DESIGN AND NON-LINEAR ANALYSIS OF A PARABOLIC LEAF SPRING

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Abstract: *Tapered cantilever beams, traditionally termed as leaf springs, undergo much larger deflections in comparison to a beam of constant cross-section that takes their study in the domain of geometric nonlinearity. This paper studies response of a leaf spring of parabolic shape, assumed to be made of highly elastic steel. Numerical simulation was carried out using both the small and large deflection theories to calculate the stress and the deflection of the same beam. Non-linear analysis is found to have significant effect on the beam's response under a tip load. It is seen that the actual bending stress at the fixed end, calculated by nonlinear theory, is 2.30-3.39 % less in comparison to a traditional leaf spring having the same volume of material. Interestingly, the maximum stress occurs at a region far away from the fixed end of the designed parabolic leaf spring.*

Keywords: *Parabolic leaf spring, End-shortening, Geometric nonlinearity, Equilibrium Configuration Path, Varying Cross-section.*

INTRODUCTION

As pointed out by Rahman et al. (2006), structural problems, coupled with geometric non-linearity are always challenges for engineers. This is because of the fact that large deflection analysis of structures, inherently involves non-linear differential equations having no closed form solutions as pointed out by Bele'ndez et al. (2005). The problem would become even more interesting if the shape of the structure itself varies from point to point. That the topic of large deflection analysis of cantilever beams is ever interesting can be seen from a huge number of studies reported in the literature; out of those researches only a few relevant to the present study are discussed below.

Very recently Bele'ndez et al. (2005) carried out numerical simulation using Runge-Kutta-Felhberg method to find the tip deflection of a very slender beam under a combined load. The authors studied the large deflections of a uniform cantilever beam under the action of a combined load consisting of an external vertical concentrated load at the free end and a uniformly distributed load and compared the numerical results with the experimental ones.

Lee (2002) dealt with large deflection of cantilever beams made of Ludwick type material under combined loading. Governing equation was derived from shearing force formulation, which has computational advantages over the bending moment formulation for large deflection analysis. It was pointed out that numerical solution is required to determine the large deflection because the governing equation is a complex non-linear deferential equation. Numerical solution was obtained using Butcher's fifth order Runge-Kutta method.

Bratus and Posvyanskii (2000) studied problem of minimizing the elastic deflection of an elastic beam of variable cross-section and fixed volume with free supported and rigidly clamped ends respectively. In case of clamped ends, it is proved that the optimum solutions must necessarily have points inside the solution range in which the distribution of the beam thickness degenerates to zero. Qualitative analytical and numerical solutions were given.

Rahman et al. (2007) carried out extensive numerical simulation of a slender cantilever beam with opening of different shapes (circle, ellipse and square slots). It was found that the elliptic holes develop the minimum stresses and deflections. Rahman et al. (2005) also performed tests to verify the soundness of the numerical results obtained considering varying cross-section because of a circular hole. Further, investigations on non-linear bending of tapered cantilever beams were also carried out by Rahman and Kowser (2007) and Kowser (2006). Though traditional design assumes constant/uniform stress distribution all along the leaf spring's span, it was shown from non-linear analysis that stresses are actually less near the tip than those at the fixed end. It was concluded that slightly more material could be removed near the tip of a tapered cantilever beam originally designed for uniform strength. Therefore, this study designs a beam of a parabolic shape. This is accomplished by designing a leaf spring (tapered cantilever beam of uniform strength) and then removing material from it defining a parabolic geometry keeping beam's volume equal to that of a traditional leaf spring. Deflections and stresses for non-linear bending are discussed and compared with those of an original/traditional leaf spring.

Nomenclature

b_0	width of the beam at fixed end
b	width of the beam at any point on its span
E	Young's modulus of the beam material
t	height/ thickness of the beam
I	area moment of inertia at any point x
L	length of the beam
M	bending moment
P	tip load
s	curved length of beam
x	horizontal distance measured from the fixed end
X_H	upper limits of x
X_L	lower limits of x
y	Elastic curve's deflection

Greek symbols

δ	tip deflection
Δ	End
σ	Stress

This study aims at utilization of the parabolic leaf spring in a compliant structure/mechanism. It should be noted that a compliant mechanism is a single-piece flexible structure where the structural deformation is utilized to transmit force or deliver motion due to an input actuation.

MATHEMATICAL ANALYSIS

Since the beams are quite slender for the present case, only pure bending is considered for this study ignoring the effect of shearing stresses. When deflection is large with respect to the span of the beam the governing equations of the elastic curve for a cantilever beam with a point load *P* (Fig. 1), in terms of large deflection formulation is given as,

$$EI\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} - P(L-\Delta) + Px = 0 \qquad (1)$$

The above equation is highly nonlinear and has been solved numerically in the present study by Runge-Kutta method.

For calculating the end-shortening Δ , consider a cantilever beam having length *L*. When load *P* is applied, the beam bends and Δ is the amount of end-shortening. On the elastic curve, an infinitesimal segment length *ds* is given by

$$ds = \sqrt{dx^2 + dy^2}$$

Therefore, the total length of the bent beam is given by

$$S = \int_{x_L}^{x_H} \sqrt{dx^2 + dy^2} = \int_{x_L}^{x_H} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad (2)$$

Where, $X_H = L\text{-}\Delta$ and X_L=0

Δ is calculated numerically by trial. At first the elastic curve is evaluated from Eq. (1) without considering Δ. Next, assuming the value of $X_H = L\text{-}\ \Delta$, in such a way that the value of integration of Eq. (2) becomes $s \approx L$, it can be said that end-shortening is $\Delta = L\text{-}X_H$. The converging criterion was $0.998L \le s \le L$. Then putting the value of Δ in Eq. (1) of the elastic curve, deflections at corresponding loads can be found. Alternately, at first a small value of Δ is assumed and Eq. (1) is solved. Once the elastic curve is known, Eq. (2) is integrated numerically to check whether the assumed value of Δ is accurate or, needs to be improved by the next step. Therefore, in order to take into account the end-shortening, Eqs. (1) and (2) have to be solved simultaneously, as described in Kowser (2006).

Next, the variable width of the two parabolic leaf springs are calculated from the following equations, with reference to Fig. 1. The vertex of the parabola lies at the tip of the leaf spring. For specimen 1 (b_0=32.66mm, width at tip=9.02mm, *L*=145mm)

$$b = 9.02 + 23.64 \times (1\text{-}x/L)^2 \qquad (3)$$

For specimen 2 (b_0=21mm, width at tip=5.80mm, *L*=145mm)

$$b = 5.80 + 15.20 \times (1\text{-}x/L)^2 \qquad (4)$$

Other dimension of the beam taken for analysis is *t*=0.83mm.

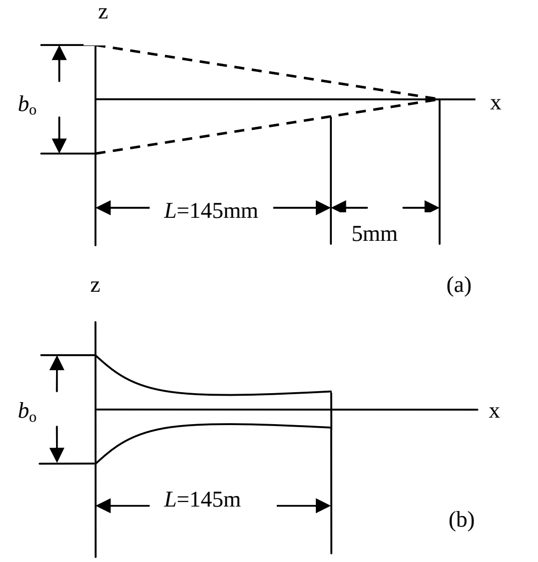

Figure 1: (a)Traditional leaf spring and (b) the proposed parabolic leaf spring of same volume (b_0=32.66 mm and 21 mm for specimens 1 and 2, respectively).

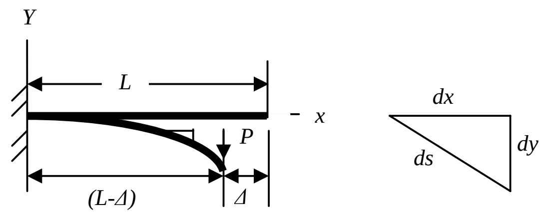

Figure 1(c): Deflection of beam under load *P* and corresponding end-shortening

It should be noted that the proposed leaf spring (*L*=145mm) and the traditional leaf spring (*L*=150mm) have same width at fixed end and same volume as well.

The problem here involves extensive numerical analysis as width of the beam changes nonlinearly with its span. In Table 1, non-linear solutions for a traditional leaf spring were obtained first and then compared with those for a proposed parabolic leaf spring.

RESULTS AND DISCUSSION

We start discussion by proving soundness of our numerical scheme by comparing few results, taking into account only geometric non-linearity, for a highly flexible cantilever beam of constant cross-section under a combined load, as treated by Bele'ndez et al. (2005). Table 2 shows the comparison. The Young's modulus for a particular load was not explicitly given by Bele'ndez et al. (2005). It was stated to be within 180-210 GPa. We used its value as 200 GPa. As seen, the numerical non-linear solution matches within an error of only 3.5% at the highest experimental load found by Bele'ndez et al. (2005). A better match would be possible with the known correct value of *E*. For example, *E* = 194.5 GPa was found to give least error as shown by Bele'ndez et al. (2005). Therefore, our numerical predictions would match even better with the experimental results with *E* = 194.5 GPa. Anyway, it is now proven that the present numerical scheme, as used here, is capable of predicting the elastic curve with

Table 1: Comparison of the proposed parabolic and traditional leaf springs for two different cases.

(b_0=32.66mm and 21mm for specimens 1 and 2, respectively)

Specimen No.	P (N)	Design Stress (MPa)	Non-linear solutions with end-shortening				
			Traditional leaf spring (L = 150mm, t = 0.83 mm)		Parabolic leaf spring of same volume (L = 145 mm, t = 0.83 mm)		
			Fixed end Stress (MPa)	Tip deflection (mm)	Fixed end stress (MPa)	Max Stress (MPa)	Tip Deflection (mm)
1	10	400	370	50.56	357.43	399.71	46.22
2	8	500	444	59.62	433.62	484.91	54.07

Maximum stresses occur at x=53mm and 51mm, for specimen 1 and specimen 2, respectively.

Table 2: Comparison of experimental results of Bele'ndez et al. (2005) with the numerical results generated by the present study using convergence criterion, $L \geq s \geq 0.998L$

(Thickness, h=0.0004m; length, L=0.40m; uniform weight, w=0.758 N/m)

P (N)	Tip deflections (mm)			Error (%)	
	Experiment from Bele'ndez et al. (2005)	Numerical solutions with Δ (present study)			
		Linear	Non-linear	Linear	Non-linear
0	89±1	81.80	84.48	8.80	5.35
0.098	149±1	132.50	142.65	12.45	4.45
0.196	195±1	166.22	184.71	17.31	5.57
0.294	227±1	188.38	210.96	20.50	7.6
0.396	251±1	211.96	236.28	18.41	5.22
0.490	268±1	229.67	254.57	16.68	5.27
0.588	281±1	236.84	271.77	18.64	3.39

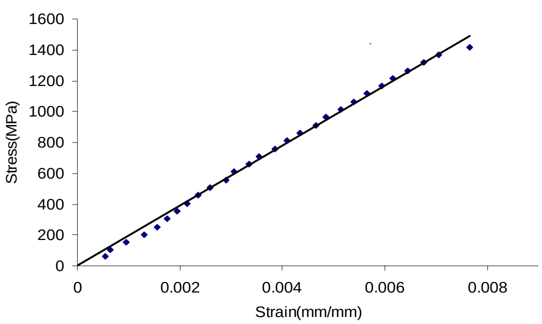

Figure 2: Tensile stress-strain curve of beam material (high strength steel).

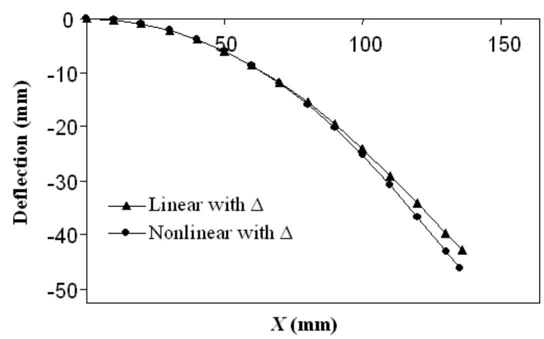

Figure 3(b): Deflected shape of parabolic leaf spring (specimen 1).

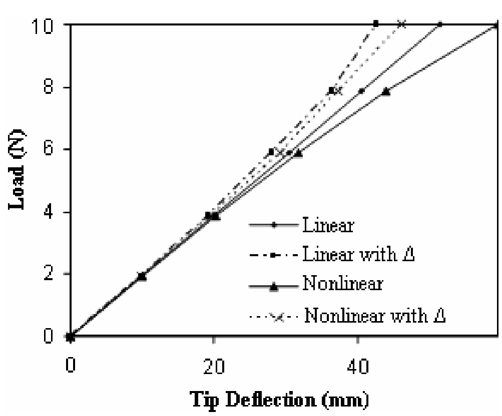

Figure 3(a): Tip deflection of parabolic leaf spring (specimen 1).

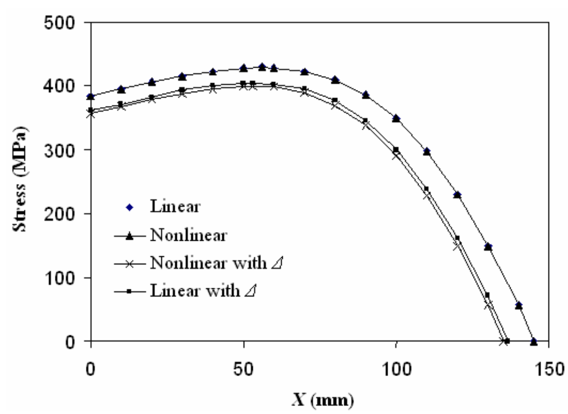

Fig. 4: Stress distribution for parabolic leaf spring (specimen 1).

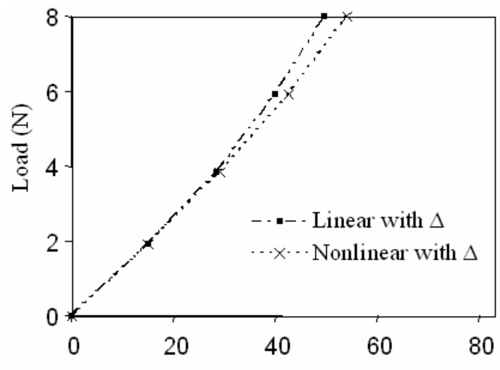

Figure 5(a): Tip deflection of parabolic leaf spring (specimen 2).

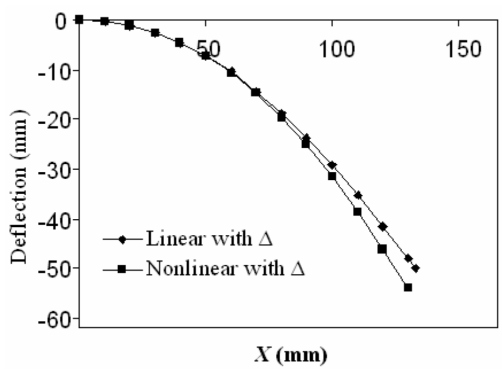

Figure 5(b): Deflected shape of parabolic leaf spring (specimen 2).

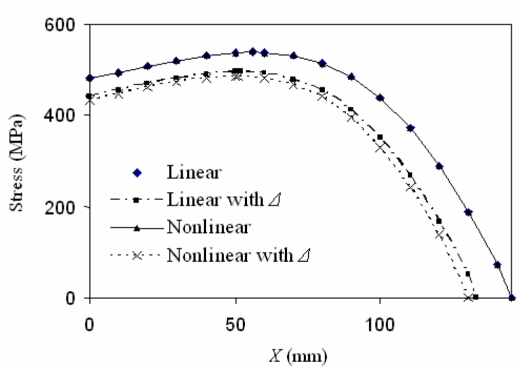

Figure 6: Stress distribution of parabolic leaf spring (specimen 2).

reasonable accuracy by large deflection theory taking into account the associated end-shortening even under a combined load. Fig. 2 shows tensile stress–strain diagram that we performed for a highly elastic steel sheet. We assume the beam can be made of this steel sheet and its Young's modulus as 194.573GPa (Fig. 2).

From Fig. 3(a) we can see the nature of the load-deflection curves (equilibrium configuration paths) of the steel beam of parabolic shape (specimen 1) obtained from numerical analysis. The tip deflection at load of 10N for the specimen is 51.68mm and 59.68 mm for linear and nonlinear analysis, respectively. But with end-shortening the tip deflection is found to 42.85mm and 46.22mm for linear and nonlinear analysis respectively, which clearly shows importance of inclusion of end-shortening for a slender cantilever beam. Fig. 3(b) shows the deflected shape of the parabolic beam for specimen 1 by linear and nonlinear analysis with end-shortening. As seen from Table 1, the tip deflections with end-shortening are 7.84% less for linear analysis and 8.58% less for nonlinear analysis compared with the values of a traditional leaf spring (length=150mm and width at fixed end bo=32.66mm).

Regarding the presentation style of the stresses for specimens 1 and 2, respectively, we prefer to use x on the abscissa for Figs.4 and 6. x has been defined in Fig. 1 as the horizontal distance of a point on the elastic curve, measured from the fixed end. For clarity, $x=X_H$ corresponds to the projection of the deflected elastic curve's tip on x axis. The advantage of stress versus x curve is that, one can get an idea of end-shortening directly. As seen, nonlinear solutions with end-shortening predict the stresses that are significantly smaller than those predicted without taking into account end-shortening. Linear solutions slightly over predict the stresses in comparison to the nonlinear solutions. Another distinguishing feature is the smooth rise and fall of stress along the beam span, the highest stress being near the middle in contrast to a classical leaf spring that develops maximum stress at the fixed end that continuously decreases all along its span (Kowser 2006).

Fig. 4 shows the stress distribution along the beam-span of parabolic leaf spring by different solution schemes. Non-linear solutions with end-shortening predict the minimum bending stresses. The maximum stresses occur at a point far away from the fixed end unlike a tapered traditional leaf spring. But, there is no abrupt change in the stresses along the span. Fixed end stresses by linear and non-linear theories with end-shortening being taken into account, are 361.45MPa and 357.43MPa, respectively which are slightly less when compared with a traditional leaf spring (Table 1).

In Fig. 5(a), tip deflections with end-shortening at a load of 8N for specimen 2, are 49.80mm and 54.07mm by linear and nonlinear analyses, respectively. Corresponding results show that these tip deflections with end-shortening are 7.76% less for linear analysis and 9.31% less for nonlinear analysis in comparison to their counterpart (Table 1). The deflected shape of the parabolic beam for specimen 2 by linear and nonlinear analysis with end-shortening is shown in Fig. 5(b).

Fig. 6 shows the stress distribution along the beam-span for specimen 2, which is similar to that of specimen 1 as shown in Fig. 4. According to non-linear theory with end-shortening the maximum stresses are 399.71MPa and 484.91MPa, respectively for specimens 1 and 2. Interestingly, these maximum stresses occur near the middle span of the beam (Figs. 4, 6).

The main objective of a beam with variable cross section is to make the best use of material, in terms of economy. From detailed non-linear analyses it is obvious more material can be removed from a traditional leaf spring (by inscribing a parabolic shape, for example, as done here) with insignificant change in the response of the beam, in terms of stress and deflections.

CONCLUSIONS

An innovative parabolic leaf spring has been designed and analyzed solving highly non-linear differential equations. The effects of two vitally important factors, namely, the end-shortening and geometric nonlinearity, on the response of parabolic shaped variable cross section, have been demonstrated by numerical analysis. Nonlinear solution plays vital role in determining the true stresses in highly flexible structures.

Of course, manufacturing difficulty may arise for such a proposed contour of the parabolic leaf spring. But, it is found that the response, in terms of stress and deflection, of the proposed parabolic leaf spring is not significantly changed from that of a traditional leaf spring. Therefore, it justifies the use of such a parabolic contour, especially, in terms of economy and light weight of the leaf spring.

Results of another research group have been compared with those generated by present numerical scheme and found to be close. It verifies the soundness of the numerical scheme for the proposed parabolic leaf spring.

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