

## APPLICATION OF MECHANICAL ADVANTAGE AND INSTANT CENTERS ON SINGULARITY ANALYSIS OF SINGLE-DOF PLANAR MECHANISMS

Soheil Zarkandi\*

Department of Mechanical Engineering  
Babol University of Technology, Mazandaran, Iran

\*Corresponding email: slzarkandi@yahoo.com

**Abstract:** *Instantaneous kinematics of a mechanism becomes undetermined when it is in a singular configuration; this indeterminacy has undesirable effects on static and motional behavior of the mechanism. So these configurations must be found and avoided during the design, trajectory planning and control stages of the mechanism. This paper presents a new geometrical method to find singularities of single-dof planar mechanisms using the concepts of mechanical advantage and instant centers.*

**Key Words:** Planar mechanisms; Singularity; Instant center; Angular velocity; Mechanical advantage.

### INTRODUCTION

The concept of instantaneous center of rotation (instant center) was introduced by Johann Bernoulli<sup>1</sup>. Instant centers are useful for velocity analysis of planar mechanisms and for determining motion transmission between links<sup>2</sup>. The method has proved to be efficient in finding the input-output velocity relationships of complex linkages<sup>3</sup>.

Another application of instant centers is singularity analysis of planar mechanisms. Different approaches have been adopted in dealing with singularity analysis of planar mechanisms; considering a mechanism as an input-output device, Gosselin and Angeles<sup>4</sup> identified three types of singularities:

Type (I) singularities (inverse kinematic singularities) occur when inverse instantaneous kinematic problem is unsolvable. This type of singularities occurs when at least one out of the input-variable rates can be different from zero even though all the output-variable rates are zero. In one-dof mechanisms, such singularities occur when the output link reaches a dead center, i.e. when an output variable reaches a border of its range; also for this type of mechanisms, in type (I) singularities mechanical advantage becomes infinite because in this configurations, at least one component of output torque (force), applied to the output link, is equilibrated by the mechanism structure without applying any input torque (force) in the actuated joints.

Type (II) singularities (direct kinematic singularities) occur when direct instantaneous kinematic problem is unsolvable. This type of singularities occurs when at least one out of the output-variable rates can be different from zero even though all the input-variable rates are zero. In one-dof mechanisms, such configurations occur when the input link reaches a dead center. In type (II) singularities, a (finite or infinitesimal) output torque

(force), applied to the output link, need at least one infinite input torque (force) in the actuated joints to be equilibrated, which in one-dof mechanisms, corresponds to a zero mechanical advantage.

Type (III) singularities (combined singularities) occur when both the inverse and the direct instantaneous kinematic problems are unsolvable, i.e. when two previous singularities occurs simultaneously; In this type of singularities the input-output instantaneous relationship, used out of such singularities, holds no longer and the mechanism behavior may change. In one-dof mechanisms, these singularities lead to one or more additional uncontrollable dofs.

Many articles have been presented for singularity analysis of single<sup>5-6</sup> or multi<sup>7-10</sup> dof planar mechanisms; some of these articles have geometrically addressed the singularity analysis of planar mechanisms using instant centers. For instance, Daniali<sup>10</sup> classified singularities of 3-dof planar parallel manipulators through instant centers. Di Gregorio<sup>6</sup> presented an exhaustive analytical and geometrical study about the singularity conditions occurring in single-dof planar mechanisms, which is based on the instant centers. He also<sup>9</sup> found singular configurations of multi dof planar mechanisms, considering the n dof mechanisms as the union of n one dof planar mechanisms and using the principle of superposition.

There is a close relation between singularities of a one-dof mechanism and its stationary configurations; in this case, Yan and Wu<sup>11, 12</sup> gave a geometric criterion to identify which instant centers coincide at a stationary configuration<sup>11</sup> and developed a geometric methodology to generate planar one-dof mechanisms in dead center positions<sup>12</sup>. Here the author presents a method for singularity analysis of one-dof planar mechanisms using the concepts of mechanical advantage and instant centers.

This paper is organized as follows: some notations are presented in section 2; section 3 shows how instant centers can be used to calculate mechanical advantage of single dof planar mechanisms; in section 4, singularities of single dof planar mechanisms are analyzed using the results obtained in previous section; in section 5, two illustrative examples are presented to show the method; and finally section 6 presents some conclusions of this research activity.

**NOTATION**

As it is shown in the next section, mechanical advantage of a single dof mechanism can be computed using only the instant centers of the relative motions among the four links: input link (i), output link (o), reference link (1) used to evaluate the rate of the input and output variables and link (k) which is an arbitrary link except the three previous links. The input (output) variable is a geometric parameter that defines the pose (position and orientation) of link ‘‘i’’ (‘‘o’’) with respect to link ‘‘1’’.

For the four links mentioned above, the different relative motions are six, and they will be denoted ‘‘li’’, ‘‘ok’’, ‘‘ik’’, ‘‘lk’’, ‘‘oi’’ and ‘‘lo’’ (the second letter indicates the link from which the motion of the link, denoted by the first letter, is observed). The instant centers of these six motions will be denoted as  $C_{mn}$ , where  $mn \in \{ip, ok, ik, kp, oi, op\}$ .  $C_{mn}$  will denote the position vector that locates the instant center  $C_{mn}$  in the plane of motion (it is meant that all the position vectors are defined in a unique reference system fixed to the plane of motion). According to the Aronhold–Kennedy theorem, these instant centers must lie on the straight lines shown in Fig. 1.

In this paper, we just consider a single input-single output mechanism (SISO mechanism), however a one dof single input-multiple output mechanism (SIMO mechanism) can be considered as  $n$  independent SISO mechanisms working in parallel.

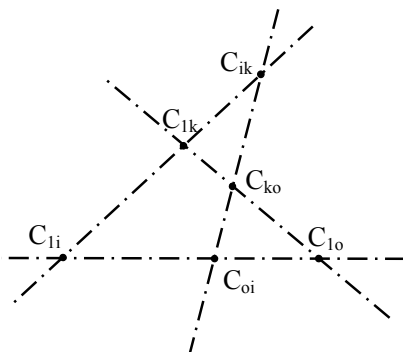


Figure 1. Straight lines the six instant centers lie on.

**MECHANICAL ADVANTAGE AND INSTANT CENTERS**

If we assume that a mechanism is a conservative system (i.e. energy losses due to friction, heat, etc., are negligible compared to the total energy transmitted by the system), and if we assume that there are no inertia forces, input power ( $P_{in}$ ) is equal to output power ( $P_{out}$ ), i.e.,

$$P_{in} = P_{out}$$

Also, if we consider a single-dof mechanism as an input-output device, the input and output powers can be calculated in different ways according to type of input and output variables.

If the variable is rotational, then the power is

$$P_* = T_* \omega_* = r_* F_* \omega_* \tag{1a}$$

And if the variable is translational, then the power is

$$P_* = F_* V_* \tag{1b}$$

Where (\*) can be ‘‘in’’ or ‘‘out’’ which are the summarized form of the words ‘‘input’’ and ‘‘output’’, respectively and  $\alpha_*$ ,  $\alpha_* \in \{P, T, r, F, \omega, V\}$ , denotes the input or output part of parameter ‘‘ $\alpha$ ’’.  $P_*$  is referred to as power.  $T_*$  is representative of input and output torques,  $\omega_*$  is angular velocity of the correspondent input and output links,  $r_*$  is the torque’s arm,  $F_*$  is the magnitude of force applied on the input and output links and finally  $V_*$  is velocity of input and output links.

It is worth noting that in the case, the variable is translational (i.e. Eq. (1b)), the direction of force  $F_*$  is considered to be parallel to the direction of motion of the correspondent link.

According to the type of input or output variables, we can classify single-dof mechanisms in four groups: mechanisms in which (i) both the input and the output variables are rotational (rot–rot mechanisms), (ii) the input variable is a rotational angle and the output variable is a translational (rot–tra mechanisms), (iii) the input variable is a translational and the output variable is a rotational angle (tra–rot mechanisms), and, finally, (iv) both input and output variables are translational (tra–tra mechanisms). In the following subsections, mechanical advantage is obtained for the each group.

**Rot–rot mechanisms**

In this group of mechanisms both the input and output variables are rotational, so following relations can be written.

$$P_{in} = T_{in} \omega_{in} = T_{out} \omega_{out} = P_{out} \tag{2}$$

or

$$\frac{T_{out}}{T_{in}} = \frac{\omega_{in}}{\omega_{out}} \quad (3)$$

Therefore, mechanical advantage is obtained as follows.

$$M.A. = \frac{F_{out}}{F_{in}} = \left(\frac{r_{in}}{r_{out}}\right)\left(\frac{T_{out}}{T_{in}}\right) = \left(\frac{r_{in}}{r_{out}}\right)\left(\frac{\omega_{in}}{\omega_{out}}\right) \quad (4)$$

Figure 1 shows the geometric relationship among the six instant centers. With reference to Fig. 1, velocity of instant center  $C_{io}$ ,  $\dot{C}_{io}$ , can be written as

$$\dot{C}_{io} = |C_{li} - C_{io}| \omega_{in} = |C_{lo} - C_{io}| \omega_{out} \quad (5)$$

which results in

$$\left(\frac{\omega_{in}}{\omega_{out}}\right) = \frac{|C_{lo} - C_{io}|}{|C_{li} - C_{io}|} \quad (6)$$

So

$$M.A. = \left(\frac{r_{in}}{r_{out}}\right) \frac{|C_{lo} - C_{io}|}{|C_{li} - C_{io}|} \quad (7)$$

**Rot-tra mechanisms**

In this group of mechanisms, the input variable is rotational and the output variable is translational and following relation can be written.

$$P_{in} = T_{in} \omega_{in} = r_{in} F_{in} \omega_{in} = F_{out} V_{out} = P_{out} \quad (8)$$

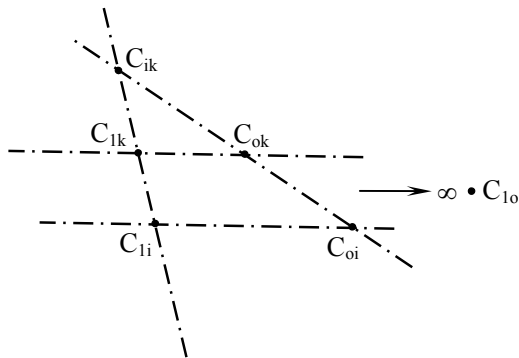


Figure 2. Straight lines the instant centers lie on when  $C_{lo}$  is at infinity.

According to Fig. 2, instant center  $C_{lo}$  is located at infinity. Velocity of the output link is equal to the Velocity of instant center  $C_{io}$ . So

$$V_{out} = |C_{li} - C_{io}| \omega_{in} \quad (9)$$

Introducing Eq. (9) into the Eq. (8) leads to,

$$r_{in} F_{in} \omega_{in} = F_{out} |C_{li} - C_{io}| \omega_{in} \quad (10)$$

So mechanical advantage for this group of mechanisms is

$$M.A. = \frac{F_{out}}{F_{in}} = \frac{r_{in}}{|C_{li} - C_{io}|} \quad (11)$$

**Tra-rot mechanisms**

In this type of mechanisms, the input variable is translational and the output variable is rotational and following relation can be written.

$$P_{in} = F_{in} V_{in} = T_{out} \omega_{out} = r_{out} F_{out} \omega_{out} = P_{out} \quad (12)$$

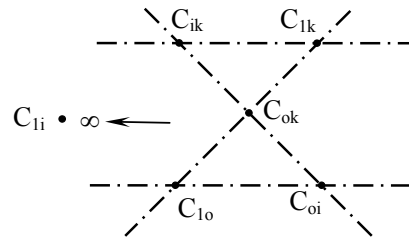


Figure 3. Straight lines the instant centers lie on when  $C_{li}$  is at infinity.

As it is shown in Fig. 3 instant center  $C_{li}$  is located at infinity. Velocity of the input link is equal to the velocity of instant center  $C_{io}$  so

$$V_{in} = |C_{lo} - C_{io}| \omega_{out} \quad (13)$$

Introducing Eq. (13) into the Eq. (12) leads to

$$F_{in} |C_{lo} - C_{io}| \omega_{out} = r_{out} F_{out} \omega_{out} \quad (14)$$

So mechanical advantage is obtained as;

$$M.A. = \frac{F_{out}}{F_{in}} = \frac{|C_{lo} - C_{io}|}{r_{out}} \quad (15)$$

**Tra-tra mechanisms**

In this group of mechanisms, both the input and output variables are translational and following relation can be written.

$$P_{in} = F_{in} V_{in} = F_{out} V_{out} = P_{out} \quad (16)$$

With reference to Fig. 4, the instant centers  $C_{1i}$ ,  $C_{1o}$  and  $C_{io}$  are located at infinity, so different method must be used to compute the mechanical advantage.

Every point on input link has the same velocity as the instant center  $C_{ik}$ , so

$$V_{in} = |C_{1k} - C_{ik}| \omega_k \tag{17}$$

Also every point on output link has the same velocity as the instant center  $C_{ok}$ , so

$$V_{out} = |C_{1k} - C_{ok}| \omega_k \tag{18}$$

Where  $\omega_k$  is angular velocity of link k. Introducing Eq. (17) and (18) into the Eq. (16) results in

$$F_{in} |C_{1k} - C_{ik}| \omega_k = F_{out} |C_{1k} - C_{ok}| \omega_k \tag{19}$$

So mechanical advantage is

$$M.A. = \frac{F_{out}}{F_{in}} = \frac{|C_{1k} - C_{ik}|}{|C_{1k} - C_{ok}|} \tag{20}$$

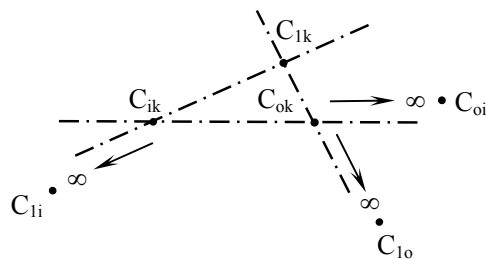


Figure 4. Straight lines the instant centers lie on when  $C_{1i}$  and  $C_{1o}$  are at infinity.

**SINGULARITY ANALYSIS OF PLANAR MECHANISMS WITH ONE DEGREE OF FREEDOM**

According to the classification presented in [4], there exist three types of singularities:

- (1) Type (I) singularities in which mechanical advantage becomes infinite.
- (2) Type (II) singularities in which mechanical advantage theoretically<sup>1</sup> become zero.
- (3) Type (III) singularities in which two previous singularities occur simultaneously.

In the following subsections, the above conditions will be applied on Eqs. (7), (11), (15) and (20) and geometric conditions identifying singularities will be given for all groups of mechanisms.

<sup>1</sup> Note that long before mechanical advantage becomes zero, the mechanism breaks down.

**Rot-rot mechanisms**

The analysis of Eq. (7) brings to the conclusion that an inverse kinematic singularity ( $M.A.=\infty$ ) occurs when (Fig. 5).

$$C_{1i} = C_{io} \tag{21}$$

On the other side, Eq. (7) shows that a direct kinematic singularity ( $M.A.=0$ ) occurs when (Fig. 6).

$$C_{1o} = C_{io} \tag{22}$$

Finally, combined singularity occurs when Eqs. (21) and (22) are satisfied simultaneously for the same configuration of the mechanism.

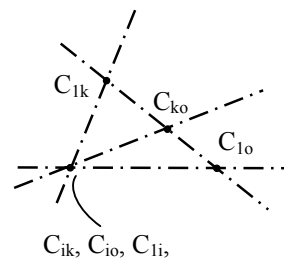


Figure 5. An example of type-(I) singularities of the rot-rot mechanisms in which condition (21) is matched.

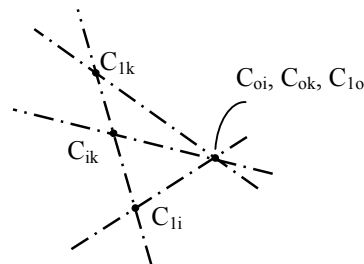


Figure 6. An example of type-(II) singularities of the rot-rot mechanisms in which condition (22) is matched.

**Rot-tra mechanisms**

Since denominators in Eqs. (7) and (11) are identical, geometric condition (21) identifies the inverse kinematic singularities for this case too. However, in the rot-tra mechanisms, the configuration of mechanism corresponding to Eq. (21) is different from the one of the same equation in the rot-rot mechanisms. For Example, Fig. 5 becomes Fig. 7 in the rot-tra mechanisms.

On the other side, Eq. (11) brings to the conclusion that direct kinematic singularity ( $M.A.=0$ ) occur when  $C_{1o}$  locates at infinity. Note that, according to Aronhold-Kennedy theorem, instant centers  $C_{io}$ ,  $C_{1o}$  and  $C_{1i}$  lie on the same line; therefore this type of singularities occurs when the following geometric condition is matched (See Fig. 8).

$$C_{io} = C_{lo} = \infty \tag{23}$$

Finally, combined singularity occurs when Eq. (21) together with Eq. (23) are satisfied for the same configuration of the mechanism.

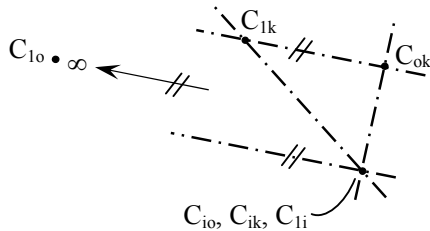


Figure 7. An example of type-(I) singularities of the rot-tra mechanisms in which condition (21) is matched.

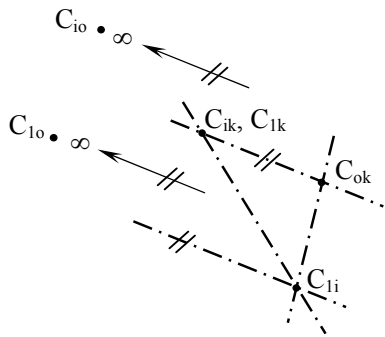


Figure 8. An example of type-(II) singularities of the rot-tra mechanisms in which condition (23) is matched.

**Tra-rot mechanisms**

The analysis of Eq. (15) brings to the conclusions that an inverse kinematic singularity occurs when  $C_{io}$  locates at infinity. Again, according to Aronhold-Kennedy theorem, this condition leads to (Fig. 9).

$$C_{io} = C_{li} = \infty \tag{24}$$

Comparing Eq. (15) and (7), one can see that geometric condition (22) identifies the direct kinematic singularities for this case too. However, in the tra-rot mechanisms, the graphic representations of Eq. (22) are different from the ones of the same equations in the rot-rot mechanisms. For instance, Fig. 6 becomes Fig. 10 in the tra-rot mechanisms.

Finally, type-(III) singularities occur when Eq. (22) together with Eq. (24) are satisfied for the same configuration of the mechanism.

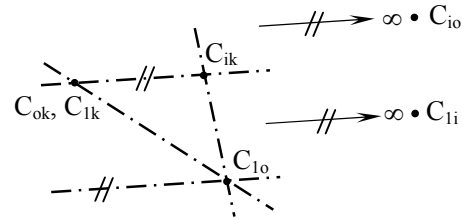


Figure 9. An example of type-(I) singularities of the tra-rot mechanisms in which condition (24) is matched.

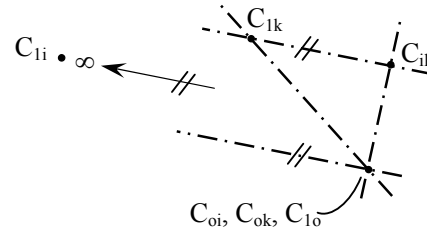


Figure 10. An example of type-(II) singularities of the tra-rot mechanisms in which condition (22) is matched.

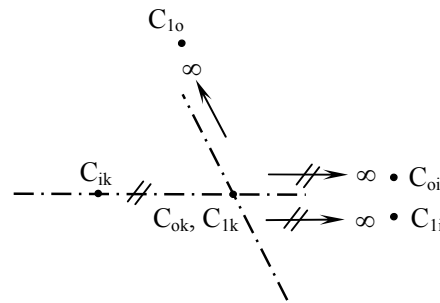


Figure 11. An example of type-(I) singularities of the tra-tra mechanisms in which condition (25) is matched.

**Tra-tra mechanisms**

Considering Eq. (20), inverse kinematic singularity occurs when (Fig. 11).

$$C_{1k} = C_{ok} \tag{25}$$

and direct kinematic singularity occurs when (Fig. 12).

$$C_{1k} = C_{ik} \tag{26}$$

Finally, combined singularities occur when both conditions (25) and (26) are satisfied for the same configuration of the mechanism.

Now singularities of single-dof planar mechanisms can be found using the above conditions.

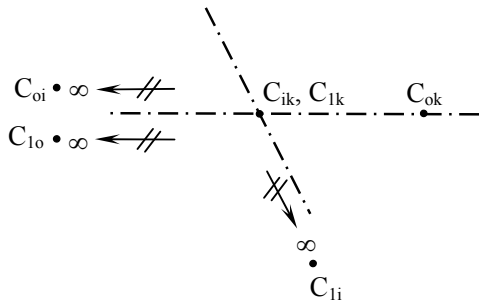


Figure 12. An example of type-(II) singularities of the tra-tra mechanisms in which condition (26) is matched.

**ILLUSTRATIVE EXAMPLES**

In the following subsections, singularities of two single-dof planar mechanisms are analyzed to show the method presented.

**Singularity analysis of an intersecting four bar mechanism**

This mechanism is shown in Fig. 13. With reference to Fig. 13, Link 2 is the input link; link 4 is the output link and link 1 is the reference link used to evaluate the rate both of the input variable and of the output variable.  $\theta_{21}$  and  $\theta_{41}$  are the input and output variables, respectively, so “i” = 2, “o” = 4 and the mechanism is in the group of rot-rot mechanisms.

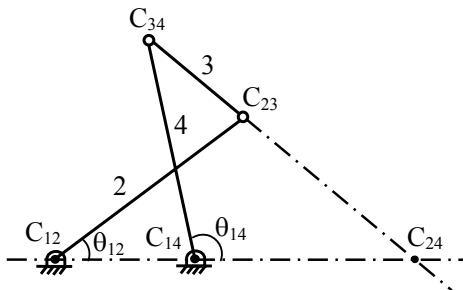


Figure 13. The intersecting four bar mechanism at a generic configuration with its instant centers.

According to the condition (21), type (I) singularities occur when  $C_{i_o}$  coincides with  $C_{i_i}$ , i.e. when  $C_{24}$  coincides with  $C_{12}$ . An example of this type of singularities is shown in Fig. 14, in which links 2 and 3 are collinear.

The condition (22) shows that type (II) singularities occur when  $C_{24}$  coincides with  $C_{14}$ . Figure 15 shows an example of this type of singularities for the mechanism under study in which links 3 and 4 are collinear.

Type (III) singularities occur when two previous singularities occur simultaneously which is equivalent to coincidence of  $C_{12}$ ,  $C_{24}$  and  $C_{14}$ ; this condition is satisfied when  $C_{24}$  is not determined and identifies a configuration in which the mechanism is flattened, see Fig. 16.

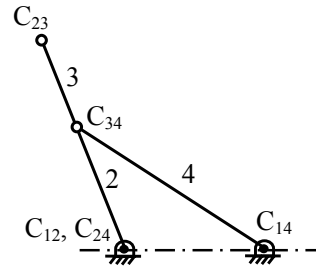


Figure 14. The intersecting four bar mechanism at a type (I) singularity.

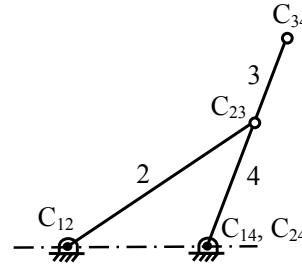


Figure 15. The intersecting four bar mechanism at a type (II) singularity.

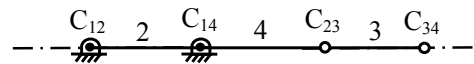


Figure 16. The intersecting four bar mechanism at a type (III) singularity.

**Singularity analysis of a six bar single-dof mechanism**

Figure 17 shows a six bar single-dof mechanism together with its instant centers. Link 2 is the input link; link 6 is the output link; link 1 is the reference link used to evaluate the rate both of the input and output variables. Arbitrary point A is fixed to link 1.  $\theta_{21}$  is the input variable;  $S_{61}$  is the output variable; Therefore, “i” = 2, “o” = 6, and the mechanism is included in the group of the rot-tra mechanisms.

Singularity condition (21) brings to the conclusion that type-(I) singularities occur when  $C_{26}$  coincides with  $C_{12}$ . Considering Fig. 17 one can see that coincidence of these two instant centers occur when link 2 is perpendicular to link 4 which is the known geometric condition identifying the dead-center positions of  $S_{61}$ . Fig. 18 shows the mechanism at a type-(I) singularity.

According to condition (23), type (II) singularities occur when  $C_{26}$  coincides with  $C_{16}$ . With reference to Fig. 17,  $C_{26}$  coincides with  $C_{16}$  when the direction of motion of link 6 is parallel to link 4 and  $\theta_{12}$  is at its dead-center position, See Fig. 19.

Finally type (III) singularities occur when both previous singularities occur simultaneously, i.e.  $C_{26}$  must coincide with  $C_{12}$  and  $C_{16}$  simultaneously; this condition is satisfied when the position of  $C_{26}$  is not

determined and mechanism gains one degree of freedom, see Fig. 20.

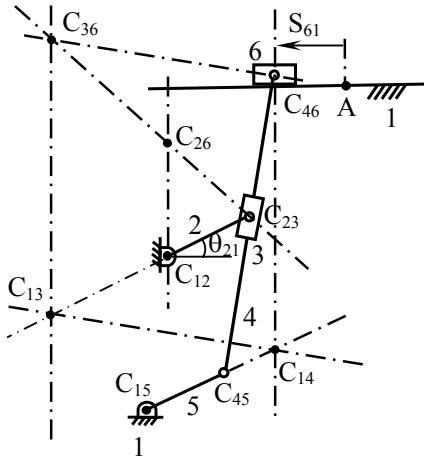


Figure 17. The six bar mechanism at a generic configuration with its instant centers.

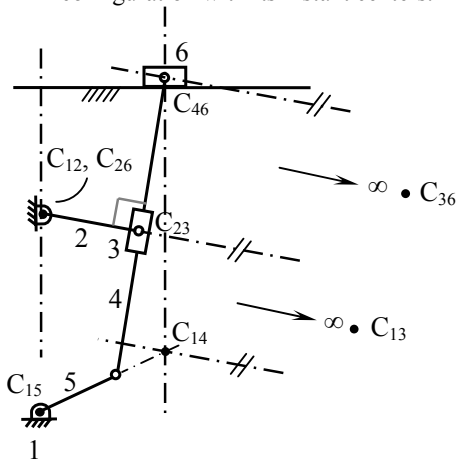


Figure 18. The six bar mechanism at a type (I) singularity.

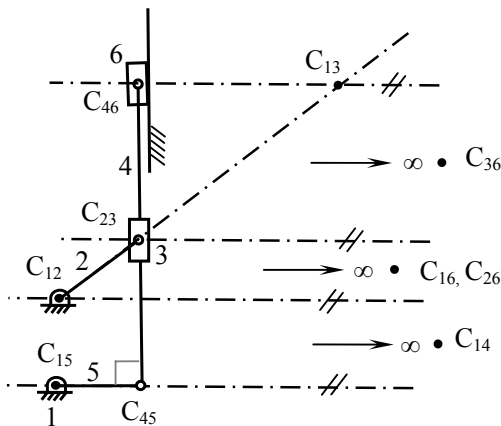


Figure 19. The six bar mechanism at a type (II) singularity.

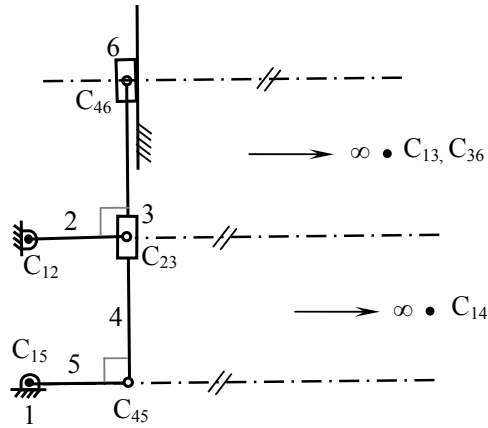


Figure 20. The six bar mechanism at a type (III) singularity.

**CONCLUSION**

A geometric method for singularity analysis of single dof planar mechanisms was presented. First, one dof planar mechanisms were classified into four groups, based on the type of input and output variables; then mechanical advantage for each group of the mechanisms was obtained using the concept of instant centers and geometric conditions corresponding to different types of singularities were found for each group. Finally two illustrative examples were presented to show the method.

The method is simple and comprehensive and can be used to find singular configurations of all types of single dof planar mechanisms.

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