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APPLICATION OF THE SEIFERT-VAN KAMPEN THEOREM TO CERTAIN SUMS OF TOPOLOGICAL SPACES

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ABSTRACT

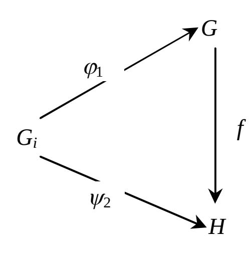
Seifert-Van Kampen theorem for the sum $X + Y$ and external sum $X \oplus Y$ of topological spaces is studied and the fundamental groups of these sums have been determined.

Key words: Sum, External sum, Fundamental group, Commutative diagram

INTRODUCTION

A number of sums of topological spaces have been studied by Hossain (2007), Hossain and Majumder (2010) and Majumder and Asaduzzaman (2001). The authors established some characterization theorems and a number of results for their defined sums in their respective articles. Two particular sums $X+Y$ and $X \oplus Y$ of topological spaces was studied by Hossain (2007), Hossain and Majumdar (2010). In order to determine the fundamental group of these two kinds of sums the important and well-known theorem of Seifert and Van Kampen is applied here. For convenience of the readers the definitions, terminologies and notions of Hossain (2007), Hossain and Majumder (2010) and Majumder and Asaduzzaman (2001) have been used frequently in here. For the clear concept of fundamental group and the Siefert-Van Kampen theorem, the author referred to Massey (1967) and recalled some essential definitions. The free product of groups and free products with an amalgamated subgroups are defined as follows:

Let $\{G_i : i \in I\}$ be a collection of groups, and assume there is given for each index *i*, a homomorphism φ_i of G_i into a fixed group G. The author says that G is the free product of the groups G_i with respect to the homomorphisms φ_i if and only if the following condition holds: for any group H and any homomorphisms $\varphi_i : G_i \to H, i \in I,$ the following diagram is commutative:



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Let A_α be groups, where α ranges over a set of indices, and let a proper subgroup B_α be chosen in every A_α such that all these subgroups are isomorphic to a fixed group B. By φ_α the author denotes a specific isomorphic mapping of B_α onto B; then $\psi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1}$ is an isomorphic mapping of B_α onto B_β.

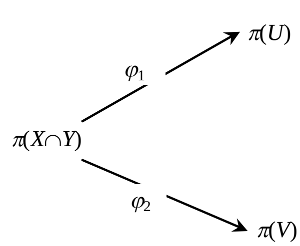
The free product of the groups A_α with the amalgamated subgroup B is defined as the factor group G of the free product of the groups A_α with respect to the normal subgroup generated by all elements of the form $b_\alpha b_\beta^{-1}$, where $b_\beta = b_\alpha \psi_{\alpha\beta}$. Here b_α ranges over the whole subgroup B_α, and α and β are all possible index pairs. In other words, if every group A_α is given by a system of generators M_α and a system of defining relations Φ_α then G has as a system of generators the union of all sets M_α, as a system of defining relations the union of the sets Φ_α, and in addition, all relations obtained by identifying those elements of different subgroups B_α and B_β which are mapped by the isomorphisms φ_α and φ_β onto one and same element of B. The subgroups B_α are amalgamated, as it were, in accordance with the isomorphisms $\varphi_{\alpha\beta}$.

SUM $X + Y$: Given two topological spaces (X, T_1) and (X, T_2) such that $X \cap Y$ is open in both X and Y, $X \cup Y$ is a topological space with topology $T = \{U \cup V \mid U \in T_1, V \in T_2\}$. $X \cup Y$ is called the sum of X and Y and is denoted by $X + Y$. In this situation X and Y are said to be compatible with each other. This definition is due to Hossain and Majumdar (2010). Almost similar definition occurs in Bourbaki (1966) and Dugundji (1989).

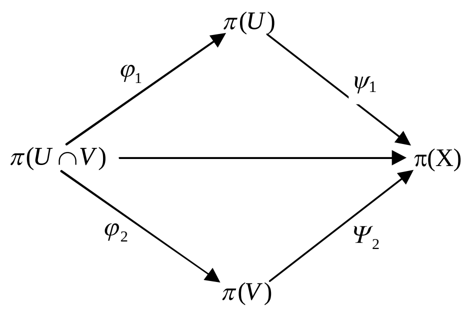
If $X \cap Y = \phi$, $X + Y$ is called a direct sum of X and Y, for a detailed study of sum and direct sum, is referred to (Hossain and Majumder (2010). Clearly, the topologies T_1, T_2 on X and Y are the same as the topologies as subspaces of $X + Y$.

FUNDAMENTAL GROUP OF $X + Y$

The author shall now apply the Seifert Van Kampen theorem to obtain the fundamental group of the sum of two compatible spaces in terms of those of the summands. This theorem is used to get the fundamental group of arcwise connected space. The theorem of Seifert and Van Kampen asserts that, if U and V are both open sets of arcwise-connected space X so that $X = U \cup V$, and $U \cap V$ is nonempty connected, then the fundamental group $\pi(X)$ of X is completely determined by the following diagram of groups and homomorphisms:

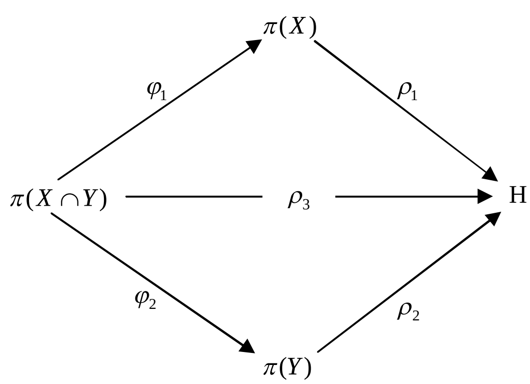


Here φ_1 and φ_2 are induced by inclusion maps and the above diagram can be completed by forming the following commutative diagram:

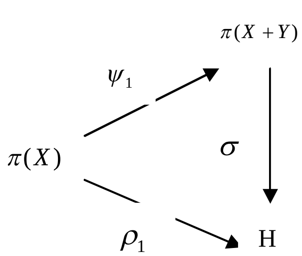


So the Seifert-Van Kampen theorem for the sum $X + Y$ of two compatible spaces X and Y will take the following form.

Theorem 1: Let X and Y be path connected compatible spaces such that $X \cap Y \neq \phi$ and let $X \cap Y$ be path connected also. If H be a group and ρ_1, ρ_2, ρ_3 are any homomorphisms such that the diagram



is commutative where the homomorphisms φ_1 and φ_2 are induced by the inclusion maps. Then there exists a unique homomorphism $\sigma : \pi(X+Y) \to H$ such that the following three diagrams are commutative:



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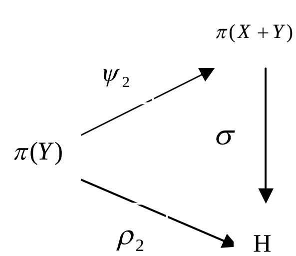

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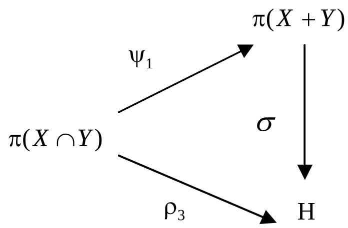

where ψ_1, ψ_2, ψ_3 are also homomorphisms induced by inclusion maps.

Proof. From the construction of the sum it is seen that *X* and *Y* are open subsets of *X+Y* and they have the nonempty intersection with each other. Since *X* and *Y* are path connected compatible spaces such that $X \cap Y$ is also path connected, thus the result follows immediately from the application of famous theorem of Seifert and Van Kampen in general case.

For the above results, the following corollary is obtained :

Corollary. If $\pi(X \cap Y) = 1$ i.e., if $X \cap Y$ is simply connected, then π (*X+Y*) is the free product of π (X) and π (Y) under the homomorphisms ψ_1, ψ_2 i.e., π (X+Y) $= \pi$ (X)* π (Y), where * denotes the free product.

External Sum $X \oplus Y$ **:** Let X_1 and X_2 be two disjoint topological spaces and let there be two non-empty closed sets F_1 and F_2 in X_1 and X_2, respectively such that F_1 and F_2 are homeomorphic. Let $f : F_1 \to F_2$ be a homeomorphism. The author now defines a relation *R* on the direct sum of X_1 and X_2 (direct sums studied in (Hossain 2007) and also occur in Dugundji (1989) as follows:

(i) For each $x_1 \in X_1 - F_1$, $x_1 R z$ and $z R x_1$ if and only if $z = x_1$,

(ii) for each $x_2 \in X_2 - F_2$, $x_2 R z$ and $z R x_2$ if and only if $z = x_2$,

(iii) for each $x_1 \in F_1$, $x_1 R z$ and $z R x_1$ if and only if $z = x_1$ or $z = f^{-1}(x_1)$,

(iv) for each $x_2 \in F_2$, $x_2 R z$ and $z R x_2$ if and only if $z = x_2$ or $z = f^{-1}(x_2)$.

Then *R* is an equivalence relation. The quotient space $\frac{X_1 \otimes X_2}{R}$ will be called an external sum and will be denoted by $X_1 \oplus_F X_2$ or simply by $X_1 \oplus X_2$ where $F=F_1=F_2$ (after identification). A study of such sum has been made by Majumder and Asaduzzaman (2001).

If *X* and *Y* are subspaces of a topological space *Z*, then one may choose the subspace topology on $X \cup Y$ and obtain a space which is called the usual sum, written $X \oplus_z Y$. If *X* and *Y* are disjoint, then the external sum may be defined $X \oplus Y$ as before and call it the usual external sum.

Investigation in section fundamental group of $X + Y$ suggests that the fundamental group of the external sum $X \oplus Y$ can be obtained also by similar procedure for the sum in that section by the application of Seifert-Van Kampen theorem.

The fundamental group $\pi(X + Y)$ of the particular sum $X + Y$ is obtained through the Theorem 1. From the expression of the theorem shown by the diagrams, it can be stated that the group $\pi(X + Y)$ is generated by the union of the images $\psi_\lambda[\pi(X_\lambda)]$ where $\lambda = 1, 2$ with $X_1 = X$ and $X_2 = Y$.

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