{3}{2\lambda_2(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2 - \lambda_3)}, & r_2 &= \frac{3}{2\lambda_3(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_2 + 2\lambda_3)} \\ r_3 &= \frac{1}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)}, & r_4 &= \frac{1}{2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)} \end{aligned} \quad (24)$$

Substituting the values of n , j and $f^{(0)}$ in Eq. (5) and according to our restrictions, we obtain four equations for A_1 , A_2 , A_3 and u_1 whose solutions are respectively given by the following equations

$$\begin{aligned} A_1 &= -\frac{(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{a_1^3}{(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)} + \frac{6a_1 a_2 a_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)}, \\ A_2 &= -\frac{(2\lambda_2 - \lambda_1 - \lambda_3)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{3a_1^2 a_2}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2 - \lambda_3)} + \frac{3a_2^2 a_3}{2\lambda_2(2\lambda_2 + \lambda_3 - \lambda_1)}, \\ A_3 &= -\frac{(2\lambda_3 - \lambda_1 - \lambda_2)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} + \frac{3a_1^2 a_3}{(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 - \lambda_2)} + \frac{3a_2 a_3^2}{2\lambda_3(2\lambda_3 + \lambda_2 - \lambda_1)}, \end{aligned} \quad (25)$$

and

$$\begin{aligned} u_1 &= \frac{3a_1 a_2^2}{2\lambda_2(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2 - \lambda_3)} + \frac{3a_1 a_3^2}{2\lambda_3(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_2 + 2\lambda_3)} \\ &\quad + \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)} + \frac{a_3^2}{2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)}. \end{aligned} \quad (26)$$

Also substituting Eq. (25) into Eqs. (17) - (18) and according to authors' restrictions, they obtain three equations for B_1 , B_2 and B_3 whose solutions are respectively given by the following equations

$$\begin{aligned}
B_1 &= \frac{9}{(\lambda_1 + \lambda_2 + 2\lambda_3)(\lambda_1 + 2\lambda_2 + \lambda_3)} \left[\frac{1}{2\lambda_2(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2 - \lambda_3)} \right. \\
&+ \frac{1}{2\lambda_3(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_2 + 2\lambda_3)} \left. \right] a_1 a_2^2 a_3^2 + \frac{(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1'^2 a_1}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2} \\
&- \frac{\lambda_1' a_1^3}{(3\lambda_1 - \lambda_2)^2(3\lambda_1 - \lambda_3)^2} - \frac{6\lambda_1' a_1 a_2 a_3}{(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_3)^2} - \frac{\lambda_1'' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \\
B_2 &= \frac{3a_2^3 a_3^2}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)(3\lambda_2 + \lambda_3)(3\lambda_3 + 2\lambda_2 - \lambda_1)} \\
&+ \frac{18a_1^2 a_2^2 a_3}{4\lambda_2(\lambda_1 + \lambda_2)^2(\lambda_1 + 2\lambda_2 + \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_3)} + \frac{(2\lambda_2 - \lambda_1 - \lambda_3)\lambda_2'^2 a_2}{(\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2} \\
&- \frac{3\lambda_2' a_1^2 a_2}{(\lambda_1 + \lambda_2)^2(2\lambda_1 + \lambda_2 - \lambda_3)^2} - \frac{3\lambda_2' a_2^2 a_3}{4\lambda_2^2(2\lambda_2 + \lambda_3 - \lambda_1)^2} - \frac{\lambda_2'' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \\
B_3 &= \frac{a_2^2 a_3^3}{2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)(\lambda_2 + 3\lambda_3)(2\lambda_2 + 3\lambda_3 - \lambda_1)} \\
&+ \frac{3a_1^2 a_2 a_3^2}{2\lambda_3(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2 + 2\lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_2)} + \frac{(2\lambda_3 - \lambda_1 - \lambda_2)\lambda_3'^2 a_3}{(\lambda_3 - \lambda_1)^2(\lambda_3 - \lambda_2)^2} \\
&- \frac{3\lambda_3' a_1^2 a_3}{(\lambda_1 + \lambda_3)^2(2\lambda_1 + \lambda_3 - \lambda_2)^2} - \frac{3\lambda_3' a_2 a_3^2}{4\lambda_3^2(\lambda_2 + 2\lambda_3 - \lambda_1)^2} - \frac{\lambda_3'' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.
\end{aligned} \tag{27}$$

Authors are not interested to determine the correction term u_2 as it has no such effect on the solution. But it is too much complicated to solve, laborious and tedious work. So they can ignore it. Now substituting the values of A_1 , A_2 , B_1 and B_2 from Eq. (25) and Eq. (27) into Eq. (4), they obtain the following equations

$$\begin{aligned}
\dot{a}_1 &= \lambda_1 a_1 + \varepsilon \left(-\frac{(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{a_1^3}{(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)} + \frac{6a_1 a_2 a_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)} \right) \\
&+ \varepsilon^2 \left[\frac{9}{(\lambda_1 + \lambda_2 + 2\lambda_3)(\lambda_1 + 2\lambda_2 + \lambda_3)} \left[\frac{1}{2\lambda_2(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2 - \lambda_3)} \right. \right. \\
&+ \frac{1}{2\lambda_3(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_2 + 2\lambda_3)} \left. \right] a_1 a_2^2 a_3^2 + \frac{(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1'^2 a_1}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2} \\
&- \frac{\lambda_1' a_1^3}{(3\lambda_1 - \lambda_2)^2(3\lambda_1 - \lambda_3)^2} - \frac{6\lambda_1' a_1 a_2 a_3}{(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_3)^2} - \frac{\lambda_1'' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \left. \right]
\end{aligned} \tag{28}$$

$$\begin{aligned}
\dot{a}_2 = & \lambda_2 a_2 + \varepsilon \left(-\frac{(2\lambda_2 - \lambda_1 - \lambda_3)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{3a_1^2 a_2}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2 - \lambda_3)} + \frac{3a_2^2 a_3}{2\lambda_2(2\lambda_2 + \lambda_3 - \lambda_1)} \right) \\
& + \varepsilon^2 \left[\frac{3a_2^3 a_3^2}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)(3\lambda_2 + \lambda_3)(3\lambda_3 + 2\lambda_2 - \lambda_1)} \right. \\
& + \frac{18a_1^2 a_2^2 a_3}{4\lambda_2(\lambda_1 + \lambda_2)^2(\lambda_1 + 2\lambda_2 + \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_3)} + \frac{(2\lambda_2 - \lambda_1 - \lambda_3)\lambda_2'^2 a_2}{(\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2} \\
& \left. - \frac{3\lambda_2' a_1^2 a_2}{(\lambda_1 + \lambda_2)^2(2\lambda_1 + \lambda_2 - \lambda_3)^2} - \frac{3\lambda_2' a_2^2 a_3}{4\lambda_2^2(2\lambda_2 + \lambda_3 - \lambda_1)^2} - \frac{\lambda_2'' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \right] \quad (29)
\end{aligned}$$

For a damped nonlinear system, substituting $\lambda_1 = -\lambda(\tau)$, $\lambda_{1,2} = -\mu(\tau) \pm \omega(\tau)$ and $a_1 = a$, $a_2 = \frac{1}{2}be^{i\varphi}$, $a_3 = \frac{1}{2}be^{-i\varphi}$ into Eq. (26) and Eqs. (28) - (29) and then simplifying them, they obtain the following equations for the amplitudes, phase variables and the correction terms as the forms

$$\begin{aligned}
\dot{a} = & -\lambda(\tau)a + \varepsilon(l_0 a + l_1 a^3 + l_2 ab^2) + \varepsilon^2(l_3 a + l_4 a^3 + l_5 ab^2 + l_6 ab^4), \\
\dot{b} = & -\mu(\tau)b + \varepsilon(m_0 b + m_1 a^2 b + m_2 b^3) + \varepsilon^2(m_3 b + m_4 a^2 b + m_5 b^3 + m_6 a^2 b^3 + m_7 b^5), \quad (30) \\
\dot{\varphi} = & \omega(\tau) + \varepsilon(n_0 + n_1 a^2 + n_2 b^2) + \varepsilon^2(n_3 + n_4 a^2 + n_5 b^2 + n_6 a^2 b^2 + n_7 b^4),
\end{aligned}$$

and

$$u_1 = ab^2(c_2 \cos 2\varphi + d_2 \sin 2\varphi) + b^3(c_3 \cos 3\varphi + d_3 \sin 3\varphi), \quad (31)$$

where

$$\begin{aligned}
l_0 = & -\frac{2(\lambda - \mu)\lambda'}{((\lambda - \mu)^2 + \omega^2)^2}, \quad l_1 = \frac{1}{(3\lambda - \mu)^2 + \omega^2}, \quad l_2 = \frac{3}{2((\lambda + \mu)^2 + \omega^2)}, \\
l_3 = & -\frac{2(\lambda - \mu)\lambda'^2 - ((\lambda - \mu)^2 + \omega^2)\lambda''}{((\lambda - \mu)^2 + \omega^2)^2}, \quad l_4 = \frac{\lambda'}{((3\lambda - \mu)^2 + \omega^2)^2}, \quad l_5 = \frac{3\lambda'}{2((\lambda + \mu)^2 + \omega^2)^2}, \\
l_6 = & \frac{-9(\mu((\lambda + \mu)^2 - 3\omega^2) - 4\omega^2(\lambda + \mu))}{16(\mu^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)((\lambda + \mu)^2 + 9\omega^2)((\lambda + 3\mu)^2 + \omega^2)}, \\
m_0 = & \frac{(3\mu'\omega - \omega'(\lambda - \mu)(\lambda - \mu) - \omega(\mu'(\lambda - \mu) + 3\omega\omega'))}{2\omega((\lambda - \mu)^2 + \omega^2)}, \quad m_1 = \frac{3(\lambda(\lambda + \mu) - \omega^2)}{2(\lambda^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)} \\
m_2 = & \frac{-3(\mu(\lambda - 3\mu) + \omega^2)}{8(\mu^2 + \omega^2)((\lambda - 3\mu)^2 + \omega^2)}, \\
m_3 = & \frac{-((\lambda - \mu)((\lambda - \mu)^2 + 5\omega^2)(\mu'^2 - \omega'^2) + 2\mu'\omega\omega'((\lambda - \mu)^2 - 3\omega^2) + 2\omega(\mu''\omega + \omega''(\lambda - \mu))((\lambda + \mu)^2 + \omega^2))}{4\omega^2((\lambda - \mu)^2 + \omega^2)^2},
\end{aligned}$$

$$\begin{aligned}
m_4 &= \frac{3(\mu'((\lambda^2 - \omega^2)((\lambda + \mu)^2 - \omega^2) - 4\lambda\omega^2(\lambda + \mu)) + 2\omega\omega'((\lambda + \mu)(\lambda^2 - \omega^2) + \lambda((\lambda + \mu)^2 - \omega^2)))}{4(\lambda^2 + \omega^2)^2((\lambda + \mu)^2 + \omega^2)^2}, \\
m_5 &= \frac{3(\mu'((\mu^2 - \omega^2)((\lambda - 3\mu)^2 - \omega^2) + 4\mu\omega^2(\lambda - 3\mu)) + 2\omega\omega'(\mu((\lambda - 3\mu)^2 - \omega^2) - (\mu^2 - \omega^2)(\lambda - 3\mu)))}{16(\mu^2 + \omega^2)^2((\lambda - 3\mu)^2 + \omega^2)^2}, \\
m_6 &= \frac{-9(\mu((\lambda + \mu)^2 - \omega^2) - 2\omega^2(\lambda + \mu))((\lambda + \mu)(\lambda + 3\mu) - 3\omega^2) - \omega^2(4\lambda + 10\mu)((\lambda + \mu)(\lambda + 3\mu) - \omega^2)}{8(\mu^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)((\lambda + 3\mu)^2 + \omega^2)((\lambda + \mu)^2 + 9\omega^2)}, \\
m_7 &= \frac{-3((2(\mu(\lambda - 3\mu) + 3\omega^2)(\mu^2 - \omega^2) - 5\mu\omega^2(\lambda - 6\mu))(\lambda - 5\mu) + \omega^2(2(\lambda - 6\mu)(\mu^2 - \omega^2) + 5\mu(\mu(\lambda - 3\mu) + 3\omega^2)))}{128(\mu^2 + \omega^2)(\mu^2 + 4\omega^2)(4\mu^2 + \omega^2)((\lambda - 3\mu)^2 + 9\omega^2)((\lambda - 5\mu)^2 + \omega^2)}, \\
n_0 &= -\frac{(\mu'(\lambda - \mu) + 3\omega\omega')(\lambda - \mu) + \omega(3\mu'\omega - \omega'(\lambda - \mu))}{2\omega((\lambda - \mu)^2 + \omega^2)}, n_1 = \frac{3\omega(2\lambda + \mu)}{2(\lambda^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)}, \\
n_2 &= \frac{-3\omega(\lambda - 4\mu)}{8(\mu^2 + \omega^2)((\lambda - 3\mu)^2 + \omega^2)}, \\
n_3 &= \frac{-(\omega((\lambda - \mu)^2 - 3\omega^2)(\mu'^2 - \omega'^2) - 2\mu'\omega'(\lambda - \mu)((\lambda - \mu)^2 + 5\omega^2) + 2\omega(\mu''(\lambda - \mu) - \omega\omega''))((\lambda - \mu)^2 + \omega^2)}{4\omega^2((\lambda - \mu)^2 + \omega^2)^2}, \\
n_4 &= \frac{3(2\mu'\omega((\lambda + \mu)(\lambda^2 - \omega^2) + \lambda((\lambda + \mu)^2 - \omega^2)) - \omega'(((\lambda + \mu)^2 - \omega^2)(\lambda^2 - \omega^2) - 4\lambda\omega^2(\lambda + \mu)))}{4(\lambda^2 + \omega^2)^2((\lambda + \mu)^2 + \omega^2)^2}, \\
n_5 &= \frac{3(2\mu'\omega(\mu((\lambda - 3\mu)^2 - \omega^2) - (\mu^2 - \omega^2)(\lambda - 3\mu)) - \omega'(\mu^2 - \omega^2)((\lambda - 3\mu)^2 - \omega^2) + 4\mu\omega^2(\lambda - 3\mu))}{16(\mu^2 + \omega^2)^2((\lambda - 3\mu)^2 + \omega^2)^2}, \\
n_6 &= \frac{-9\omega(((\lambda + \mu)(\lambda + 3\mu) - \omega^2)^2 + (4\lambda + 10\mu)(\mu((\lambda + \mu)^2 - \omega^2) - 2\omega^2(\lambda + \mu)))}{8(\mu^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)((\lambda + 3\mu)^2 + \omega^2)((\lambda + \mu)^2 + 9\omega^2)}, \\
n_7 &= \frac{-3\omega((2(\lambda - 6\mu)(\mu^2 - \omega^2) + 5\mu(\mu(\lambda - 3\mu) + 3\omega^2))(\lambda - 5\mu) - 2(\mu^2 - \omega^2)(\mu(\lambda - 3\mu) + 3\omega^2) + 5\mu\omega^2(\lambda - 6\mu))}{128(\mu^2 + \omega^2)(\mu^2 + 4\omega^2)(4\mu^2 + \omega^2)((\lambda - 3\mu)^2 + 9\omega^2)((\lambda - 5\mu)^2 + \omega^2)}, \tag{32}
\end{aligned}$$

and

$$\begin{aligned}
 c_2 &= \frac{3(-\mu(\lambda + \mu)^2 + \omega^2(4\lambda + 7\mu))}{4(\mu^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)((\lambda + \mu)^2 + 9\omega^2)}, \\
 d_2 &= \frac{3\omega((\lambda + \mu)(\lambda + 5\mu) - 3\omega^2)}{4(\mu^2 + \omega^2)((\lambda + \mu)^2 + \omega^2)((\lambda + \mu)^2 + 9\omega^2)}, \\
 c_3 &= \frac{\mu^2(\lambda - 3\mu) + \omega^2(-2\lambda + 15\mu)}{16(\mu^2 + \omega^2)(\mu^2 + 4\omega^2)((\lambda - 3\mu)^2 + 9\omega^2)}, \\
 d_3 &= \frac{-3\omega(\mu(\lambda - 4\mu) + 2\omega^2)}{16(\mu^2 + \omega^2)(\mu^2 + 4\omega^2)((\lambda - 3\mu)^2 + 9\omega^2)}.
 \end{aligned} \tag{33}$$

Thus the second order approximate solution of Eq. (20) is obtained by

$$x(t, \varepsilon) = a + b \cos \varphi + \varepsilon u_1 \tag{34}$$

where a , b and φ are the solutions of Eq. (30) and u_1 is given by Eq. (31).

RESULTS AND DISCUSSION

The authors have solved two simultaneous differential equations for the amplitude(s) and phase variables, and a partial differential equation for u_1 involving three independent variables, amplitude(s) and phase. Also they are able to solve all the equations of A_j and B_j , $j = 1, 2, 3$ including u_1 by a unified formula. In particular case, they are forced to assume that $\lambda(\tau)$, $\mu(\tau)$ are constants and $\omega(\tau) = \omega_0 e^{-h\tau}$ is varying slowly with time t , where ω_0 is constant. The amplitudes and phase variables change slowly with time t . The behavior of amplitudes and phase variables characterizes the oscillating processes and they keep an important role to the non-linear dynamical systems. The amplitudes tend to zero as $t \rightarrow \infty$ (i.e. when time is very large) in presence of damping. Figures are drawn to compare between the approximate solutions obtained by the perturbation method and those obtained by the numerical procedure for several damping.

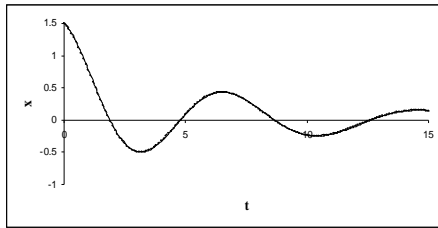


Fig. 1. (a) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (—solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions [$x(0) = 1.50838$, $\dot{x}(0) = -0.38079$, $\ddot{x}(0) = -0.97857$] or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

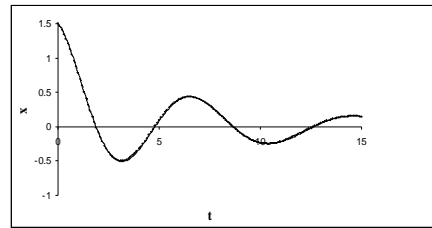


Fig. 1. (b) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (—solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions [$x(0) = 1.50838$, $\dot{x}(0) = -0.37974$, $\ddot{x}(0) = -0.98498$] or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

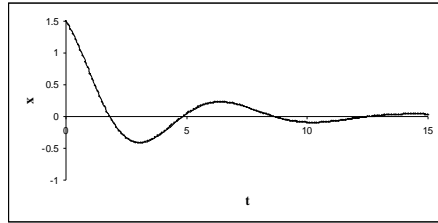


Fig. 2. (a) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (-solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50848$, $\dot{x}(0) = -0.58074$, $\ddot{x}(0) = -0.84169]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

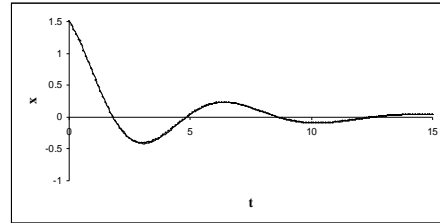


Fig. 2. (b) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (-solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838$, $\dot{x}(0) = -0.57933$, $\ddot{x}(0) = -0.84885]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

In Figs. (1)- (2), we observe that the analytical approximate solutions show good agreement with those obtained by the numerical procedure in presence of strong damping with slowly varying coefficients and the analytical approximate solutions deviate from the numerical solution when the damping effect is small (Fig. 3).

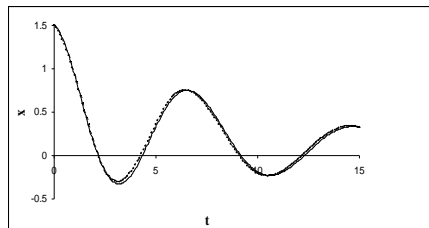


Fig. 3. (a) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (-solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50578$, $\dot{x}(0) = -0.17768$, $\ddot{x}(0) = -1.05083]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

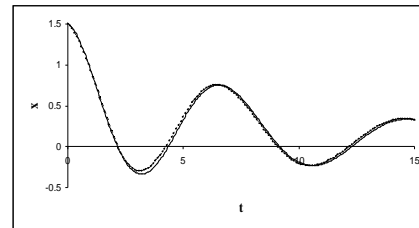


Fig. 3. (b) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (-solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50578$, $\dot{x}(0) = -0.17715$, $\ddot{x}(0) = -1.05717]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

Moreover, this method is able to give the required result when the coefficients of the given nonlinear system become constants ($h = 0$, Fig. 4). The limitation of the presented method is that it is valid only for weakly nonlinear system with strong damping and converges rapidly to the numerical solution otherwise it deviates from the numerical solution. Most of the authors did not discuss this limitation of the unified KBM method.

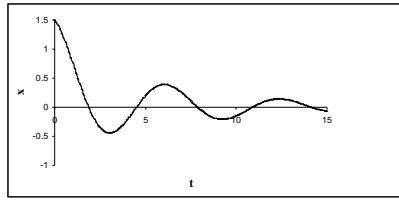


Fig. 4. (a) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (-solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838$, $\dot{x}(0) = -0.41556$, $\ddot{x}(0) = -0.95255]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

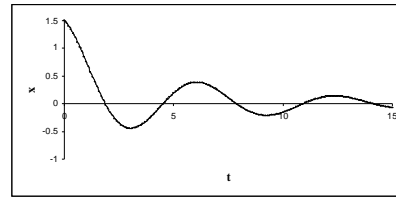


Fig. 4. (b) First approximate solution (—•— dotted lines) of Eq. (20) is compared with the corresponding numerical solution (-solid line) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25$, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838$, $\dot{x}(0) = -0.41611$, $\ddot{x}(0) = -0.94904]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

According to the theory of nonlinear oscillations, higher order approximate solutions give the better results. In practice, however, a few terms are sufficient for good agreement to the numerical solution. In the present study, it is seen that the first order approximate solutions lead to high accuracy in this case. As a result the graphs for the first and second order approximate solutions are almost same.

CONCLUSION

Usually, it is so much difficult to formulate the unified KBM method for obtaining the higher order approximate solutions of a third order nonlinear differential systems. The authors have presented a general formula for the second order approximate solutions by the unified KBM method for obtaining the transient's response of a third order nonlinear differential systems with slowly varying coefficients in presence of strong damping.

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