STEADY FLOW INTO A CIRCULAR WELL AT THE CENTER OF A CONFINED ELLIPTICAL DRAINAGE SYSTEM

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Abstract

Thermal recovery in many heavy oil reservoirs ***requires*** *high steam injectivity. In very viscous oil-bearing formation, it can be achieved only by parting the formation. The parting (fracture) creates an elliptical drainage system. During the production phase, the inner boundary of the flow problem is bracketed by two extreme conditions: an open fracture or circular wellbore. Existing petroleum literature presents solutions with vertical fracture at the wellbore. A solution, not previously found in petroleum literature, for steady saturated flow to a fully penetrating circular well in the elliptical drainage system, is presented.*

Introduction

Many of the fractures created during steam injection processes or hydraulic fracturing would close in the production phase. These flow situations along with directional permeability inherent in anisotropic or naturally fractured reservoirs are actually more appropriately described by elliptical flow towards a circular well at the center. A homogeneous and isotropic reservoir with an elliptical boundary will also have a similar flow domain.

Although analytical flow equations for an elliptical flow domain with vertical fracture at the wellbore are abundant in the petroleum literature[1,2,3,4,5,6,7,8] no reference is found with circular wellbore at the center.

van der Ploeg et al.[9] developed a closed-formed solution for steady saturated flow into a fully penetrating well in elliptical flow geometry. van der Ploeg et al.'s work was related to water flow in a confined elliptical aquifer. Steady state solutions were developed for various well locations using gravity flow. Results and flow nets were presented for several cases. The essence of the approach was to derive orthonormal functions for the specific problems using methods of Powers et al.[10]. Although van der Ploeg et al. presented solution for different well locations, only the solution for a well at the center is considered here.

Well at the Center of the Ellipse

van der Ploeg et al.[9] developed a closed-form solution for steady saturated flow into a fully penetrating well in an elliptical confined aquifer assuming only gravity drainage. They considered the aquifer to be isotropic and homogeneous. The geometry of the problem is shown in Fig 1. The free surface of the fluid height at the boundary is assumed to be constant. For a steady drawdown, the problem is to find the expression for the hydraulic head ϕ.

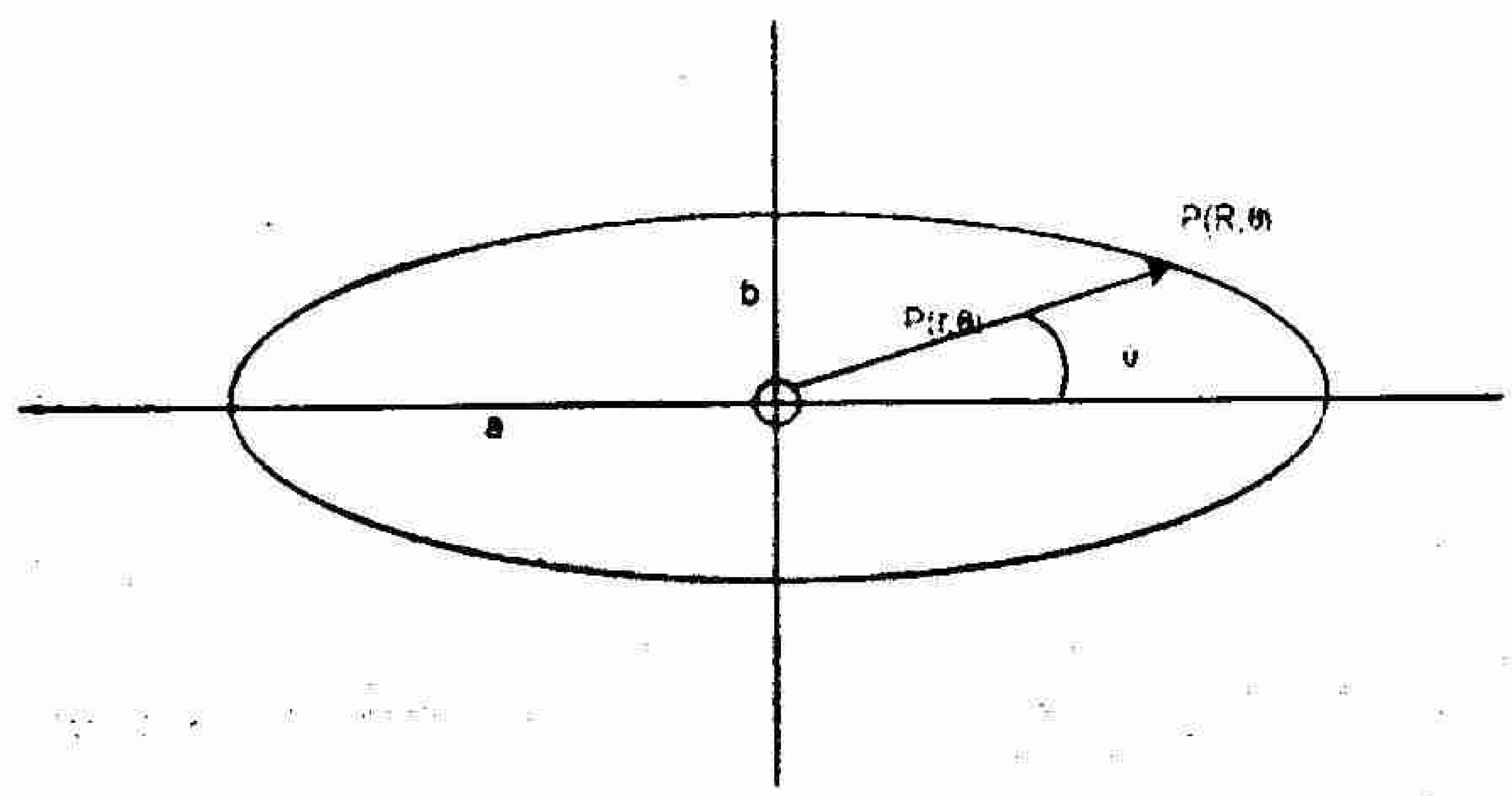

Fig 1: Elliptic Flow Geometry in Polar Coordinate

The equation of ellipse in a rectangular co-ordinate is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

Using polar co-ordinates (R,θ), x and y can be written as,

$x = R\cos\theta$ and $y = R\sin\theta$.

Now, for a point $P(R,\theta)$ on the boundary, Eqn (1) can be written in the following form –

$$R^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1 \qquad (2)$$

Boundary conditions:

a. $\phi = 0$ for $r = r_w$ $\quad 0 \le \theta \le \pi/2$

b. $\phi = 1$ for $r = R$ $\quad 0 \le \theta \le \pi/2$

c. $\partial\phi/\partial\theta = 0$ for $\theta = 0$ $\quad r_w < r < a$

d. $\partial\phi/\partial\theta = 0$ for $\theta = \pi/2$ $\quad r_w < r < a$

Laplace's equation in polar co-ordinates is,

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \qquad (3)$$

The solution of this problem should give an expression of ϕ which should satisfy the specified boundary conditions and the Laplace's equation.

Solution

The authors used Gram-Schmidt method as modified by Powers et al.[10] to determine the solution as –

$$\phi=\sum_{m=0}^{N}A_{Nm}u_m(r,\theta) \qquad (4)$$

where, $m = 0, 1, 2, \ldots\ldots, N$;
$N = 0, 1, 2, \ldots\ldots, \infty$,

and $$u_m(r,\theta)=\frac{\left(\frac{r}{a}\right)^{2m}-\left(\frac{r_w^2}{ar}\right)^{2m}}{1-\left(\frac{r_w^2}{a^2}\right)^{2m}}\cos(2m\theta) \qquad (5)$$

On the boundary of the ellipse, $r = R$ and using Eqn (2) to express R in terms of θ, Eqn (5) becomes,

$$u_m(\theta)=\frac{\left(\frac{R^2}{a^2}\right)^{m}-\left(\frac{r_w^2}{a^2}\frac{r_w^2}{R^2}\right)^{m}}{1-\left(\frac{r_w^2}{a^2}\right)^{2m}}\cos(2m\theta) \qquad (6)$$

Now the equation of hydraulic head may be written as,

$$\phi_{r=R}=\sum_{m=0}^{N}A_{Nm}u_m(\theta) \qquad 0\le\theta\le\pi/2 \qquad (7)$$

To satisfy the boundary condition (b), Eqn (7) becomes,

$$1=\sum_{m=0}^{N}A_{Nm}u_m(\theta) \qquad 0\le\theta\le\pi/2 \qquad (8)$$

Powers et al.[10] derived a table of orthonormal functions to solve potential flow problems like seepage of steady state rain through soil bedding. In accordance with their method, the two constants to determine A_{Nm} in this problem are,

$$w_m=\int_0^{\pi/2}(1)u_m(\theta)d\theta, \qquad m=0,1,\ldots,N \qquad (9)$$

$$u_{mn}=\int_0^{\pi/2}u_m(\theta)u_n(\theta)d\theta,\ m=0,1,\ldots,N \quad n\le m \qquad (10)$$

where, $u_m(\theta)$ and $u_n(\theta)$ are obtained from Eqn (6).

When $N\to\infty$, boundary condition (b) is satisfied exactly. It is difficult to determine the above integration analytically. Any numerical integration method may be employed to evaluate these functions. It may be seen that the terms with zero subscripts produce indeterminate forms. Using L'Hospital rule, u_o would be,

$$u_o(\theta)=\frac{\ln(r/r_w)}{\ln(a/r_w)} \qquad (11)$$

Therefore, the hydraulic head may be expressed as,

$$\phi=A_{N0}\frac{\ln(r/r_w)}{\ln(a/r_w)}+\sum_{m=1}^{N}A_{Nm}\frac{(r/r_w)^{2m}-(r_w^2/ar)^{2m}}{1-(r_w^2/a)^{2m}} \qquad (12)$$

After the values of w_m and u_{mn} are found, all A_{Nm} values can be calculated using Table 1 and 2 of Powers et al.[10]

Table 1. A_{Nm} **(N = 4) coefficient for the flow configuration of a =1, b = 0.5 and r_w = 0.01**

N	A_{N0}	A_{N1}	A_{N2}	A_{N3}	A_{N4}
0	1.093003				
1	1.096468	-0.079403			
2	1.096561	-0.079400	-0.013218		
3	1.096565	-0.079401	-0.013214	-0.002941	
4	1.096566	-0.079401	-0.013215	-0.002939	-0.000742

Table 2. Coefficient A_{NO}, dimensionless flow Q_D and % error in flow with different eccentricity of ellipse for the radial configuration of r_w=1.0 and r_e = 500

a	b	e	A_{N0}	Q_D	% error
500	500.0	0.000	1.000	1.011	0.000
510	490.2	0.276	1.003	1.011	0.003
525	476.2	0.421	1.008	1.011	0.019
575	434.8	0.654	1.024	1.012	0.157
675	370.4	0.836	1.056	1.018	0.720
725	344.8	0.880	1.071	1.022	1.126
800	312.5	0.921	1.100	1.028	1.748
900	277.8	0.951	1.124	1.038	2.709
1000	250.0	0.968	1.153	1.048	3.734
1500	166.7	0.994	1.283	1.102	9.029

Sequential Formulas Presented by Powers et al.,[10]

$$D_m = (u_m u_m) - \sum_{n=0}^{m-1} c_{mn}^2 D_n \tag{13}$$

$$c_{mn} = \frac{(u_m u_n) - \sum_{r=0}^{n-1} J_{nr}(u_m u_r)}{D_n} \tag{14}$$

$$J_{m0} = c_{m0} - \sum_{n=1}^{m-1} c_{mn} J_{n0} \tag{15}$$

$$J_{mn} = c_{mn} - \sum_{r=n+1}^{m-1} c_{mr} J_{rn} \tag{16}$$

$$G_m = w_m - \sum_{n=0}^{m-1} c_{mn} G_n \tag{17}$$

$$E_m = G_m / D_m \tag{18}$$

$$A_{Nm} = E_m - \sum_{p=m+1}^{N} E_p J_{pm} \tag{19}$$

Well Flow Rate

The well discharge q can be calculated by using the expression for the potential function ϕ in Darcy's equation. For a reservoir with thickness, h, discharge, q, can be written as,

$$q = -Kh\Delta\phi \int_0^{2\pi} \left(\frac{\partial \phi}{\partial r}\right)_{r=r_w^-} r d\theta \qquad r_w \to 0 \tag{20}$$

where, K is the hydraulic conductivity. Integrating the above equation, total flow rate at the wellbore can be found as,

$$q = -\frac{2\pi Kh A_{NO} \Delta\phi}{\ln(a/r_w)} \tag{21}$$

Ignoring the minus sign and rearranging the above equation, a dimensionless flow equation may be found to be

$$Q_D = \frac{q}{Kh\Delta\phi} = \frac{2\pi A_{N0}}{\ln\left(\frac{a}{r_w}\right)} \tag{22}$$

The hydraulic conductivity K used in groundwater flow is related to permeability, density and viscosity used in petroleum engineering as $K = k\rho g/\mu$. As density is assumed to be constant, the piegeometric head is $\phi = p/\rho g + z$. Substituting this value in Eqn (21), the flow equation can be expressed as

$$q = -\frac{2\pi kh A_{N0} \Delta(p + \rho g z)}{\mu \ln(a/r_w)} \tag{23}$$

Results

If one were concerned about flow at the wellbore only, a few values of A_{Nm} are needed for quick convergence. These values are used to determine A_{NO}, which is the only coefficient required in the flow equation. Table 1 shows how the A_{Nm} varies as N increases. The early coefficients quickly converge to a constant and the later ones approach zero.

Using any arbitrary ratio between a, b and r_w, it can be easily shown that the constants $U_m(\theta)$, $U_n(\theta)$ and $U_O(\theta)$ are independent of wellbore radius or the size of the ellipse. In other words, A_{Nm} is a geometric factor only.

Creation of elliptical flow domain in cyclic steam stimulation (CSS) operation in heavy oil reservoirs is well recognized[11,12,13]. Depending on the number of cycles or how the permeability is treated, the eccentricity of the elliptical domain may vary from 0.4 to 0.99[11]. Settari[14] and Settari et. al.[15] developed the concept of shear failure where it was shown that high-pressure steam injection in heavy oil reservoirs would create an elliptical zone of high permeability and porosity[11] observed that during production, the primary as well as the shear fractures would close. Despite the closure of these fractures, the residual porosity and permeability would not return to their original values and will remain significantly higher even after a long production period. So, if it is assumed that the fractures created during injection period closes in production phase, use of any solution that assumes a vertical fracture at the wellbore would produce erroneous results. If the eccentricity of the elliptical drainage system were not very high, radial flow equation assuming the same drainage area would result in less than one percent error. On the other hand, eccentricity above 0.9 may produce as high as nine percent error in flow calculation. Table 2 shows the extent of errors in dimensionless flow calculation assuming radial flow system for elliptical systems of equal drainage area. Fig 2 is showing the graphical representation of these results.

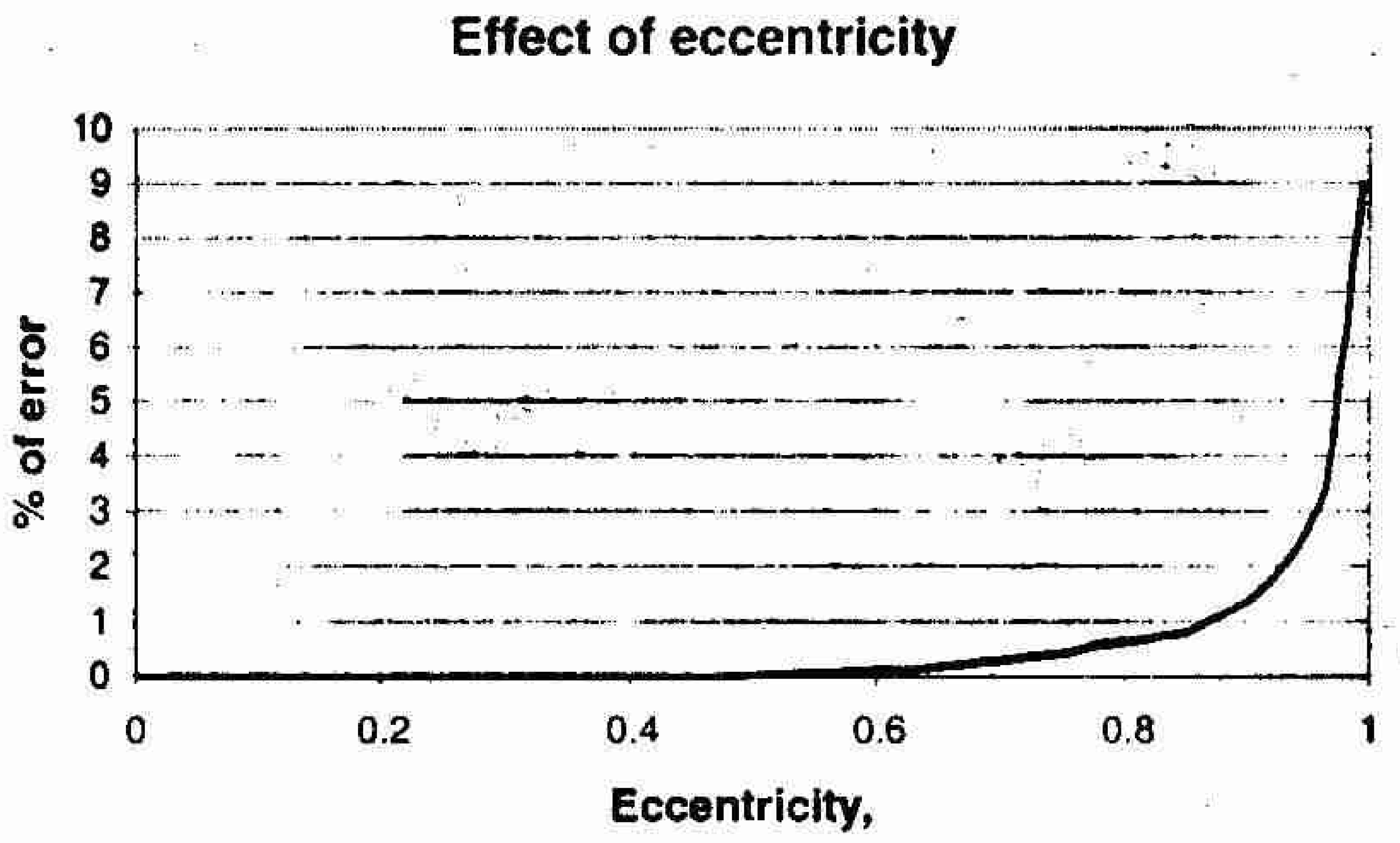

Fig 2: Effect of Eccentricity

Tamim and Rahman[16] in their analytical predictive model of CSS, combined the concepts of shear failure zone and pseudo-relative permeability with the flow equation of a circular wellbore in the middle of an elliptical flow domain.

Conclusion

Elliptical flow problems arise in many reservoir engineering applications, notably in CSS. The significance of flow towards a circular wellbore at the center of an elliptical drainage system is discussed and a slightly modified analytical solution of this problem has been presented, as reported in ground water literature.

Nomenclature

a	major axis of the ellipse (m)
A_{Nm}	coefficient in elliptical equation
A_{N0}	coefficient in elliptical equation
b	minor axis of the ellipse (m)
e	eccentricity of ellipse (fraction)
g	acceleration due to gravity (ms^{-2})
h	formation thickness (m),
k	absolute permeability (m^2)
K	hydraulic conductivity ($m\ s^{-1}$)
p	pressure (kPa)
q	discharge, fluid flow rate ($m^3\ s^{-1}$)
Q_D	dimensionless flow
r	radius (m)
R	boundary radius on the ellipse in polar coordinates
z	gravity head (m)

Greek Symbols

ϕ	hydraulic head (m)
μ	dynamic viscosity (Pa.s)
θ	angle
ρ	density (kg m^{-3})

Subscripts

e	external
w	wellbore, inner

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