THERMAL BUCKLING AND POSTBUCKLING CHARACTERISTICS OF EXTENSIONAL SLENDER ELASTIC RODS

Sumon Saha*
Department of Mechanical Engineering, The University of Melbourne, Victoria 3010, Australia
Abu Rayhan Md. Ali
Department of Mechanical Engineering,
Bangladesh University of Engineering and Technology, Dhaka-1000, Bangladesh
*Corresponding email: s.saha2@pgrad.unimelb.edu.au

Abstract: *This paper presents an exact mathematical model for the postbuckling of a uniformly heated slender rod with axially immovable simply supported ends on the basis of geometrically nonlinear theory of extensible rods. The material is assumed linear elastic and its thermal strain-temperature relationship is considered nonlinear. Two approaches have been used in this study. The first approach is based on the extensible elastica theory. The governing equations are derived and solved analytically for the exact closed form solutions that include the equilibrium configurations of the rod, equilibrium paths, and temperature gradients. The exact solutions take the form of elliptic integrals of the first and second kinds. In the second approach, the multisegment integration technique is employed to solve a set of nonlinear differential equations with the associated boundary conditions. The equations are integrated by using the Runge-Kutta algorithm. A comparison study between the analytical elliptic integral solutions and the numerical multisegment integration technique solutions show excellent agreement of results. Special features of the solutions in the form of determination of buckling temperature, effects of slenderness ratio and nonlinear strain-temperature coefficients on the buckling and postbuckling behavior as a function of temperature are also discussed extensively.*

Keywords: *Thermal postbuckling, elliptic integral, multisegment integration, critical buckling temperature.*

INTRODUCTION

In many practical cases, the postbuckling of slender components needs to be considered for design purposes, such as in mechanical systems (robotic arms, optical fibers, and satellite tethers) or when a component is subjected to thermal loads (railroad tracks, concrete road pavements, and pipelines). It is often possible to associate the buckling of slender components with the buckling of a simple slender rod. However, the problem of elastic stability of rods subjected to thermal loads and mechanical compressive loads is substantially different and in fact not as many articles have been published regarding thermal buckling of rods. Compared with the study for the postbuckling of the rods subjected to mechanical loads, little was found in a search of literature on the thermal postbuckling of rods or beams.

The discussion on linear problems of thermal elastic stability for straight rods can be found in literature[1,2]. Based on the simplified geometric and equilibrium equations, the solution with elliptic integral form for thermally expansive buckling of simply supported rods was obtained by Jankang and Rupeng[3]. Jekot[4] examined the thermal postbuckling of a beam made of physically nonlinear thermo-elastic material. The range for safe buckling temperature was determined and some comparisons between the nonlinear and linear postbuckling behaviors were discussed. However, the geometric nonlinearity due to the central axis curvature was not considered and a simplified form of the nonlinear axial strain was used. Coffin and Bloom[5] gave an elliptic integral solution for the symmetric postbuckling response of a linear-elastic and hygrothermal beam with two ends pinned. They assumed a linear thermal strain–temperature

Nomenclature

A	Cross-sectional area of the rod
E	Young's modulus of elasticity
I	Cross-sectional second moment of inertia
k	Curvature of the deformed curve
K	Dimensionless curvature of the deformed curve
L	Undeformed length of the rod
L^*	Deformed length of the rod
L_o	Material nonlinearity constant with temperature
$\overline{L}$	Nondimensional deformed length of the rod
l,m,n	Murnaghan's constants
M	Internal bending moment
N	Axial internal force
p	Dimensionless axial compressive force
P	Axial compressive force due to expansion
s	Arc length of deflection curve
S	Dimensionless arc length of deflection curve
ΔT	Uniform temperature rise
u, w	Displacement along x- and y- direction
U, W	Nondimensional displacement
x, y	Orthogonal coordinate system
X, Y	Curvilinear coordinate system
$\overline{X}, \overline{Y}$	Nondimensional curvilinear coordinate system
Greek symbols	
α	Coefficient of thermal expansion
μ	Stretch ratio of axial line
θ	Rotation angle of the cross-section
ε	Strain along central axis
υ	Poisson's ratio
ξ	Nondimensional undeformed neutral axis
τ	Nondimensional temperature
γ	Nonlinear thermal strain coefficient
λ	Rod slenderness ratio
β	Rotation angle of the end cross-section
σ	Stress state in the rod
π	Constant pi number
Subscript	
c	Compressive
cr	Critical
max	Maximum
T	Thermal

relationship and solved the set of differential equations for the undeformed configuration; hence, two coupled integral elliptic equations needed to be simultaneously solved. Based on the exact nonlinear geometric theory of an extensible rod and using a shooting method, a computational analysis for the thermal postbuckling behavior of rods with axially immovable pinned-pinned ends as well as fixed-fixed ends was given by Li and Cheng[6]. More recently, Li et al.[7] presented a mathematical model for the postbuckling of an elastic rod with pinned–fixed ends when a quasi-static increasing temperature was applied. Using the shooting method in conjunction with the concept of analytical continuation, the nonlinear boundary value problem consisting of ordinary differential equations was numerically solved. The results showed that the critical buckling temperature and the postbuckled rod configuration were sensitively influenced by the slenderness ratio.

Cisternas and Holmes[8] included thermal expansion effects in the extensible rod theory, focusing their study on the bifurcations of the resulting equilibrium equations under both traction and displacement boundary conditions and determined sub-critical and supercritical pitchfork bifurcations. Finally, Vaz and Solano[9,10] developed a closed-form analytical solution via uncoupled elliptical integrals for the postbuckling analysis of slender elastic rods subjected to uniform thermal loads with non-movable hinged ends[9] and with non-movable hinged at one end and at the other end constrained by a linear spring[10]. The thermal strain–temperature relationship was considered nonlinear and the material was assumed to be linearly elastic, homogeneous, and isotropic. Most recently, Zhao et al.[11] explained the characteristics of thermal postbuckling equilibrium paths of the FGM rod with different gradient index in the uniform temperature field. Analysis of thermal postbuckling behaviors of FGM rod was made by using shooting method.

Thermal post-buckling of uniformly heated axially extensible elastic rods with both ends hinged or pinned is discussed in this work. Material of the slender rod is considered linear elastic and its strain–temperature relationship may be considered nonlinear. First, through introducing basic unknown function s(x), the arc length of the deformed axial line of the rod, an exact mathematical model of the problem are established on the basis of the nonlinear geometric theory for an extensible rod. The analytical solution is obtained by uncoupled elliptic integrals, which are derived from the governing equations in the deformed configuration, hence completely defining the shape of the rod. This study may be qualitatively expanded to pipes and other slender structures such as beams subjected to thermal loads. Also, by using multi-segment integration technique, the nonlinear system of ordinary differential equations with two-point boundary values are solved numerically. The corresponding secondary equilibrium paths and the post-buckled configurations have been presented.

DEFINITION OF THE PROBLEM

Let us consider a uniformly heated slender rod of initial undeformed length L, made of physically linear elastic material, and with the Young's modulus E and a coefficient of thermal expansion α. Assume that the end support conditions are immovably simply supported, that is, the rod is pinned at both ends so that the axial movements of the two ends are prohibited. A uniform temperature increase ΔT is applied to the rod from its natural state. When the temperature is increased, the rod will tend to expand; however, the constraints at the ends completely restrain its axial expansion. Initially, the rod remains in its undeformed straight state; at the same time an axial compressive force develops. If the temperature rise is cover the critical value, the undeformed state becomes unstable and a buckling deformation takes place. Our main purpose is to seek the post-buckling response of the rod. For convenience in formulating the buckling deformation, we assume that Kirchhoff's hypothesis (the cross sections remain plane and are perpendicular to the deformed central axis) holds and that the cross-sectional area remains constant during the deformation.

As is shown in Figure 1, we choose Cartesian orthogonal coordinate systems (x, y) and (X, Y) to position the points at the material and the deformed configurations of the central axis of the rod, respectively, where x coincides with the nonstressed central axis. Here P is the compressive load arising from the expansion constraint and L^* is the deformed rod length.

An arbitrary material point of the central axis is denoted by A: (x, y) = (x, 0) where $0 \le x \le L$. When the rod is buckled, the point A moves to point B: (X, Y) = (x + u, w), in which u(x) and w(x) are the displacements of the point A(x, 0) along x and y direction respectively. Here, we presume that the deformed central axis of the rod is still in x-y plane and is symmetric about the middle point of the rod. On the basis of nonlinear geometric theory[5-7] for an axial extensible rod with large deflections, an exact mathematical model of the problem are derived from the geometrical compatibility, equilibrium of forces and moments, constitutive equations, and strain-displacement relation, which are presented next.

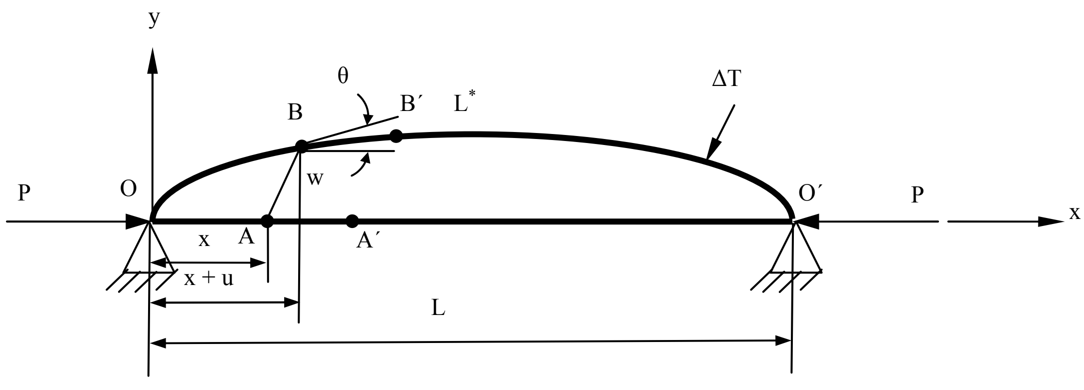

Figure 1. Schematic of Rod Element with Coordinates Axis

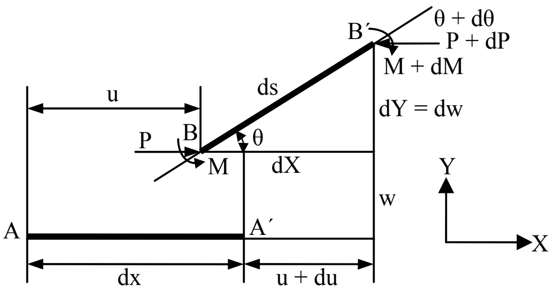

Figure 2. Infinitesimal Element of the Deflected Rod

MATHEMATICAL MODEL

Geometrical Compatibility:

By analyzing the geometric relationship of the deformation of element dx into element ds as shown in Figure 2, it is easy to derive the geometric relations,

$$\frac{dX}{ds} = \cos\theta \qquad \frac{dY}{ds} = \sin\theta \tag{1}$$

$$\frac{ds}{dx} = \mu \qquad \frac{du}{dx} = \mu\cos\theta - 1 \qquad \frac{dw}{dx} = \mu\sin\theta \tag{2}$$

in which s(x) is the arc length of the deformed central axis ($0 \le s \le L^*$), μ(x) is called the stretching of the central axis and can be expressed in terms of u(x) and w(x) by

$$\mu = \sqrt{\left(1+\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} \tag{3}$$

and θ(x) is the angle formed by the curve tangent with the longitudinal X-axis, also called the rotation angle of the cross-section.

The strain ε(x) along the central axis is defined as,

$$\varepsilon = \frac{ds - dx}{dx} = \mu - 1 \tag{4}$$

The general definition of the curvature k is

$$k = \frac{d\theta}{ds} = \frac{1}{\mu}\frac{d\theta}{dx} \tag{5}$$

Constitutive Relations:

Assuming linear elastic, homogeneous and isotropic materials (constitutive relations given by Hooke's law), and considering the state of pure bending, results in

$$N = \iint_A \sigma\, dA = EA\varepsilon_c \tag{6}$$

$$M = \iint_A \sigma\, y\, dA = -EI\frac{d\theta}{ds} = -EIk \tag{7}$$

where $I = \iint_A y^2 dA$ is the moment of inertia about the neutral axis and A is the cross-sectional area of the rod.

Equilibrium of Forces and Moments:

In the post-buckled state, the rod is assumed to be in static equilibrium. A schematic of the internal forces and moments in the deformed infinitesimal element of the rod is presented in Figure 2. The global equilibrium equations can be derived by considering the part of length s(x) of the deformed rod. The sum of the projections of the forces in the normal of the cross-section yields

$$N + p\cos\theta = 0 \tag{8}$$

where N(x) is the axial internal force. Again equilibrium of forces in the X-direction ($\sum F_X = 0$) results in a constant compressive load P along the rod. Therefore,

$$\frac{dP}{ds} = \frac{dP}{dx} = 0 \tag{9}$$

Note that for double-hinged ends there is no component of reaction forces in the Y-axis. The equilibrium of moments at B′ ($\sum M_{B'} = 0$), for instance, yields

$$\frac{dM}{ds} = p\sin\theta \tag{10}$$

or, $-M + Pw = 0$ (11)

where M(x) is the internal bending moment. Therefore, substituting Eq. (7) into Eq. (10) results in the following:

$$\frac{dk}{ds} = -\frac{P}{EI}\sin\theta \tag{12}$$

Strain-Displacement Relation:

For an infinitesimal element, the specific linear strain ε (or relative elongation) is defined as being the relation between the elongations suffered by the element, when passing to the deformed configuration and its initial length:

$$\frac{dx}{ds} = \frac{1}{1+\varepsilon} \tag{13}$$

When a slender rod is subjected to a temperature gradient ΔT, it tends to expand and consequently, a compressive load P appears if movement of the ends is restricted. Hence, the total strain is given by the addition of the thermal strain and the strain due to the compressive load ($\varepsilon = \varepsilon_T + \varepsilon_C$):

$$N = EA\left(\varepsilon - \alpha\Delta T - \frac{L_o}{E}\alpha^2\Delta T^2\right) \tag{14}$$

$$\varepsilon = \alpha\Delta T + \frac{L_o}{E}\alpha^2\Delta T^2 - \frac{P}{EA}\cos\theta \tag{15}$$

where α is the thermal expansion coefficient,

$$L_o = l(1-2\upsilon) - 2m(\upsilon^2 - 1) + n\upsilon^2 \tag{16}$$

l, m, n are Murnaghan's constants; and υ is Poisson's ratio. The first two terms on the right-hand side of Eq. (15) define the thermal strain for materials whose strain-temperature dependence is nonlinear, as proposed by Smith[12]. Note that, for metals, L_o may only assume negative values. Substituting Eq. (15) into Eq. (13), and rearranging, yields

$$\frac{dx}{ds} = \frac{1}{1+\alpha\Delta T + \frac{L_o}{E}\alpha^2\Delta T^2 - \frac{P}{EA}\cos\theta} \tag{17}$$

From Eqs. (4), (5), (7), (11) and (15) we get the following equations,

$$\mu = 1 + \alpha\Delta T + \frac{L_o}{E}\alpha^2\Delta T^2 - \frac{P}{EA}\cos\theta \tag{18}$$

$$\frac{d\theta}{dx} = -\frac{\mu}{EI}Pw \tag{19}$$

NON-DIMENSIONAL GOVERNING EQAUTIONS

Upon introducing the following dimensionless quantities:

$$\xi = \frac{x}{L} \quad \overline{X} = \frac{X}{L} \quad \lambda^2 = L^2\left(\frac{A}{I}\right)$$
$$S = \frac{s}{L} \quad \overline{Y} = \frac{Y}{L} \quad p = \frac{PL^2}{EI}$$
$$U = \frac{u}{L} \quad K = kL \quad \tau = \lambda^2\alpha\Delta T$$
$$W = \frac{w}{L} \quad \overline{L} = \frac{L^*}{L} \quad \gamma = \frac{L_o}{E} \tag{20}$$

The governing equations are transformed into the non-dimensional forms,

$$\frac{d\overline{X}}{dS}=\cos\theta \qquad \frac{d\overline{Y}}{dS}=\sin\theta \tag{21}$$

$$\frac{dS}{d\xi}=\mu \qquad \frac{dU}{d\xi}=\mu\cos\theta-1 \qquad \frac{dW}{d\xi}=\mu\sin\theta \tag{22}$$

$$\frac{d\theta}{d\xi}=-\mu pW \qquad \frac{d\theta}{dS}=K \qquad \frac{dK}{dS}=-p\sin\theta \tag{23}$$

$$\frac{dp}{d\xi}=\frac{d\tau}{d\xi}=0 \tag{24}$$

$$\frac{d\xi}{dS}=\frac{1}{\mu}=\frac{1}{1+\varepsilon} \tag{25}$$

where,

$$\varepsilon=\gamma\frac{\tau^2}{\lambda^4}+\frac{\tau}{\lambda^2}-\frac{p}{\lambda^2}\cos\theta \tag{26}$$

$$\mu=1+\gamma\frac{\tau^2}{\lambda^4}+\frac{(-p\cos\theta+\tau)}{\lambda^2} \tag{27}$$

The constant λ is the rod slenderness ratio and γ is the nonlinear thermal strain coefficient. By using the symmetry of the post-buckled configuration of the rod, it is enough to consider $\xi \in [0, 1/2]$. Then we have considered the following boundary conditions for numerical solutions,

$$S(0)=U(0)=W(0)=\theta(0)-\beta=U(1/2)=\theta(1/2)=0 \tag{28}$$

where β is the rotation angle of the end cross-section of the simply supported (hinged or pinned) ends.

CRITICAL BUCKLING TEMPERATURE

The determination of the critical buckling load is found by applying the equilibrium equations to the rod element in an infinitesimal slightly deformed configuration. Because the rotation θ is assumed small compared to unity, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. So, the governing equation may be reduced to,

$$\frac{d^4\overline{Y}}{d\overline{X}^4}+p\frac{d^2\overline{Y}}{d\overline{X}^2}=0 \tag{29}$$

The general solution for the homogeneous differential equation (29) with constant coefficients is quickly found:

$$\overline{Y}=C_1\sin\left(\sqrt{p}\overline{X}\right)+C_2\cos\left(\sqrt{p}\overline{X}\right)+C_3\overline{X}+C_4 \tag{30}$$

The boundary conditions in dimensionless forms for the solution of the above equations are as follows:

$$\begin{aligned}&\overline{X}(0)=\overline{Y}(0)=\overline{Y}''(0)=K(0)=0\\&\overline{X}(\overline{L})-1=\overline{Y}(\overline{L})=\overline{Y}''(\overline{L})=K(\overline{L})=0\end{aligned} \tag{31}$$

Application of boundary conditions for rods with double-hinged ends yields

$$C_2=C_3=C_4=C_1\sin\left(\sqrt{p}\right)=0$$

and to avoid trivial solution, C_1 must be different from zero, which can be satisfied if

$$\sin\left(\sqrt{p}\right)=0 \text{ and } \sqrt{p}=n\pi$$

where n is a positive integer. The smallest eigen value in this case corresponds to n = 1, that is, the first buckling mode corresponds to

$$p_{cr}=\pi^2 \tag{32}$$

subjected to a uniform temperature increase, the rod tends to expand, but until it reaches the critical buckling load, its strain is zero ($\varepsilon = 0$); hence, from Eq. (26)

$$\gamma\frac{\tau^2}{\lambda^2}+\tau-p=0 \tag{33}$$

Equation (32) can be substituted in Eq. (33) to find the critical buckling temperature

$$\tau_{cr}=\frac{\lambda^2}{2\gamma}\left[-1+\sqrt{1+\frac{4\pi^2\gamma}{\lambda^2}}\right] \tag{34}$$

Equation (34) indicates that two parameters control the critical buckling temperature; the rod slenderness ratio λ and the nonlinear strain coefficient γ. In case of linear strain-temperature relationship (i.e. $\gamma = 0$), the critical temperature equals to the value of critical compressive load,

$$\tau_{cr}=p_{cr}=\pi^2 \tag{35}$$

The physical and geometrical rod properties should be carefully selected to ensure practical and real meaning of the analysis, as well as to avoid nonconformity with the assumptions from the mathematical formulation. Therefore, high temperatures and strains above 3% should not be considered. Furthermore, the parametric study was conducted for rod slenderness ratios λ = 50, 100, 150 and 200, and nonlinear thermal strain coefficient γ = –5 and 0. The critical buckling temperature with the variation of the rod slenderness ratios, considering linear (γ = 0) and nonlinear (γ = –5) strain-temperature relationships, are presented in Figure 3.

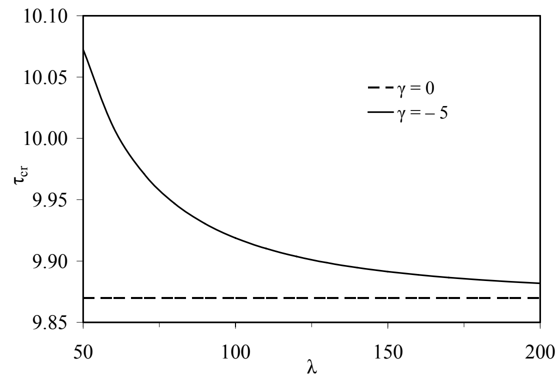

Figure 3. Critical Temperature as a Function of Slenderness Ratio

ANALYTICAL POSTBUCKLING SOLUTION

A closed-form analytical solution for the thermal postbuckling of a slender elastic rod uniformly heated is developed next via complete elliptical integrals derived from the governing equations in the deformed configuration, following similar work developed by Vaz and Solano[9,10]. The material is assumed linearly elastic and its thermal-strain temperature relationship is nonlinear. Furthermore, the boundary conditions are assumed hinged-hinged with non-movable ends. It is more convenient to work with the slope angle θ, so the non-dimensional differential equations (23) yield

$$\frac{d^2\theta}{dS^2}=-p\sin\theta \tag{36}$$

It so happens that this nonlinear equation can be easily solved by employing elliptic integrals. The solution was given in 1859 by Kirchhoff, who noticed that it is mathematically identical to the equation that describes large pendulum oscillations, which had been earlier solved by Lagrange (a kinetic analogy of columns). Integrating Eq. (36) and applying the boundary conditions at the ends of the rod yield

$$\frac{1}{2}\left(\frac{d\theta}{dS}\right)^2 = p\cos\theta + B \tag{37}$$

where B = –pcosβ and θ(0) = –θ($\overline{L}$) = β. Hence,

$$\frac{d\theta}{dS} = \sqrt{2p(\cos\theta - \cos\beta)} \tag{38}$$

After returning to familiar trigonometric identities to rewrite Eq. (38), separating and changing variables,

$$\sin\frac{\theta}{2} = \sin\frac{\beta}{2}\sin\varphi = c\sin\varphi$$

where, $c = \sin\frac{\beta}{2}$

and after some algebraic manipulation followed by an integration, the deformed rod length is found:

$$\overline{L} = \frac{2}{\sqrt{p}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}} \tag{39}$$

The integral appearing in Eq. (39) is known as a complete elliptic integral of the first kind and is dependent only on c. The x- and y-coordinates for the slender rod deflected configuration may be obtained from the non-dimensional Eqs. (21):

$$\overline{X} = \frac{1}{\sqrt{p}}\int_{\varphi_0}^{\pi/2}\frac{1-2c^2\sin^2\varphi}{\sqrt{1-c^2\sin^2\varphi}}d\varphi \tag{40a}$$

$$\overline{Y} = \frac{2c}{\sqrt{p}}\cos\varphi_0 \tag{40b}$$

where, $-\pi/2 \le \varphi_0 \le \pi/2$ and the integral in Eq. (40a) is the complete elliptic integral of the second kind.

Since $k = -p\overline{Y}$, the rod curvature k at the deformed configuration may now be readily obtained:

$$k = -\frac{2pc}{\sqrt{p}}\cos\varphi_0 \tag{41}$$

Symmetry implies that the point of maximum displacement occurs for $\overline{X}(\overline{L}/2) = 1/2$, so one may calculate p for this condition as a simple application of Eq. (40a):

$$p = \left[2\int_0^{\pi/2}\frac{1-2c^2\sin^2\varphi}{\sqrt{1-c^2\sin^2\varphi}}d\varphi\right]^2 \tag{42}$$

Therefore, for each deformed configuration (which is related to a temperature gradient), that is, for a given slope β, c = sin (β/2) is calculated and, consequently, p from Eq. (42). Finally, it is possible to find the coordinates $(\overline{X},\overline{Y})$ and curvature k along the rod from Eqs. (40a), (40b) and (41).

The temperature gradient associated with the deformed configuration may be obtained by considering Eq. (26). Thus,

$$\tau = \frac{\lambda^2}{2\gamma}\left[-1+\sqrt{1+\frac{4A_o\gamma}{\lambda^2}}\right] \tag{43}$$

where

$$A_o = \lambda^2(\overline{L}-1) + 2\sqrt{p}\int_0^{\pi/2}\frac{(1-2c^2\sin^2\varphi)^2}{\sqrt{1-c^2\sin^2\varphi}}d\varphi \tag{44}$$

Equation (44) also corresponds to the non-dimensional temperature based on linear strain-temperature relationship (γ = 0). This expression may be readily evaluated once p and $\overline{L}$ are known.

MULTISEGMENT INTEGRATION TECHNIQUE

It is very cumbersome to obtain analytical solutions to the nonlinear boundary value problems. For the symmetric buckling response of a uniformly heated rod, an elliptical integral solution is possible, but due to the limit of the elliptical integral to the boundary conditions, only the case of pinned-pinned ends is considered. So, we also apply a multisegment integration technique to seek numerical solutions of the problem, which can be further extended for the solution of thermal postbuckling of rods with fixed-fixed and pinned-fixed ends.

The fundamental set of nonlinear equations (22-24) together with the boundary conditions (28) has to be integrated over a finite range of the independent variable ξ. It is not difficult to see that it is a two-point boundary value problem with strong nonlinearity, which contains four basic unknown functions S(ξ), U(ξ), W(ξ), θ(ξ) and two unknown load parameters τ and p. We take constant β as the control parameter of the thermal post-buckled configurations of the rods with the simply supported boundary condition. Eqs. (22) and (24) may be regarded as exact mathematical model of the problem under Kirchhoff's hypothesis. For a prescribed value of β, a thermal buckled configuration can be determined. Thus, the nonlinear governing equations of thermal postbuckling problem have been solved here by using the method of multisegment integration developed by Kalnins and Lestingi[13]. The solution is implemented through a computational program developed in the mathematical software *Mathematica* and *Mathcad.*

For the sake of convenience, the differential equations (22-24) are represented here in matrix notation as follows:

$$df/d\xi = F(\xi, f_1, f_2, f_3, f_4, f_5, f_6) \tag{45}$$

where,

$f(\xi)=[f_1\ f_2\ f_3\ f_4\ f_5\ f_6]^T = [S\ U\ W\ \theta\ \tau\ p]^T$

and F represents the six functions of eqns (22-24) arranged in column matrix form. Solutions of eqns (45) by the method of multisegment integration in the interval $\xi_1 \ge \xi \ge \xi_{M+1}$, where ξ_1 corresponds to symmetric end point (ξ = 0) and ξ_{M+1} corresponds to the centre of the buckled rod at which the boundary conditions (28) are applicable, consists of the following steps:

(i) Division of the given interval of ξ into M_o sufficiently small segments so that the length of each segment is less than the critical meridional length as defined by Sepetoski *et al.*[14]
(ii) Integration of Eq.(45) over each of the M_o segments as an initial value problem. The initial values used for starting in each segment are arbitrary.
(iii) Integrations of six additional initial-value problems in each segment for which the variables are the derivatives of the six fundamental variables S, U, W, θ, τ and p with respect to each of their initial values. The necessary equations for these integrations may be derived by differentiating eqns (22-24) with respect to the initial values of each of the six fundamental variables. The initial values for these six initial value problems are the columns of a 6 × 6 unit matrix.
(iv) Solution of a system of M_o matrix equations, which ensures continuity of the variables at the end points of the segments.

Table 1. Comparison of Results for Simply Supported Extensible Heated Rod with $\gamma = 0$ and $\lambda = 120$.

β	p/p_{cr}			τ/τ_{cr}		
	Complete elliptical integral	Multisegment integration technique	Shooting method (Li and Cheng[6])	Complete elliptical integral	Multisegment integration technique	Shooting method (Li and Cheng[6])
2	0.9995	0.9995	0.9950	1.4426	1.4438	1.4438
4	0.9982	0.9982	0.9982	2.7768	2.7764	2.7764
6	0.9959	0.9958	0.9959	5.0033	5.0018	5.0019
8	0.9927	0.9927	0.9927	8.1242	8.1265	8.1265
10	0.9886	0.9885	0.9886	12.157	12.159	12.159
12	0.9836	0.9836	0.9836	17.114	17.112	17.113
14	0.9777	0.9777	0.9777	23.003	23.001	23.001
16	0.9710	0.9709	0.9710	29.840	29.842	29.842
18	0.9633	0.9633	0.9634	37.655	37.656	37.656
20	0.9549	0.9548	0.9549	46.470	46.467	46.468

(v) Repetition of steps (2), (3) and (4) until the conditions of continuity of the variables at the end points of the segments are satisfied. In each pass the improved values of the variables obtained in step (4) are used as their initial values in step (2). The convergence of the solution is achieved when the values of the variables at the end point of a segment as obtained from the initial-value integrations of eqns (22-24) match with their initial values of the next segment obtained from the solutions of matrix equations in step (4).

The convergence studies carried out for the postbuckling load ratio (p/p_{cr}) and postbuckling temperature ratio (τ/τ_{cr}) of simply supported rod for λ = 120 are tabulated in Table 1. Buckling load and corresponding temperature obtained by analytical and numerical method are compared with the results of Li and Cheng[6]. These comparisons make the present numerical method an excellent one for studying nonlinear postbuckling response of slender heated rod or beam.

RESULTS AND DISCUSSION

In the present analysis, the most significant results regarding the phenomenon of thermal postbuckling of the rods are presented for typical values of slenderness ratio: deformed configuration, maximum deflection, maximum inclination angle, maximum curvature, compressive load and total deformation. The results are presented for $\gamma = 0$ and $\gamma = -5$ (metallic materials), which respectively correspond to materials with linear and nonlinear strain-temperature relationships. The coefficient $\gamma = -5$ is relatively large for steels, but it is assumed to emphasize nonlinear effects.

The initial equilibrium of the rod is stable and in unbuckled state when $\tau < \tau_{cr}$ and when $\tau > \tau_{cr}$, the rod is in the post-buckled state or possesses the secondary equilibrium configurations. When $\tau > \tau_{cr}$ the post buckling deformations are dependent on λ, which is a feature of nonlinear theory of the rod with axial extension. Because thermal buckling of a rod is due to the thermally axial expansion, the whole deformation depends on the elongation of the axial line and the geometric size. It is different from the post buckling of rods subjected to mechanical loads that the thermal post buckling of a rod is a course which develops slowly and monotonously along with the increasing of the temperature. It is obvious that τ_{cr} is not dependent on the slenderness λ, which is a feature of linear theory, but using non-linear theory of strain-temperature relationship, τ_{cr} is a function of both slenderness ratio λ and strain-temperature coefficient γ.

Figure 4 presents the deformed configurations of the rods as a function of inclination angles. The rising nature of inclination angle and the corresponding reverse trends of compressive load variation with the change of maximum rod deflection are shown in Figure 5.

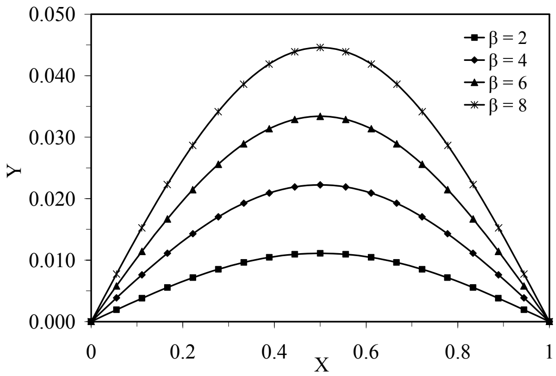

Figure 4. The Buckled Configurations of the Rod for Some Values of β

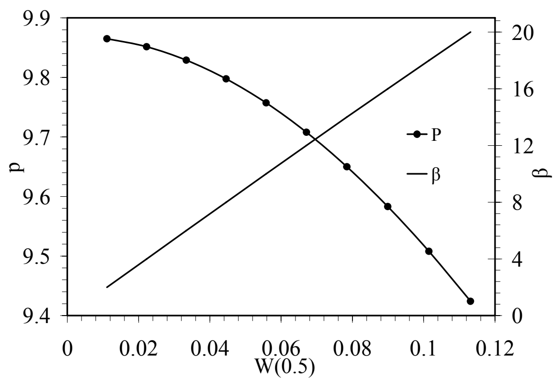

Figure 5. Compressive Load and Maximum Angle as a Function of Mid-Span Deflection

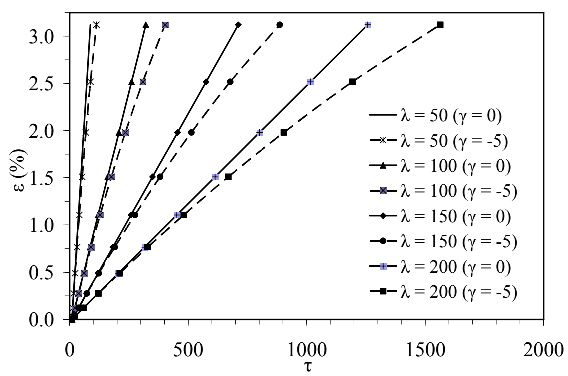

Figure 6. Total Deformation as a Function of the Temperature

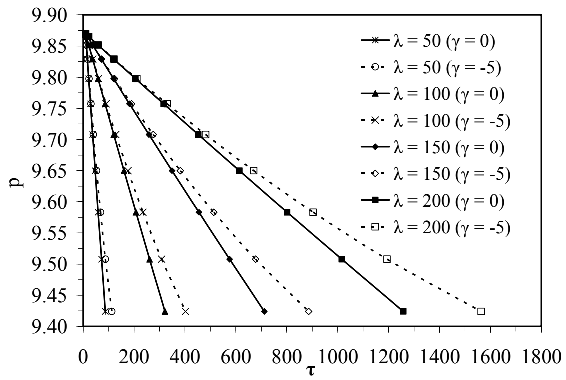

Figure 7. Compressive Load Beyond Bifurcation as a Function of the Temperature

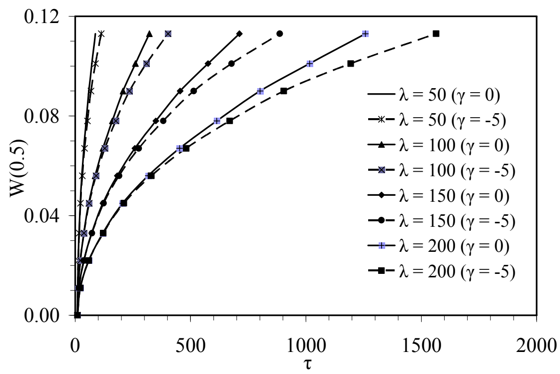

Figure 8. Maximum Deflection as a Function of the Temperature

The rod is initially straight, deprived of initial imperfections or initial temperature-induced strain. The temperature is evenly increased and the total rod deformation ε is monitored with the objective of establishing the maximum allowable temperature for each slenderness ratio. The temperature at which total deformation becomes substantially at the ends of the rod, considering linear (γ = 0) and nonlinear (γ = –5) strain-temperature relationships, are shown in Figure 6. The deformation begins when the critical buckling temperature is reached and, as expected, it rises with temperature.

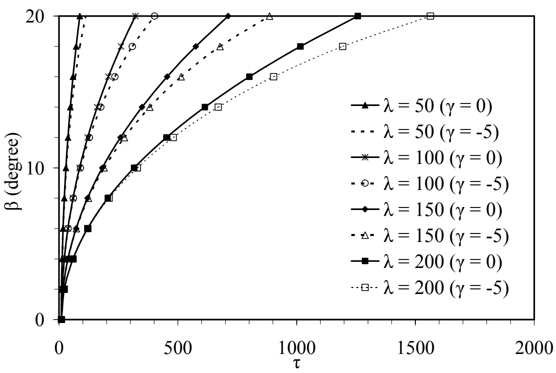

Figure 9. Maximum Angle as a Function of the Temperature

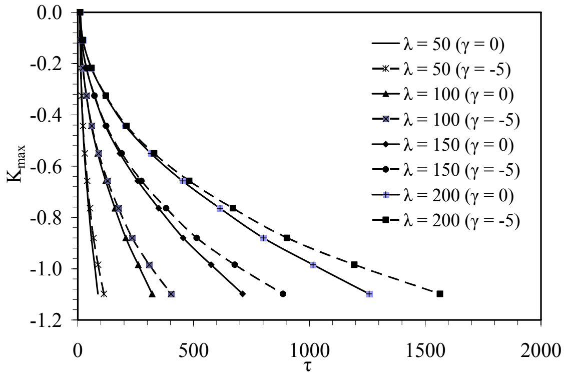

Figure 10. Maximum Curvature as a Function of the Temperature

Once the critical buckling load is reached and temperature is progressively increased, the compressive force arising in the boundaries falls considerably. This is clearly observable in Figure 7, where the variation of the compressive load versus the temperature gradient for rods subjected to linear and nonlinear thermal strain-temperature relationship is presented. For higher slenderness ratios, linear (γ = 0) and nonlinear (γ = –5) analyses differ more significantly.

Figures 8 to 10, respectively, present the results for maximum deflection, maximum inclination angle, and maximum curvature in the rod, as a function of the temperature. The maximum rod deflection which occurs at X = 0.5, increases with temperature as shown in Figure 8. The maximum inclination angle also increases with temperature but it occurs at the rod ends. The maximum curvature occurs at the middle of the rod and also increases, in modulus, as temperature is progressively increased. The characteristics curves are all monotone functions of the dimensionless temperature rising τ. The interaction points of them with coordinate axis τ are all the same and the value is just equal to the critical temperature, τ_{cr}.

The buckled deformation depends on the slenderness λ and strain-temperature coefficient γ. From the results above, we can see that W and β are all monotonously increasing functions of τ. Otherwise, the end constrained force p decreases gradually from its maximum value $p = p_{cr}$ along with the increase of temperature τ.

CONCLUSIONS

Both thermal critical buckling and postbuckling of slender rods with pinned-pinned ends and subjected to uniform temperature rise are presented. The extensibility of the rod due to axial thermal expansion at the immovable ends is taken into account. Boundary value problem for the nonlinear ordinary differential equations are solved effectively by using the multisegment integration technique. Also analytical closed-form elliptic integral solutions are obtained for comparison. Characteristics curves of the critical buckling temperature versus the slenderness ratio are plotted corresponding to the linear and nonlinear variation of the strain-temperature relation. Thermal postbuckling equilibrium paths of extensible rods with temperature rising are analyzed. From the present results, the following conclusions are reached.

(i) The present paper reports the first attempt in applying the multisegment integration analog to the thermal postbuckling of uniformly heated rods. Multisegment integration method is seen to be capable of determining all possible buckling temperatures and their corresponding deflections of the rods.
(ii) Thermal buckling of a rod differs from the buckling of rods subjected to mechanical loads.
(iii) Thermal buckling is due to the thermal expansion of an axially constrained rod, so the axial extensibility must be considered.
(iv) The nonlinear strain-temperature coefficient has a sharp influence on the critical buckling temperature and postbuckling response of the heated rod.
(v) The dimensionless buckling parameters are also sensitive to the slenderness ratio of the rod and show the consistent behavior for uniform heating conditions.
(vi) The thermal post-buckling of a rod develops slowly and monotonously along with increase of the temperature.
(vii) The magnitude of the axial compressive force arrives at its maximum at the onset of buckling and decreases as the temperature increases in the post buckling regime.

REFERENCES:

[1] Nowinski, J. L., 1978, “Theory of Thermal Elasticity with Applications”, Sijihoff and Noordhoff Int. Publishers, Groningen, The Netherlands, pp. 547–564.

[2] Boley, B. A., Winner, J. H., 1997, “Theory of Thermal Stresses”, Dover, New York, pp. 415–418.

[3] Jankang, C., Rupeng, W., 1994, “Analysis of the Characteristic of Thermally Expansive Buckling of Elastic Rods”, Mechanics and Practice, Vol. 16(3), pp. 23–26.

[4] Jekot, T., 1996, “Nonlinear Problems of Thermal Postbuckling of a Beam”, J. Thermal Stresses, Vol. 19, pp. 359–367.

[5] Coffin, D. W., Bloom, F., 1999, “Elastica Solution for the Hygrothermal Buckling of a Beam”, Int. J. Nonlinear Mech., Vol. 34, pp. 935–947.

[6] Li, S.-R., Cheng, C.-J., 2000, “Analysis of Thermal Postbuckling of Heated Elastic Rods”, Appl. Math. Mech. (English ed.), Vol. 21(2), pp. 133–140.

[7] Li, S.-R., Zhou, Y.-H., Zheng, X., 2002, “Thermal Postbuckling of a Heated Elastic Rod with Pinned-Fixed Ends”, J. Thermal Stresses, Vol. 25, pp. 45–56.

[8] Cisternas, J., Holmes, P., 2002, “Buckling of Extensible Thermoelastic Rods”, Math. Comput. Model, Vol. 36, pp. 233–243.

[9] Vaz, M. A., Solano, R. F., 2003, “Postbuckling Analysis of Slender Elastic Rod Subjected to Uniform Thermal Loads”, J. Thermal Stresses, Vol. 26, pp. 847–860.

[10] Vaz, M. A., Solano, R. F., 2004, “Thermal Postbuckling of Slender Elastic Rods with Hinged Ends Constrained by a Linear Spring”, J. Thermal Stresses, Vol. 27, pp. 367–380.

[11] Zhao, F.-Q., Wang, Z.-M., Liu, H.-Z., 2007, “Thermal Postbuckling Analyses of Functionally Graded Material Rod”, Appl. Math. Mech. (English ed.), Vol. 28(1), pp. 59–67.

[12] Smith, R.T., Stern, R., Stephens, P.W., 1966, “Third Order Elastic Moduli of Polycrystalline Metals from Ultrasonic Measurements”, J. Acoust. Soc. Amer., Vol. 40, pp. 1002–1008.

[13] Kalnins, A., Lestingi, J.F., 1967, “On Nonlinear Analysis of Elastic Shells of Revolution”, J. Appl. Mech., Ser. E, Vol. 34(1), pp. 59-64.

[14] Sepetoski, W.K., Pearson, C.E., Dingwell, I.W., Adkins, A.W., 1962, “A Digital Computer Program for the General Axially Symmetric Thin Shell Problem”, J. Applied Mech., Ser. E., Vol. 29, pp. 655-661.