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ERROR ANALYSIS OF THE SECOND ORDER TIME DEPENDENT PARABOLIC EQUATION BY USING DISCONTINUOUS GALERKIN METHOD

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ABSTRACT

The paper is offered a mathematical study to establish the error estimate of the numerical solution by applying discontinuous Galerkin (DG) finite element method of the second order time dependent parabolic equation. The DG method is an exciting numerical method with much mass compensation and more flexible meshing than other numerical methods. This study is stated a general introduction and discuss about the discontinuous Galerkin Method for the time dependent parabolic scheme. The method is well suited for large scale time-dependent computations in which high accuracy is required. The discontinuous Galerkin (DG) method has been extensively studied and applied to a wide range of parabolic problems. The main objective of this study is to theoretically explore the convergence of the solution as well as to adjust the error estimate of the methods and display the validity of the results. Two numerical experiments are shown that validate the efficiency of the method.

Keywords: *Time Dependent, Parabolic equation, Discontinuous Galerkin, Finite element method.*

1. INTRODUCTION

This paper provides a theoretical perception to approximate the error of the solutions of second order time dependent parabolic differential equation. This study is focused on the weak formulation of the discontinuous Galerkin (DG) method. Finite element methods (FEM) have been proven valued in the numerical approximation of solutions to parabolic equation. Beatrice Riviere (Riviere, 2008), offered the Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations; covered theory, implementation and other information of this method. Hesthaven and Warburton (Hesthaven *et al, 2008*), described the Nodal Discontinuous Galerkin Methods; Algorithms, Analysis and also described various applications of the method. Lewis and Ward (Lewis *et al*, 1991), provided the general introduction of the Finite Element Method. Arnold (Arnold, 1982), presented an interior penalty finite element method with discontinuous elements. Becker, Hansbo, and, Larson (Becker *et al*, 2003), provided the energy norm in the case of a posteriori error estimation for discontinuous Galerkin methods. Carstensen, Gudi, and Jensen (Carstensen *et al, 2009*), included the error estimate with discontinuous Galerkin(DG) FEM to unifying the theory of a posteriori error approximation.

Cockburn (Cockburn *et al*, 1999), published a book on Discontinuous Galerkin methods for convection-dominated problems and showed in case of higher-order, the methods has been existed vast information. Cockburn, Karniadakis, and Shu (Cockburn *et al*, 1999), explained this DG method in the perception of the theory, computation and applications of the problem. Georgoulis (Georgoulis, 2003), comprised the shape-regular meshes on discontinuous Galerkin(DG) FEM. Sjodin and Bjorn (Sjodin *et al*, 2016), demonstrated the conceptual difference among FEM, FDM and, FVM to give the clear idea about the FEM, FDM, and FVM. Cockburn, Karniadakis and, Shu (Cockburn *et al*, 2000), represented the theoretical and computational framework of the DG method. Babuˇska (Babuˇska, 1973), provided the mathematical validation of the DG method by applying the Lagrangian multipliers. Brenner and Scottfor (Brenner *et al*, 1994), established the mathematical structure of Finite Element Methods. Also, Cockburn, Kanschat, and, Schötzau (Cockburn *et al*, 2003), symbolized the local discontinuous Galerkin method for the Oseen equations. Finally, Lax and Milgram (Lax *et al, 1954*), offered the discontinuous Galerkin (DG) FEM for the parabolic equations in their book. The focus of this study is to theoretically explore the convergence of the solution as well as to amend the error estimate of the methods and shown the validity of the results.

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2. TIME DEPENDENT PARABOLIC PROBLEM

Let Ω be a bounded polygonal domain $\mathbb{R}^d. d = 1,2\ or\ 3$ and $(0, T)$ be a time interval. For $f \in L^2\big(0, t;\ L^2(\Omega)\big)$, in $L^2(0,\ t;\ H^{\frac{1}{2}}(\partial\Omega))$ and $u_0 \in L^2(\Omega)$. Consider the parabolic problem with Dirichlet boundary condition.

$$\frac{\partial u}{\partial t} - \nabla.(\nabla u) + cu = f\ in\ (0, T) \times \Omega \ldots\ldots (1)$$

With boundary condition,

$$u = g\ \ in\ (0, T) \times \partial\Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$u = u_0\ on\ \{0\} \times \Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (3)$$

This problem reproduction the conduction of heat in Ω over the time period $[0, T]$ with u being the body temperature. This problem also models the diffusion of a chemical species of concentration u in a porus medium. A strong solution of the parabolic problem belongs to $C^2([0, T] \times \partial\Omega)$ and satisfies (1) to (3) point wisely. A weak solution of the parabolic problem belongs to the space $L^2\big(0,\ T;\ H^1(\Omega)\big) \cap H^1\big(0,\ T;\ L^2(\Omega)\big)$ and satisfied the variational formulation.

$$\left(\frac{\partial t}{\partial t}, v\right)_\Omega + (\nabla u, \nabla v)_\Omega = (f, v)_\Omega\ ;\ \ \forall\ t > 0, \forall\ v \in H_0^1(\Omega),$$

$$(u(0),\ v)_{\ \Omega} = (u_0,\ v)_\Omega\ ;\ \ v \in H_0^1(\Omega).$$

2.1 SEMI DISCRETE SOLUTION

This section is approximated the solution $u(t)$ by a function $U_h(t)$ that belongs to the finite dimensional space $D_k(\varepsilon_h)$ for all $t \geq 0$ .The solution U_h is referred to as the semidiscrete solution or sometimes as the continuous in time solution. Let $v \in H^s(\varepsilon_h)$ for $s > \frac{3}{2}$, multiply (1) by v , integrate over one mesh elements, use Green's theorem and sum over all elements to obtain,

$$\begin{aligned}\forall\ t > 0, \int_\Omega \frac{\partial u}{\partial t} v + \sum_{E \in \varepsilon_h} \int_E \nabla u(t).\nabla v - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla u(t).\boldsymbol{n}_e\}[v] + \varepsilon \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla u.\boldsymbol{n}_e\}[v(t)] \\ + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\delta_e^0}{|e|^{\gamma_0}} \int_e [u(t)][v] + \int_\Omega cuv \\ = L(t; v) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .\ldots (4)\end{aligned}$$

Where, $\quad L(t;\ v) = \int_\Omega f(t)v + \sum_{e \in \partial\Omega} \int_e g(t)(\varepsilon(\nabla v.\boldsymbol{n}_e) + \frac{\delta_e^0}{|e|^{\gamma_0}} v)$

Define the energy norm for the parabolic problem

$$\|v\|_\varepsilon = \left(\sum_E \|\nabla v\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\delta_e^0}{|e|^{\gamma_0}} \|v\|_{L^2(e)}^2\right)^{\frac{1}{2}}$$

Denote the bilinear form by b_ε

$$b_\varepsilon(w, v) = \sum_{E \in \varepsilon_h} \int_E \nabla w.\nabla v - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla v.\boldsymbol{n}_e\}[v] + \varepsilon \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla v.\boldsymbol{n}_e\}[w] + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\delta_e^0}{|e|^{\gamma_0}} \int_e [w][v]$$

And, assume the coercivity of b_ε holds true for some $\tau > 0$.

$$\tau\|v\|_\varepsilon^2 \leq b_\varepsilon(v, v);\ \ \forall\ v \in D_k(\varepsilon_h), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .\ldots\ldots\ldots\ldots .. \ldots .. (5)$$

Thus, the semidiscrete variational is as follows: For all $t \geq 0$, find $U_h(t) \in D_k(\varepsilon_h)$ such that

$$\forall\ t > 0, \forall\ v \in D_k(\varepsilon_h), \qquad \left(\frac{\partial U_h}{\partial t}, v\right) + b_\varepsilon(U_h(t), v) = L(t; v) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (6)$$

$$\forall\, v \in D_k(\varepsilon_h), \quad (U_h(0), v)_\Omega = (\tilde{u}_0, v)_\Omega \ldots\ldots\ldots\ldots (7)$$

The initial condition $\tilde{u}_0$ can be chosen to be u_0 if u_0 belongs to the discrete space $D_k(\varepsilon_h)$, or it can be chosen to be $\tilde{u}_0(0)$,where $\tilde{u}$ is an approximation of u to be specified later. Using the global basis function, expand the semidiscrete solution

$$U_h(t,x) = \sum_{E\in\varepsilon_h}\sum_{i=1}^{N_{loc}} \eta_i^E(t)\zeta_i^E(x)\,; \;\; \forall\, t \in (0,T), \forall\, x \in \Omega \ldots\ldots\ldots\ldots (8)$$

The degree of freedom $\eta_i^{E\prime}s$ are functions of time. Let N_{e1} denote the number of elements in the mesh. Rename the basis functions and the degree of freedom such that

$$\{\zeta_i^E: 1 \le i \le N_{loc}, \quad E \in \varepsilon_h\} = \{\tilde{\zeta}_j: 1 \le j \le N_{loc}N_{e1}\},$$

$$\{\eta_i^E: 1 \le i \le N_{loc}, \quad E \in \varepsilon_h\} = \{\tilde{\eta}_j: 1 \le j \le N_{loc}N_{e1}\}$$

Plugging (8) into (6-7) yields a linear system of ordinary differential equations with the vector of unknowns $\tilde{\eta} = (\tilde{\eta}_j)_j$:

$$M\frac{d\tilde{\eta}}{dx}(t) + A\tilde{\eta}(t) = F(t)$$

$$M\tilde{\eta}(0) = \widetilde{U}_0$$

The matrices $M = (M_{ij})_{ij}$, $A = (A_{ij})_{ij}$ are called the mass and stiffness matrices, and they are defined by

$$M_{ij} = (\tilde{\zeta}_j, \tilde{\zeta}_i)_\Omega\,, \;\; A_{ij} = a_\varepsilon(\tilde{\zeta}_j, \tilde{\zeta}_i); \;\; \forall\, 1 \le i,\, j \le N_{loc}\, N_{e1} \ldots\ldots\ldots\ldots (9)$$

From (5) the matrix A is positive definite. In fact, it is the matrix resulting from the DG method applied to a parabolic problem. The matrix M is block diagonal, symmetric positive definite and thus it is invertible. The vectors $F(t)$ and $\widetilde{U}_0$ have components $\left(L(t;\, \tilde{\zeta}_i)\right)_i$ and $\left((\tilde{u}_0,\, \tilde{\zeta}_i)_\Omega\right)_i$. The existence and uniqueness of $\bar{\eta}$ is obtained from the theory of ordinary differential equations.

2.2 STABILITY OF THE SOLUTION

This section derived a stability bounds for the numerical solution. Choosing $v = U_h(t)$ in (6) and using the coercivity result (5) to have:

From Cauchy-Schwarz's inequality, the right-hand side is bounded by

$$\left|L\big(t; U_h(t)\big)\right| \le \|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} + \sum_{e\epsilon\partial\Omega}\left(\|\nabla U_h(t).\boldsymbol{n}_e\|_{L^2(\mathrm{e})} + \frac{\delta_e^0}{|e|^{\gamma_0}}\|U_h(t)\|_{L^2(\mathrm{e})}\right)\|g(t)\|_{L^2(\mathrm{e})}$$

Next, use the trace inequality and Young's inequality. As usual the constant C is independent of the mesh size h . So,

$$\left|L\big(t;\, U_h(t)\big)\right| \le \|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} + \frac{\tau}{2}\|U_h(t)\|^2{}_\varepsilon + C\sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g(t)\|^2{}_{L^2(\Omega)} \ldots\ldots\ldots\ldots (10)$$

Therefore, obtain the intermediate result

$$\frac{1}{2}\frac{d}{dt}\|U_h(t)\|^2{}_{L^2(\Omega)} + \frac{\tau}{2}\|U_h(t)\|^2{}_\varepsilon \le \|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} + C\sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g(t)\|^2{}_{L^2(\mathrm{e})} \ldots\ldots\ldots (11)$$

. Applying Gronwall's inequality, simply bound

$$\|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} \le \frac{1}{2}\|f(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|U_h(t)\|^2_{L^2(\Omega)}$$

Multiply the equation by 2 and integrate from 0 to t.

$$\|U_h(t)\|^2_{L^2(\Omega)} + \tau\int_0^t \|U_h(s)\|^2_{\varepsilon} \le \int_0^t \|f(s)\|^2_{L^2(\Omega)} + \int_0^t \|U_h(s)\|^2_{L^2(\Omega)} + \|U_h(0)\|^2_{L^2(\Omega)} + C\sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\int_0^t \|g(s)\|^2_{0,e}$$

Then, by the continuous Gronwall's inequality, conclude that

$$\|U_h(t)\|^2_{L^2(\Omega)} + \tau\int_0^t \|U_h(s)\|^2_{\varepsilon} \le C\left(\int_0^t \|f(s)\|^2_{L^2(\Omega)} + \|U_h(0)\|^2_{L^2(\Omega)} + \sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\int_0^t \|g(s)\|^2_{0,e}\right) \ldots\ldots (12)$$

The constant C grows exponentially in time; observe that this approach is valid for all primal DG methods with zero penalties. By using Poincare's in inequality and Young inequality to bound $\|U_h(t)\|_{L^2(\Omega)}$, to have ,

$$\frac{1}{2}\frac{d}{dt}\|U_h(t)\|^2_{L^2(\Omega)} + \frac{\tau}{2}\|U_h(t)\|^2_{\varepsilon} \le \frac{\tau}{4}\|U_h(t)\|^2_{\varepsilon} + C\|f(t)\|_{L^2(\Omega)} + C\sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g(t)\|^2_{L^2(\mathrm{e})}$$

After multiplying by 2 and integrating from 0 to t it is obtained:

$$\|U_h(t)\|^2_{L^2(\Omega)} + \frac{\tau}{2}\int_0^t \|U_h(s)\|^2_{\varepsilon} \le \left\|\widetilde{U_0}\right\|^2_{L^2(\Omega)} + C\int_0^t \|f(s)\|^2_{L^2(\Omega)} + C\sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\int_0^t \|g(s)\|^2_{L^2(\mathrm{e})}$$

Which is the same inequality as (12) modulo some multiplicative contains. However, here the constant C is independent of time. This approach is valid if the penalty value δ_e^0 is positive for all faces e. The final result is stated in the following lemma.

Assume that $\gamma_0 \ge (d-1)^{-1}$. There exists a positive constant C independent of h such that

$$\|U_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_0^T \|U_h\|^2_{\varepsilon} \le C\left\|\widetilde{U_0}\right\|^2 + C\|f\|^2_{L^2(0,T;L^2(\Omega))} + C\sum_{e\epsilon\partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g(s)\|^2_{L^2(0,T;L^2(\Omega))} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

It is noticed that, the last term on the right hand-side of (13) blows up as the mesh size h tends to zero. The space of test function is then defined as

$$D_k^0(\varepsilon_h) = \{V \in D_k(\varepsilon_h):\ V = 0 \ on\ \partial\Omega\}$$

In that case, the stability estimates are

$$\|U_h\|^2_{L^\infty(0,\ T;\ L^2(\Omega))} + \int_0^T \|U_h\|^2_{\varepsilon} \le C\left\|\widetilde{U_0}\right\|^2 + C\|f\|^2_{L^2(0,\ T;\ L^2(\Omega))}$$

And the solution is equal to $U_h + g_h$, where $g_h \in D_k(\varepsilon_h)$ is an interpolant of a lift of the Dirichlet boundary condition g.

3. ERROR ANALYSIS

This section has derived error estimates for the numerical error $u - U_h$ in the

$L^\infty(0,\ T;\ L^2(\Omega))$ and $L^2(0,\ T;\ H^1(\varepsilon_h))$ norms. Define the parabolic projection $\tilde{u}$ of the exact solution u:

$$a_\varepsilon(u(t) - \tilde{u}(t), v) = 0\,;\ \ \forall\, t \geq 0\,,\ \ \forall\, v \in D_k(\varepsilon_h), \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (14)$$

Now if u belongs to $L^2\big(0,\ T;\ H^1(\varepsilon_h)\big)$ for $s > \frac{3}{2}$ the following error estimates hold:

$$\|u(t) - \tilde{u}(t)\|_\varepsilon \leq Ch^{\min(k+1,s)-1} |||u(t)|||_{H^s(\varepsilon_h)}\,;\ \ \forall\, t \geq 0\ \ \dots \dots \dots \dots \dots \dots \dots \dots \dots (15)$$

In addition, if Ω is convex, error estimates in L^2 norm are

$$\forall\, t \geq 0\,, \qquad \|u(t) - \tilde{u}(t)\|_{L^2(\Omega)} \leq Ch^{\min(k+1,s)} |||u(t)|||_{H^s(\varepsilon_h)}\ for\ SIPG \dots \dots \dots \dots \dots (16)$$

$$\forall\, t \geq 0\,, \qquad \|u(t) - \tilde{u}(t)\|_{L^2(\Omega)} \leq Ch^{\min(k+1,s)-1} |||u(t)|||_{H^s(\varepsilon_h)}\ for\ NIPG\ and\ IIP \dots (17)$$

Under some conditions such as superpenalization $(\gamma_0 \geq 3(d-1)^{-1})$ the estimates in L^2 norm are optimal.

What can be saying about the time derivatives of $u(t) - \tilde{u}(t)$ using linearity of the bilinear form to have:

$$a_\varepsilon \frac{d}{dt}\big((u(t) - \tilde{u}(t), v)\big) = 0\,;\ \ \forall\, v \in D_k(\varepsilon_h)$$

The time derivative of the parabolic projection is the parabolic projection of the time derivative.

Now, the following theorem is introduced to state the error analysis.

3.1 THEOREM 1

Assume that, u belongs to $H^1\big(0,\ T;\ H^s(\varepsilon_h)\big)$ and that U_0 belongs to $H^s(\varepsilon_h)$ for $s > \frac{3}{2}$. Assume that $\gamma_0(d-1) \geq 1$ and t δ_e^0 is sufficiently large for all e. Then there is a constant C independent of h such that

$$\left(\int_0^T \|u(t) - U_h(t)\|^2{}_\varepsilon\right)^{\frac{1}{2}} \leq Ch^{\min(k+1,s)-1} \|u\|_{H^1(0,\ T;\, H^s(\varepsilon_h))},$$

$$\|u - U_h\|^2_{L^\infty(0\,,T;\,L^2(\Omega))} \leq Ch^{\min(k+1,s)-\mu} \|u\|_{H^1(0,\ T;\, H^s(\varepsilon_h))}$$

Proof: Since the scheme is consistent, obtain the following orthogonally equation:

$$\left(\frac{\partial(U_h - u)}{\partial t}, v\right)_\Omega + b_\varepsilon(U_h(t) - u(t), v) = 0\,;\ \ \forall\, t \geq 0\,, \qquad \forall\, v \in D_k(\varepsilon_h)$$

Defining $\Psi = U_h - \tilde{u}$, to have for all $t > 0$ and for all $v \in D_k(\varepsilon_h)$

$$\left(\frac{\partial\Psi}{\partial t}, v\right)_\Omega + b_\varepsilon(\Psi(t), v) = \left(\frac{\partial(u - \tilde{u})}{\partial t}, v\right)_\Omega + b_\varepsilon(u(t) - \tilde{u}(t), v) \dots \dots \dots \dots \dots \dots ... (18)$$

Using the definition of parabolic projection, it is obtained,

$$\left(\frac{\partial\Psi}{\partial t}, v\right)_\Omega + b_\varepsilon(\Psi(t), v) = \left(\frac{\partial(u - \tilde{u})}{\partial t}, v\right)_\Omega \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (19)$$

Choosing $v = \Psi(t)$ and using the coercivity of b_ε and the definition of the parabolic projection, to get,

$$\forall\, t > 0, \qquad \frac{1}{2}\frac{d}{dt}\|\Psi\|^2{}_{L^2(\Omega)} + \tau\|\Psi(t)\|^2{}_\varepsilon \leq \left|\left(\frac{\partial(u - \tilde{u})}{\partial t}, \Psi(t)\right)_\Omega\right|$$

If the penalty parameters δ_e^0 are positive for all e , to bound the right -hand side of the equation above as

$$\left|\left(\frac{\partial(u-\tilde{u})}{\partial t},\Psi(t)\right)_{\Omega}\right| \leq \left\|\frac{\partial(u-\tilde{u})}{\partial t}\right\|_{L^2(\Omega)} \|\Psi(t)\|_{L^2(\Omega)} \leq \frac{\tau}{2}\|\Psi(t)\|^2{}_{\varepsilon} + \frac{1}{2\tau}\left\|\frac{\partial(u-\tilde{u})}{\partial t}\right\|^2_{L^2(\Omega)}$$

Therefore, using the error estimates satisfied by the parabolic projection, it is obtained

$$\frac{1}{2}\frac{d}{dt}\|\Psi\|^2{}_{L^2(\Omega)} + \frac{\tau}{2}\|\Psi(t)\|^2{}_{\varepsilon} \leq\ Ch^{2\min(k+1,s)-2\mu}\left|\left|\left|\frac{\partial u}{\partial t}\right|\right|\right|^2_{H^s(\varepsilon_h)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(20)}$$

The parameter μ is zero unconditionally. By multiplying (20) by 2 and integrate from 0 to t.

$$\|\Psi(t)\|^2{}_{L^2(\Omega)} + \kappa\int_0^t \|\Psi(\tau)\|^2_{\varepsilon}\ \|\Psi(0)\|^2{}_{L^2(\Omega)} + Ch^{2\min(k+1,s)-2\mu}\left|\left|\left|\frac{\partial u}{\partial t}\right|\right|\right|^2_{L^2(0,T;\,H^s(\varepsilon_h))}$$

Conclude by noting that $\Psi(0) = 0$ and by using the triangle inequalities in the L^2 norm.

$$\|u(t) - U_h(t)\|_{L^2(\Omega)}\|\Psi(t)\|_{L^2(\Omega)} + \|u(t) - \tilde{u}(t)\|_{L^2(\Omega).}$$

The triangle inequalities in every norm,

$$\left(\int_0^T \|u(t) - U_h(t)\|^2{}_{\varepsilon}\right)^{\frac{1}{2}} \leq \left(\int_0^T \left\|u(t) - \widetilde{U}_h(t)\right\|^2{}_{\varepsilon}\right)^{\frac{1}{2}} + \left(\int_0^T \|\tilde{u}(t) - U_h(t)\|^2{}_{\varepsilon}\right)^{\frac{1}{2}}$$

And the error estimates satisfied by $\tilde{u}$.

Now, state the error analysis by introducing the following theorem only in the case of the symmetry of the bilinear form.

3.2 THEOREM 2

Let $\varepsilon = -1$. Under the assumptions of Theorem 1, there exists a constant C independent of h such that

$$\left\|\frac{\partial(u-U_h)}{\partial t}\right\|_{L^2(0,t;L^2(\Omega))} \leq Ch^{\min(k+1,s)}\|u\|_{H^1(0,\ T;\,H^s(\varepsilon_h))}$$

Proof: In the error equation (19), choose $v = \frac{\partial\chi}{\partial t}$

$$\left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L^2(\Omega)} + b_\varepsilon\left(\Psi(t),\frac{\partial\Psi}{\partial t}\right) = \left(\frac{\partial(u-\tilde{u})}{\partial t},\frac{\partial\Psi}{\partial t}\right)_{\Omega}$$

Thus, using the symmetry property of b_ε, to have

$$\left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}b_\varepsilon\big(\Psi(t),\Psi(t)\big) = \left(\frac{\partial(u-\tilde{u})}{\partial t},\frac{\partial\Psi}{\partial t}\right)_{\Omega} \leq \frac{1}{2}\left\|\frac{\partial(u-\tilde{u})}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L^2(\Omega)}$$

Integrating from 0 to t and using the fact that $\Psi(0) = 0$ to obtain,

$$\int_0^t \left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{2}b_\varepsilon\big(\Psi(t),\Psi(t)\big) \leq \frac{1}{2}b_\varepsilon\big(\Psi(0),\Psi(0)\big) + \int_0^t \left\|\frac{\partial(u-\tilde{u})}{\partial t}\right\|^2_{L^2(\Omega)}$$

$$\leq Ch^{2\min(k+1,s)}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;H^s(\varepsilon_h))}$$

Using coercivity of $b_{\ \varepsilon}$ and the triangle inequality,

$$\left\|\frac{\partial(u-U_h)}{\partial t}\right\|_{L^2(0,t;L^2(\Omega))} \le \left\|\frac{\partial\big(u-\widetilde{U}\big)}{\partial t}\right\|_{L^2\left(0,t;L^2(\Omega)\right)} + \left\|\frac{\partial\Psi}{\partial t}\right\|_{L^2\left(0,t;L^2(\Omega)\right)}$$

$$\le Ch^{\min(k+1,s)}\left\|\frac{\partial u}{\partial t}\right\|_{L^2\left(0,T;H^s(\Omega)\right)}$$

This, conclude the proof.

4. NUMERICAL EXPERIMENT

In this section, two numerical experiments are presented to validate the theoretical result of the discontinuous Galerkin method for the second order time dependent parabolicscheme. The mesh generation and all calculations are done by FreeFem++ (Hecht. 2012). The algorithm of (1-3) is implemented on the uniform triangular mesh system. The discrete space $D_k(\varepsilon_h)$ is assembled by using piecewise polynomials of uniformed degree. For the setting of the experiment random extending the values of c to identify the proper function f and exact solution $u(x,y)$. The error and resultant convergence rates are given in Tables-1 and Tables-2 against the mesh-size. In the numerical experiments, the convergence behavior of errors $|u-u_h|$ are presented with respect to the parameter h on uniform meshes. The boundary condition and the proper functions f, the stability functions τ, g and the constant c are chosen such that $u(x,y)$ is the exact solution. For the experiments, the time interval $1 \le t \le 20$ is considered. In the numerical test, consider the domain $\Omega = (0,\ 1)^2$.

4.1 EXPERIMENT 1

For the first experiment, we obtained data functions as $g,\ f$ and c so that the exact solution is

$$u(t,x,y) = e^{-5.0(x-0.3t)^2+(y-0.3t)^2}$$

on the domain. The stability parameter is considered as $c = 2.0$ for this experiment.

Table 1: Numerical errors and convergence rate for the experiment 1.

Mesh	$\lvert u-u_h\rvert$	Order	Mesh	$\lvert u-u_h\rvert$	Order
3	0.000986353		48	8.84292e-006	1.83432
6	0.000345648	2.10500	96	7.81416e-006	1.90149
12	6.1426e-005	1.89965	192	6.91215e-006	1.99896
24	1.61629e-005	1.89424	384	5.11821e-006	2.00000

4.2 EXPERIMENT 2

For the first experiment, we obtained data functions as $g,\ f$ and c so that the exact solution is

$$u(t,x,y) = (1+2.0t)\left(1.0+\sin\left(\frac{\pi}{8.0}(1.0+x)(1.0+y)\right)\right).$$

on the domain. The stability parameter is considered as $c = 1.0$ for this experiment.

Table 2: Numerical errors and convergence rate for the experiment 2.

Mesh	$\lvert u-u_h\rvert$	Order	Mesh	$\lvert u-u_h\rvert$	Order
3	0.000897063		48	9.01292e-006	1.86732
6	0.000335648	2.10660	96	8.08141e-006	1.92349
12	6.1426e-005	1.91965	192	7.11121e-006	1.99896
24	2.01620e-005	1.93424	384	5.67821e-006	2.00011

From experiment, it is observed that the error quantity $|u-u_h|$ also have the convergence rates as our theoretical results are described and the computed order of convergence is $O(h^{2.00000})$.

5. CONCLUSIONS

The paper has explored the error of the numerical solution by applying the Discontinuous Galerkin finite element method for the second order time dependent parabolic differential equation. It is a diverse and straightforward methodology to pursue the error analysis from all other finite element systems which are given in the literature. The numerical experiments are demonstrated the efficiency of this method. The procedure used in this paper can also be prolonged to achieve the $L^2(\Omega)$ error estimate of the higher order problems with the best order of convergence.

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