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FREE VIBRATIONAL ANALYSIS OF NON-UNIFORM DOUBLE EULER-BERNOULLI BEAMS ON A WINKLER FOUNDATION USING LAPLACE DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT

In this study, the free vibrational analysis of non-uniform Euler-Bernoulli double beams on a Winkler foundation under simply supported and fixed-fixed boundary conditions is examined. The governing equation is solved using the Laplace Differential Transform Method, the combined form of the Laplace transform and differential transform technique (DTM). The accuracy of the method used is demonstrated by comparing the natural frequencies obtained using LDTM with previously published results available in the literature. It is discovered that for non-uniform double Euler-Bernoulli beams on a Winkler foundation with fixed-fixed end conditions, the natural frequencies are higher than those of simply supported end conditions. It is also observed that as the non-uniformity of the cross section of the beam increases, the natural frequencies reduce. Hence, it is suggested that the non-uniformity of the cross-section of the beam for a simply supported end condition be between 0 and less than 0.8. While the fixed-fixed end condition should have a value between 0 and 0.95.

Keywords: *Boundary Conditions, Double Beam, Euler-Bernoulli Beam, Free Vibration, Natural Frequency, Winkler Foundation, LDTM.*

1. INTRODUCTION

The free vibrational analysis of non-uniform Euler-Bernoulli beams on Winkler foundation is a topic of significant interest in the field of structural engineering and aviation. Due to the non-uniformity of the beams, the natural frequencies and mode shapes of the structure are difficult to determine analytically. In recent years, researchers have studied the free vibrational analysis of a non-uniform Euler-Bernoulli double beam on the Winkler foundation using various numerical and analytical techniques. Shahba *et al.* (2011) investigated the free vibration behaviour and stability of axially functionally graded non-uniform Euler-Bernoulli Beams using the finite element method (FEM). Kacar *et al.* (2011) studied the free vibration of non-uniform Euler-Bernoulli beams on Winkler foundations using the differential transform method. They derived a set of governing equations for the beam and solved them numerically to obtain the natural frequencies and mode shapes. They also investigated the effect of various parameters such as the length of the beam, the stiffness of the foundation, and the taper ratio on the natural frequencies. Torabi *et al.* (2011) presented the differential transform method (DTM) to solve the free vibration problem of non-uniform Euler-Bernoulli beams. The results obtained from this method were compared with those from the literature, and the accuracy of the DQM method was verified.

Wattanasakulpong and Ungbhakorn (2013) examined the free vibration analysis of Euler-Bernoulli beams with general elastic end restraints through the use of the Differential Transformation Method (DTM). Mutman (2013) examined the free vibration analysis of an Euler beam of variable width on the Winkler Foundation using the Homotopy Perturbation Method. The results showed that the natural frequencies of the structure are affected by the geometric and material properties of the beams as well as the stiffness of the Winkler foundation. The mode shapes of the structure were also studied, and it was found that they are sensitive to changes in the non-uniformity of the beams. Garijo (2015) proposed Bernstein pseudospectral collocation to solve the free vibration analysis of non-uniform Euler-Bernoulli beams. The accuracy and efficiency of the Bernstein pseudospectral collocation method were demonstrated through numerical examples. Liu *et al.* (2016) proposed the Spline Finite Point Method (SFPM) to investigate the free transverse vibration of axially functionally graded tapered Euler-Bernoulli beams. The effect of various parameters such as the length of the beam, the stiffness of the foundation, and the taper ratio on the natural frequencies was understudied using numerical examples to illustrate the effectiveness of the proposed method.

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Kumar (2018) used the Rayleigh-Ritz method to investigate the linear dynamics, statics, and buckling behaviour of a beam. The effects of varying cross-sectional dimensions and mechanical properties of the functionally graded material are added to the evaluation of structural matrices. The flexural response of the Euler-Bernoulli Beam with different support conditions was analysed numerically; likewise, the free vibrational analysis of Euler-Bernoulli beams on Winkler foundations with prismatic cross-sections was studied by Ike (2018). The finite Fourier sine integral transformation method was used to solve the free vibration behaviour of a non-uniform Euler-Bernoulli double beam on a Winkler foundation. Celik (2018) presented a Chebyshev wavelet collocation method to solve the free vibration of a non-uniform Euler-Bernoulli beam under various supporting conditions. The accuracy and efficiency of the proposed method were demonstrated through numerical examples.

Soltani and Asgarian (2019) used the Rayleigh-Ritz method to study the free vibration and stability of axially functionally graded Euler-Bernoulli beams with different cross-sections that rest on a uniform Winkler-Pasternak foundation. The power series approximation is first adopted to solve the motion equation. The Rayleigh-Ritz method is finally employed to construct the eigenvalue problem for obtaining the critical loads. Several numerical examples of axially non-homogeneous and homogeneous beams are conducted to illustrate the correctness and convergence of the method. The effect of the non-uniformity of the beam and the stiffness of the foundation on the natural frequencies was investigated, and the results were compared with those obtained using the finite element method. Usman *et al.* (2020a, 2020b) used a series solution coupled with the finite difference method and the Fourier Sine transform method, respectively, to solve the vibration analysis of an Euler-Bernoulli beam on a variable Winkler support.

Olotu *et al.* (2021) studied the free vibration of non-uniform beams on a variable Winkler foundation using the differential transform method (DTM). The accuracy of the proposed method was validated through numerical simulations, and the effect of the foundation stiffness and the non-uniformity of the beam on the natural frequencies was investigated. Usman *et al.* (2021a, 2021b) solved the governing equation of the Euler-Bernoulli beam on Winkler support analytically. Ghannadiasl (2021) investigated the free vibration of tapered Euler-Bernoulli beams on Winkler foundations using the Quintic B-spline collocation method. Singh and Sharma (2021) examined the Free vibration analysis of an axially functionally graded tapered beam using the harmonic differential quadrature method.

It is observed from the literature that studies on double beam systems are few as compared to the single beam problem; this could be as a result of the difficulty in using a method to solve an interconnected double beam system under simply supported and fixed-fixed boundary conditions. Even the few that worked on double beams, considered only the uniform cross-sections, except for a very few authors (Celik, 2018; Soltani and Asgarian, 2019; Olotu *et al.*, 2021). However, there are real-life systems modelled as non-uniform double beams that are useful in examining the response of sandwich beams encountered in engineering problems. In this paper, therefore, the Laplace differential transform method is presented to solve the free vibration of double Euler-Bernoulli beams on Winkler foundations for symmetric boundary conditions involving simply supported and fixed-fixed boundary conditions. Numerical examples are used to illustrate the effectiveness of the method.

2. MATERIALS AND METHODS

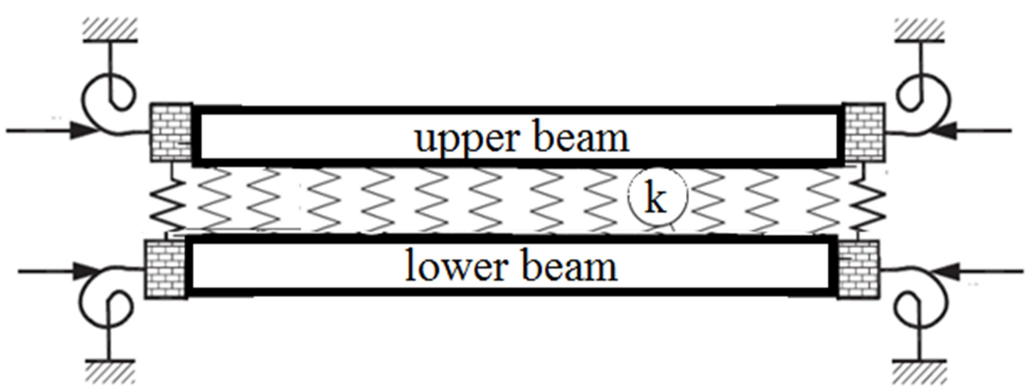

Figure 1: Schematic representation of a double non-uniform beam system on a Winkler foundation.

A non-uniform double beam system with finite length (L) as shown in Figure 1 is considered. The primary beam (upper beam) with arbitrary boundary conditions is elastically connected with the secondary beam (lower beam) through the elastic restraining springs on the coupling interface. The mechanical interaction is represented using three types of springs, namely translational spring and rotational spring and shear modulus that accounts for the shear force interaction among the springs.

The governing equation for the double Euler-Bernoulli beam are:

$$\begin{cases} \frac{\partial^2}{\partial x^2}\left(E_1 I_1(x)\frac{\partial^2 V_1(x,t)}{\partial x^2}\right) + \rho A_1(x)\frac{\partial^2 V_1(x,t)}{\partial t^2} + k(V_1(x,t) - V_2(x,t)) = 0 \\ \frac{\partial^2}{\partial x^2}\left(E_2 I_2(x)\frac{\partial^2 V_2(x,t)}{\partial x^2}\right) + \rho A_2(x)\frac{\partial^2 V_2(x,t)}{\partial t^2} + k(V_2(x,t) - V_1(x,t)) = 0 \end{cases} \quad (1)$$

where

$V_j(x,t)$ = transverse displacement of the beam of the $j^{th\ member}$
E_j = Modulus of Elasticity of the beam of the $j^{th\ member}$
$\rho A_j(x)$ = variable Mass per unit length of the beam of the $j^{th\ member}$
k = Winkler foundation
$I_j(x)$ = variable moment of inertia of the beam of the $j^{th\ member}$
x = spatial coordinate
t = time

The pertinent boundary conditions for the dynamic systems under consideration can be any of the following classical boundary conditions:

$$V_j(0,t) = 0 = \frac{\partial W_j(0,t)}{\partial x}, j = 1,2 \quad fixed\ at\ \ x = 0 \quad (2)$$

$$W_j(L,t) = 0 = \frac{\partial V_j(L,t)}{\partial x}, j = 1,2 \quad fixed\ at\ \ x = L \quad (3)$$

$$V_j(0,t) = 0 = \frac{\partial^2 V_j(0,t)}{\partial x^2}, j = 1,2 \quad simply\ supported\ at\ \ x = 0 \quad (4)$$

$$V_j(L,t) = 0 = \frac{\partial^2 V_j(L,t)}{\partial x^2}, j = 1,2 \quad simply\ supported\ at\ \ x = L \quad (5)$$

Finally, the initial conditions are:

$$V(x,0) = 0 \quad (6)$$

$$\frac{\partial V(x,0)}{\partial x} = 0 \quad (7)$$

Equation (1) is the system of equations for the vibration of double beam of finite length L

2.1 Method of Solution

The vibration of non-uniform double Euler-Bernoulli beam system on Winkler foundation is solved using Laplace Differential Transform Method (LDTM). Laplace Differential Transform Method is the combined method of Laplace Transform and Differential Transform Method.

2.1.1 Basic Idea of Laplace Differential Transform Method

To illustrate the basic idea of this method following the idea from Alqura *et al.* (2012), Kumari and Gupta (2016), Daniel and Ayodele (2020) and Daniel (2022), the general form of one-dimensional nonlinear second order homogeneous partial differential equations with variable coefficients is considered as follows:

$$\frac{d^2 V(x,t)}{dt^2} + p_n(x)RV(x,t) = 0 \qquad t > 0, x > 0, n \in \mathbb{N} \quad (8)$$

Where $\frac{d^2}{dt^2}$ is the linear differential operator of order 2, $p_n(x)$ is a the variable coefficient, $n \in \mathbb{N}$, and R is the linear operator. Equation (8) is subject to

$$V(x,0) = q_1(x), \quad V_t(x,0) = q_2(x) \quad (9)$$

$$V(0,t) = q_3(t), \quad V_x(0,t) = q_4(t) \tag{10}$$

The method involves applying a Laplace transform to equations (8) â€“ (10) and the use of the linearity property of Laplace transform

$$\mathcal{L}\left\{\frac{d^2V(x,t)}{dt^2}\right\} + \mathcal{L}\{a_n(x)RV(x,t)\} = 0$$

$$s^2\overline{V}(x,s) - s\overline{V}(x,0) - \overline{V}_t(x,0) + a_n(x)R\overline{V}(x,s) = 0 \tag{11}$$

Where $\overline{V}(x,s)$ is the Laplace transform of $V(x,t)$. Applying the initial conditions in equation (9) into equation (11) yields

$$s^2\overline{V}(x,s) - sq_1(x) - q_2(x) + p_n(x)R\overline{V}(x,s) = 0 \tag{12}$$

Subject to

$$\overline{V}(0,s) = \overline{q}_3(s), \quad \frac{d\overline{V}(0,s)}{x} = \overline{q}_4(s) \tag{13}$$

Which is second order initial value problem. Accordingly, using differential transform method, the solution of equations (12) and (13) can be written as:

$$\overline{V}(x,s) = \sum_{k=0}^{\infty} \overline{\overline{V}}(k)x^k \tag{14}$$

Where $\overline{\overline{V}}(k)$ is the differential transform of $\overline{V}(x,s)$ and $\overline{\overline{V}}(k)$ is a function of the parameters. After determining $\overline{V}(x,s)$, inverse Laplace transform is applied to equation (14) to get $V(x,t)$.

2.1.2 Method of Solution by Laplace Differential Transform Method

Simplification of equation (1) becomes

$$\begin{cases} E_1I_1(x)\frac{\partial^4V_1(x,t)}{\partial x^4} + E_1I_1'(x)\frac{\partial^3V_1(x,t)}{\partial x^3} + E_1I_1''(x)\frac{\partial^2V_1(x,t)}{\partial x^2} \\ +\rho A_1(x)\frac{\partial^2V_1(x,t)}{\partial t^2} + k(V_1(x,t) - V_2(x,t)) = 0 \\ E_2I_2(x)\frac{\partial^4V_2(x,t)}{\partial x^4} + E_2I_2'(x)\frac{\partial^3V_2(x,t)}{\partial x^3} + E_2I_2''(x)\frac{\partial^2V_2(x,t)}{\partial x^2} \\ +\rho A_2(x)\frac{\partial^2V_2(x,t)}{\partial t^2} + k(V_2(x,t) - V_1(x,t)) = 0 \end{cases} \tag{15}$$

Defining a non dimensional coordinate of:

$$\xi = \frac{x}{L}, \tau = \frac{t}{L^2}\sqrt{\frac{EI(0)}{\rho A(0)}}, I_1(\xi) = \frac{I_1(x)}{I_1(0)}, \rho A_1(\xi) = \frac{\rho A_1(x)}{\rho A_1(0)}, I_2(\xi) = \frac{I_2(x)}{I_2(0)}, \rho A_2(\xi) = \frac{\rho A_2(x)}{\rho A_2(0)}$$

the governing equation (1) becomes

$$\begin{cases} I_1(\xi)\frac{\partial^4\tilde{V}_1(\xi,\tau)}{\partial \xi^4} + I_1'(\xi)\frac{\partial^3\tilde{V}_1(\xi,\tau)}{\partial \xi^3} + I_1''(\xi)\frac{\partial^2\tilde{V}_1(\xi,\tau)}{\partial \xi^2} \\ +\frac{\rho A_1(\xi)L^4}{E_1I_1(0)}\frac{\partial^2\tilde{V}_1(\xi,\tau)}{\partial t^2} + k_1(\tilde{V}_1(\xi,\tau) - \tilde{V}_2(\xi,\tau)) = 0 \\ I_2(\xi)\frac{\partial^4\tilde{V}_2(\xi,\tau)}{\partial \xi^4} + I_2'(\xi)\frac{\partial^3\tilde{V}_2(\xi,\tau)}{\partial \xi^3} + I_2''(\xi)\frac{\partial^2\tilde{V}_2(\xi,\tau)}{\partial \xi^2} \\ +\frac{\rho A_2(\xi)L^4}{E_2I_2(0)}\frac{\partial^2\tilde{V}_2(\xi,\tau)}{\partial t^2} + k_2(\tilde{V}_2(\xi,\tau) - \tilde{V}_1(\xi,\tau)) = 0 \end{cases} \tag{16}$$

where in dimensionless form,

$$k_1 = \frac{kL^4}{E_1I_1(0)}, k_2 = \frac{kL^4}{E_2I_2(0)}$$

And the corresponding boundary conditions:

$$\tilde{V}_j(0,\tau) = 0 = \frac{\partial \tilde{V}_j(0,\tau)}{\partial \xi}, j = 1,2 \quad \text{fixed at} \quad \xi = 0 \tag{17}$$
$$\tilde{V}_j(\xi,\tau) = 0 = \frac{\partial \tilde{V}_j(\xi,\tau)}{\partial \xi}, j = 1,2 \quad \text{fixed at} \quad \xi = 0 \tag{18}$$
$$\tilde{V}_j(0,\tau) = 0 = \frac{\partial^2 \tilde{V}_j(0,\tau)}{\partial \xi^2}, j = 1,2 \quad \text{simply supported at} \quad \xi = 0 \tag{19}$$
$$\tilde{V}_j(\xi,\tau) = 0 = \frac{\partial^2 \tilde{V}_j(\xi,\tau)}{\partial \xi^2}, j = 1,2 \quad \text{simply supported at} \quad \xi = 0 \tag{20}$$

Finally, the initial conditions are:

$$\tilde{V}(\xi,0) = 0 = \frac{\partial \tilde{V}(\xi,0)}{\partial \tau} \tag{21}$$

Using Laplace Carson Integral Transform in the time integrand in lieu of equation (21), equation (16) becomes

$$\begin{cases} I_1(\xi)\frac{\partial^4 \overline{V}_1(\xi,s)}{\partial \xi^4} + I_1'(\xi)\frac{\partial^3 \overline{V}_1(\xi,s)}{\partial \xi^3} + I_1''(\xi)\frac{\partial^2 \overline{V}_1(\xi,s)}{\partial \xi^2} \\ + \frac{\rho A_1(\xi)L^4}{E_1 I_1(0)} s^2 \overline{V}_1(\xi,s) + k_1(\overline{V}_1(\xi,s) - \overline{V}_2(\xi,s)) = 0 \\ I_2(\xi)\frac{\partial^4 \overline{V}_2(\xi,s)}{\partial \xi^4} + I_2'(\xi)\frac{\partial^3 \overline{V}_2(\xi,s)}{\partial \xi^3} + I_2''(\xi)\frac{\partial^2 \overline{V}_2(\xi,s)}{\partial \xi^2} \\ + \frac{\rho A_1(\xi)L^4}{E_1 I_1(0)} s^2 \overline{V}_2(\xi,s) + k_2(\overline{V}_2(\xi,s) - \overline{V}_1(\xi,s)) = 0 \end{cases} \tag{22}$$

And the corresponding boundary conditions:

$$\overline{V}_j(0,s) = 0 = \frac{\partial \overline{V}_j(0,s)}{\partial \xi}, j = 1,2 \quad \text{fixed at} \quad \xi = 0 \tag{23}$$
$$\overline{V}_j(\xi,s) = 0 = \frac{\partial \overline{V}_j(\xi,s)}{\partial \xi}, j = 1,2 \quad \text{fixed at} \quad \xi = 0 \tag{24}$$
$$\overline{V}_j(0,s) = 0 = \frac{\partial^2 \overline{V}_j(0,s)}{\partial \xi^2}, j = 1,2 \quad \text{simply supported at} \quad \xi = 0 \tag{25}$$
$$\overline{V}_j(\xi,s) = 0 = \frac{\partial^2 \overline{V}_j(\xi,s)}{\partial \xi^2}, j = 1,2 \quad \text{simply supported at} \quad \xi = 0 \tag{26}$$

Applying differential transform method to equation (22) yields

$$\begin{aligned} &\textstyle\sum_{p=0}^{l} (p+1)(p+2)(p+3)(p+4)\overline{\overline{V}}_1(p+4)I(l-p) \\ &+2\textstyle\sum_{p=0}^{l} (l-p+1)(p+1)(p+2)(p+3)\overline{\overline{V}}_1(p+3)I(l-p+1) \\ &+\textstyle\sum_{p=0}^{l} (l-p+1)(l-p+2)(p+1)(p+2)\overline{\overline{V}}_1(p+2)I(l-p+2) \\ &-\omega^2\lambda_1^4 \textstyle\sum_{p=0}^{l} \overline{\overline{V}}_1(p)m(l-p) + k_1\left(\overline{\overline{V}}_1(l) - \overline{\overline{V}}_2(l)\right) = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &\textstyle\sum_{p=0}^{l} (p+1)(p+2)(p+3)(p+4)\overline{\overline{V}}_2(p+4)I(l-p) \\ &+2\textstyle\sum_{p=0}^{l} (l-p+1)(p+1)(p+2)(p+3)\overline{\overline{V}}_2(p+3)I(l-p+1) \\ &+\textstyle\sum_{p=0}^{l} (l-p+1)(l-p+2)(p+1)(p+2)\overline{\overline{V}}_2(p+2)I(l-p+2) \\ &-i\omega^2\lambda_1^4 \textstyle\sum_{p=0}^{l} \overline{\overline{V}}_2(p)m(l-p) + k_2\left(\overline{\overline{V}}_2(l) - \overline{\overline{V}}_1(l)\right) = 0 \end{aligned} \tag{28}$$

The corresponding boundary condition for a simply supported beams:

$$\overline{\overline{V}}_1(0) = 0, \overline{\overline{V}}_1(2) = 0, \overline{\overline{V}}_2(0) = 0, \overline{\overline{V}}_2(2) = 0 \tag{29}$$

and

$$\sum_{l=0}^{N} \overline{\overline{V}}_1(l) = 0, \sum_{l=0}^{N} l(l-1)\overline{\overline{V}}_1(l) = 0,$$
$$\sum_{l=0}^{N} \overline{\overline{V}}_2(l) = 0, \sum_{l=0}^{N} l(l-1)\overline{\overline{V}}_2(l) = 0 \tag{30}$$

The corresponding boundary condition for a fixed fixed beams:

$$\overline{\overline{V}}_1(0) = 0, \overline{\overline{V}}_1(1) = 0, \overline{\overline{V}}_2(0) = 0, \overline{\overline{V}}_2(1) = 0 \tag{31}$$

$$\sum_{l=0}^{N} \overline{\overline{V}}_1(l) = 0, \sum_{l=0}^{N} l\overline{\overline{V}}_1(l) = 0,$$
$$\sum_{l=0}^{N} \overline{\overline{V}}_2(l) = 0, \sum_{l=0}^{N} l\overline{\overline{V}}_2(l) = 0 \tag{32}$$

Equations (27) and (28) subject to equation (29) for the natural frequency ω by re-arranging the set of algebraic equations to obtain an eigenvalue problem.
The values of $\overline{\overline{V}}_1(1), \overline{\overline{V}}_1(3), \overline{\overline{V}}_2(1), \overline{\overline{V}}_2(3)$ are unknown. Setting

$$\overline{\overline{V}}_1(1) = a_1, \overline{\overline{V}}_1(3) = a_2, \overline{\overline{V}}_2(1) = a_3, \overline{\overline{V}}_2(3) = a_4 \tag{33}$$

The values of $\overline{\overline{V}}_1(4), \overline{\overline{V}}_1(5), \cdots \overline{\overline{V}}_1(N)$ and $\overline{\overline{V}}_2(4), \overline{\overline{V}}_2(5), \cdots \overline{\overline{V}}_2(N)$ can be obtained in terms of a_1, a_2, a_3, a_4.
Next, $\overline{\overline{V}}_1(0), \overline{\overline{V}}_1(1), \cdots \overline{\overline{V}}_1(N)$ and $\overline{\overline{V}}_2(0), \overline{\overline{V}}_2(1), \cdots \overline{\overline{V}}_2(N)$ are substituted into equation (30), which yields a system of four equations corresponding to the Nth term. The system of equations can be written in the matrix form as:

$$\begin{pmatrix} f_{11}^{(N)}(\omega) & f_{12}^{(N)}(\omega) & f_{13}^{(N)}(\omega) & f_{14}^{(N)}(\omega) \\ f_{21}^{(N)}(\omega) & f_{22}^{(N)}(\omega) & f_{23}^{(N)}(\omega) & f_{24}^{(N)}(\omega) \\ f_{31}^{(N)}(\omega) & f_{32}^{(N)}(\omega) & f_{33}^{(N)}(\omega) & f_{34}^{(N)}(\omega) \\ f_{41}^{(N)}(\omega) & f_{42}^{(N)}(\omega) & f_{43}^{(N)}(\omega) & f_{44}^{(N)}(\omega) \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

To get the natural frequency ω, equation (34) is evaluated by finding the eigenvalue of the following matrix:

$$\begin{pmatrix} f_{11}^{(N)}(\omega) & f_{12}^{(N)}(\omega) & f_{13}^{(N)}(\omega) & f_{14}^{(N)}(\omega) \\ f_{21}^{(N)}(\omega) & f_{22}^{(N)}(\omega) & f_{23}^{(N)}(\omega) & f_{24}^{(N)}(\omega) \\ f_{31}^{(N)}(\omega) & f_{32}^{(N)}(\omega) & f_{33}^{(N)}(\omega) & f_{34}^{(N)}(\omega) \\ f_{41}^{(N)}(\omega) & f_{42}^{(N)}(\omega) & f_{43}^{(N)}(\omega) & f_{44}^{(N)}(\omega) \end{pmatrix} = 0 \tag{35}$$

$\omega = \omega_n^{(N)}, n = 1,2$ is obtained as the Nth estimated natural frequency corresponding to Nth mode of vibration. The value of N is obtained by the convergence of natural frequency expressed by the inequality:

$$|\omega_n^{(N)} - \omega_n^{(N-1)}| \leq \epsilon \tag{36}$$

where ϵ is the error tolerance.

3. RESULTS AND DISCUSSIONS

Several numerical results of different kinds of cross-sections with simply supported boundary conditions and fixed-fixed boundary conditions are presented to demonstrate the effectiveness of the proposed method for a solution using the Laplace differential transform method. The mass per unit length of the beam and the moment of inertia of the j^{th} beam varies as follows from Chen *et al.* (2018):

$$m_j(x) = m_j(0)\left(1 - \alpha_j \frac{x}{L}\right), j = 1,2 \tag{37}$$

and

$$I_j(x) = I_j(0)\left(1 - \alpha_j \frac{x}{L}\right)^3, j = 1,2 \tag{38}$$

where $I_j(0)$ and $m_j(0)$ are the moment of inertia and the mass per unit length of the beam at the left end of the jth beam, α_j is the taper ratio for the jth beam which satisfies $0 \geq \alpha_j < 1$. When $\alpha_j = 0$, then the beam is uniform.

In a non-dimensional form, equations (37) and (38) can be written as

$$m_j(\xi) = m_j(0)\left(1 - \alpha_j \xi\right), j = 1,2 \tag{39}$$

and

$$I_j(\xi) = I_j(0)\left(1 - \alpha_j \xi\right)^3, j = 1,2 \tag{40}$$

3.1 Uniform Double Beam System with Coupling Conditions

For validation, the following modal parameters are kept the same as those used in Chen *et al.* (2018):
E_1 = 1× $10^{11}N$, $I_1(0)$ = 4× $10^{-4}m^4$, L = 10m, k = 1 × $10^5 Nm^{-2}$, $m(0)$ = $500kg/m$.
For the lower beam, the flexural stiffness and the mass per unit length are: $E_2I_2 = 2 \times E_1I_1$ and $\rho A_2 = 2 \times \rho A_1$ respectively.
With the above beam's specification, the natural frequencies are calculated and the results are shown in the Table 1.
The result reported using Fourier Series Method in Table 1 by Chen *et al.* (2018) considered the uniform double beam system with rotational spring constant equals to zero i.e $K_c = 0$ and the translational spring $k = 1 \times 10^5$. The results are compared with the results obtained using Laplace Differential Transform Method (LDTM) and show a good agreement i.e the results calculated were compared with those obtained using the Improved Fourier series method by Chen *et al.* (2018), hence validating the present method.

Table 1: Comparison of the First Four dimensionless Natural Frequency.

Frequency Parameter	Improved Fourier Series Method (Chen *et al.,* 2018)	Present Method
λ_1	3.141385	3.141592654
λ_2	4.662027	4.662078291
λ_3	6.281536	6.283184755
λ_4	6.630023	6.631144848

3.2 Single Nonuniform Beam with Different Cross-Section Parameters

The Winkler foundation was set to zero in order to verify the proposed technique in the context of varied cross-sections. As a result, the double-beam system is split into two single beams with identical vibration properties. The findings of the natural frequency calculations and comparisons can be found in Abdelghany et al. (2015) and Rao (2017). Table 2 presents the outcomes. Figures 2 and 3 show, for a simply supported end condition and a fixed-fixed end condition, respectively, the influence of taper ratio on the natural frequency at each mode (a distinct periodic pattern the beam can undergo when activated). It is discovered that, for a simply supported end condition, the natural frequency decreases as the taper ratio increases but begins to fluctuate at $\alpha = 0.8$. Regarding the fixed-fixed end situation, the natural frequency decreases with increasing taper ratio and began to incline at $\alpha = 0.95$. This suggests that for a simply supported boundary condition, the taper ratio should be between 0 and less than 0.8 i.e $0 \leq \alpha < 0.8$, in order to keep the natural frequency under control. Additionally, for a fixed-fixed end condition, the taper ratio should be within the range 0 and less than 0.95 i.e $0 \leq \alpha < 0.95$.

Table 2: Effect of Tapered ratio on double beam system with $k = 0$.

	Simply Supported			Fixed-Fixed		
λ_i	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$
1	3.141592653	2.900041541	2.503947167	4.730040746	4.401367915	3.973545532
	*3.141592654	—	—	*4.730041	—	—
	(3.1416)	(2.9287)	(2.6686)	(—)	(—)	(—)
2	6.28315305	5.847378401	5.281797423	7.853204621	7.317487965	6.651461415
	*6.283185308	—	—	*7.853205	—	—
	(6.2832)	(5.8657)	(5.3807)	(—)	(—)	(—)
3	9.42477959	8.783567659	7.991295221	10.99560783	10.25149290	9.345017005
	*9.424777962	—	—	*10.995608	—	—
	(9.4248)	(8.7966)	(8.0609)	(—)	(—)	(—)
4	12.56637061	11.71720624	10.68639202	14.13716549	13.18399357	12.0339024
	*12.56637062	—	—	*14.137165	—	—
	(—)	(—)	(—)	(—)	(—)	(—)

Results in bracket are from Abdelghany *et al.* (2015)
*Results from Rao (2017)

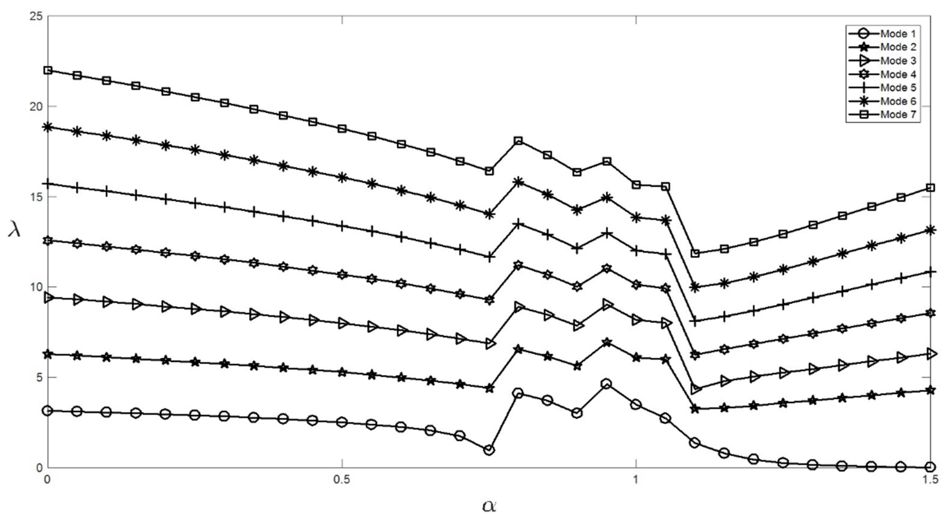

Figure 2: Effect of taper ratio on the natural frequency at each mode for simply supported end condition.

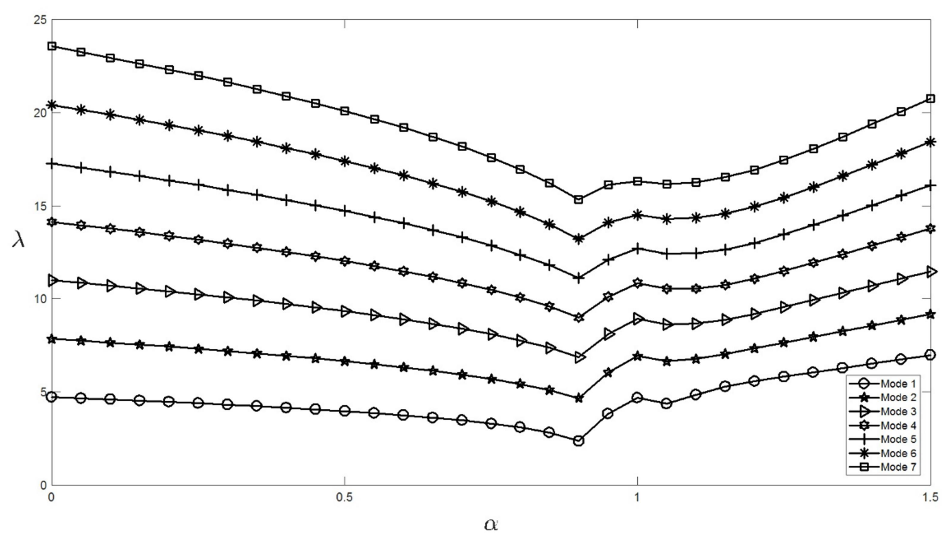

Figure 3: Effect of taper ratio on the natural frequency at each mode for fixed-fixed end condition.

3.3 Effect of Taper Ratio on the Double Beam System

The following modal parameters as used in Chen *et al.* (2018) are used:
$E_1 = 1\times 10^{11}N$, $I_1(0) = 4\times 10^{-4}m^4$, $L = 10m$, $k = 1\times 10^5 Nm^{-2}$, $m(0) = 500kg/m$.
For the lower beam, the flexural stiffness and the mass per unit length are: $E_2I_2 = 2\times E_1I_1$ and $\rho A_2 = 2\times \rho A_1$ respectively.
The effects of taper ratio of the double beam system is investigated in this subsection. The effect of taper ratio on the natural frequencies are calculated for the double beam system for $N = 30$ is presented respectively in Table 3 and 4.
Tables 3 and 4 show the effect of taper ratio on the double beam system with different cross section for simply supported and fixed-fixed boundary conditions respectively. Results show that the natural frequency decreases with an increase in the taper ratio of the beam due to the increase in the non-uniformity of the cross section as shown presented in Table 3 and graphically illustrated in Figure 4.

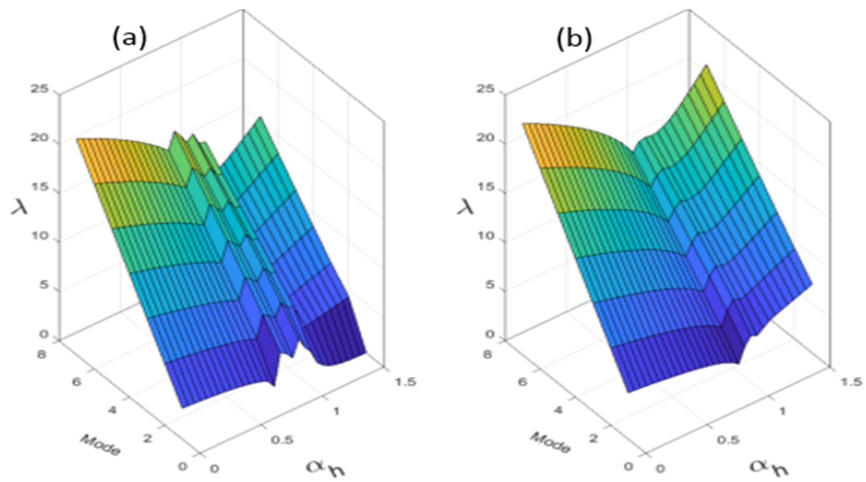

Figure 4: Effect of taper ratio on the natural frequency for (a) simply supported support (b) fixed fixed support.

Table 3: Effect of taper ratio on the simply supported double beam system.

λ_i	Simply-Supported Boundary Condition		
	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$
1	3.141592653	2.9000415407	2.503947170
2	4.662078288	4.732559584	4.848565016
3	6.283183878	5.847380592	5.281797329

Table 4: Effect of taper ratio on the Fixed-fixed double beam system.

λ_i	Fixed-Fixed Boundary Condition		
	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$
1	4.730040781	4.401367867	3.973545545
2	5.439661872	5.329083529	5.261864715
3	7.853699557	7.317654075	6.651286532

4. CONCLUSION

In this work, the free vibration of a non-uniform Euler-Bernoulli beam is taken into account. The following inference is made from the findings of this study: Euler-Bernoulli beam vibration that is unrestrained. First off, the beam with fixed-fixed end conditions has higher natural frequencies than the beam with merely supported end conditions. Second, for a simply supported end condition, the taper ratio for a non-uniform double Euler-Bernoulli system should be between 0 and less than 0.8. The value for the fixed-fixed end condition, however, should fall between 0 and 0.95. Thirdly, a double beam system with a uniform cross-section has a higher natural frequency than one with a variable cross-section. Finally, due to the increase in the cross-sectional non-uniformity of the beams, natural frequency falls as the non-uniformity of the cross-section grows. Researchers and engineers will be able to see how non-uniformity affects the free vibration of Euler-Bernoulli beams thanks to this study. Due to the increase in variation of the non-uniformity of the cross-section, which may result in a high deflection in the vibration of a non-uniform double Euler-Bernoulli beam, this study found that the natural frequency of the beam (which is a function of the stiffness and mass of the component) decreases as the taper ratio of the beam increases. Therefore, it is advised that for a simply supported end condition, the cross-sectional non-uniformity be between 0 and less than 0.8. The value for the fixed-fixed end condition, however, should fall between 0 and 0.95.

This work has shown that the Laplace differential transformation method can produce satisfactory solutions to the beam vibration problem. Therefore, utilizing the Laplace Differential Transform approach, this study can be expanded to include various boundary conditions for additional research. Additionally, as the governing equation is based on the Euler-Bernoulli beam theory, it may be assumed that shear deformation and rotating inertia effects are not taken into account and will need to be for future work.

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