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# MHD FREE CONVECTION FLOW ALONG A VERTICAL FLAT PLATE WITH THERMAL CONDUCTIVITY AND VISCOSITY DEPENDING ON TEMPERATURE

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## Abstract:

*In this present work the effects of temperature dependent viscosity and thermal conductivity on the coupling of conduction and Joule heating with MHD free convection flow along a semi-infinite vertical flat plate have been analyzed. The governing boundary layer equations with associated boundary conditions for this phenomenon are transformed to non-dimensional form using the appropriate variables. By the help of the implicit finite difference method with Keller–box scheme the resulting non-linear system of partial differential equation is then solved numerically. The purpose of this paper is to study the skin friction coefficient, the surface temperature, the velocity and the temperature profiles over the whole boundary layer for different values of the Prandtl number Pr, the magnetic parameter M, the thermal conductivity variation parameter γ, the viscosity variation parameter ε and the Joule heating parameter J. The results indicate that the flow pattern, temperature field and rate of heat transfer are significantly dependent on the above mentioned parameters. The local skin friction co-efficient and the surface temperature profiles for different values of ε are compared with previously published works and are found to be in good agreement.*

***Keywords:*** Viscosity, thermal conductivity, Joule heating, MHD, conduction, free convection.

## Nomenclature

| | |
|---|---|
| $b$, $C_p$ | plate thickness [m] and specific heat at constant pressure [JKg$^{-1}$K$^{-1}$] |
| $C_{fx}$, $C_f$ | local and general skin friction coefficient |
| $f$ | dimensionless stream function |
| $g$ , $h$ | acceleration due to gravity [ms$^{-2}$] and dimensionless temperature co-efficient |
| $H_0$ , $J$ | applied magnetic field strength and Joule heating parameter |
| $L$ , $M$ | reference length [m] and magnetic parameter |
| $p$, $Pr$ | conjugate conduction parameter and Prandtl number |
| $T$ | temperature of the interface [K] |
| $T_b$, $T_f$ | temperature outside the plate and of the fluid respectively [K] |
| $T_\infty$ | temperature of the ambient fluid [K] |
| $\bar{u}$ , $\bar{v}$ | velocity component along $\bar{x}$ , $\bar{y}$ direction [ms$^{-1}$] |
| $u$ , $v$ | dimensionless velocity component along $x$, $y$ direction |
| $\bar{x}$ , $\bar{y}$ | Cartesian co-ordinates [m] |
| $x$ , $y$ | dimensionless Cartesian co-ordinates |

### *Greek symbols*

| | |
|---|---|
| $\beta$ | volumetric coefficient of thermal expansion [K$^{-1}$] |
| $\gamma$ | thermal conductivity variation parameter |
| $\eta$ | similarity variable |
| $\theta$ | dimensionless temperature profile |
| $\kappa_s$ , $\kappa_f$ | thermal conductivity of the solid and fluid respectively [Wm$^{-1}$K$^{-1}$] |
| $\kappa_\infty$ | thermal conductivity of the ambient fluid [Wm$^{-1}$K$^{-1}$] |
| $\mu$ | dynamic viscosity of the fluid [Kgm$^{-1}$s$^{-1}$] |
| $\rho$ | density of the fluid [Kgm$^{-3}$] |
| $\tau_w$ , $\psi$ | shearing stress and stream function respectively |
| $\nu$ | kinematic viscosity of the fluid [m$^2$s$^{-1}$] |
| $\mu_f$, $\mu_\infty$ | viscosity of the fluid and ambient fluid respectively [Kgm$^{-1}$s$^{-1}$] |
| $\delta$, $\alpha$ | proportionality constant of thermal conductivity and viscosity variation parameters respectively |
| $\varepsilon$ | viscosity variation parameter |

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## 1. Introduction

The type of most common body force, which acts on a fluid, is due to gravity so that the body force can be defined as in magnitude and direction by the acceleration due to gravity. The study of fluid flows is of paramount importance in engineering. For example, the unsteady free convection flow along a vertical plate has been given considerable interest, because of its application in devices which are cooled by natural convection, as in the case of electrical heaters and transformers. Joule heating in electronics and in physics refers to the increase in temperature of a conductor as a result of moving electrons colliding with atoms, whereupon momentum is transferred to the atom, increasing its kinetic energy. Natural convection flow is often encountered in the study of the structure of stars and planets or in cooling of nuclear reactors. A considerable amount of research has been accomplished on the effects of electrically conducting fluids such as liquid metals, water mixed with a little acid and others in the presence of transverse magnetic field on the flow and heat transfer characteristics over various geometries. For the fluids, which are important in the theory of lubrication, the heat generated by the internal friction and the corresponding rise in temperature do affect the viscosity and thermal conductivity of the fluid and they can no longer be regarded as constant. Pozzi and Lupo (1988) rigorously investigated the coupling of conduction with laminar convection along a flat plate. The entire thermo-fluid-dynamic field was studied by means of two expansions. The first one, describing the field in the lower part of the plate, was a regular series the radius of convergence of which was determined by means of Pade approximant techniques. The second expansion, an asymptotic one required a different analysis because of the presence of eigensolutions.

Hossain (1992) analyzed the viscous and Joule heating effects on MHD free convection flow with variable plate temperature. In his paper, temperature varied linearly with the distance from the leading edge and in the presence of uniformly transverse magnetic field. The equations governing the flow were solved and the numerical solutions were obtained for small Prandtl numbers, appropriate for coolant liquid metal, in the presence of a large magnetic field. Merkin and Pop (1996) studied the conjugate free convection on a vertical surface. MHD free convection flow of visco-elastic fluid past an infinite porous plate was simulated numerically by Chowdhury and Islam (2000). Elbashbeshy (2000) analyzed the free convection flow along a vertical plate, taking into account the variation of the viscosity and thermal diffusivity with temperature in the presence of the magnetic field. Ahmad and Zaidi (2004) explored the magnetic effect on oberbeck convection through vertical stratum. Mamun et al. (2007) considered combined effect of conduction and viscous dissipation on MHD free convection flow along vertical flat plate.

Alim et al. (2007) investigated the Joule heating effect on the coupling of conduction with MHD free convection flow from a vertical flat plate. Rahman et al. (2008) presented effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat conduction. In his work, the numerical calculation were proceed in finite-difference method and the velocity, temperature, local skin friction co-efficient and surface temperature profiles were shown by the effect of various parameters. Alim et al. (2008) studied the combined effect of viscous dissipation & Joule heating on the coupling of conduction & free convection along a vertical flat plate. Molla et al. (2009) considered the natural convection laminar flow with temperature dependent viscosity and thermal conductivity along a vertical wavy surface. Numerical study on a vertical plate with variable viscosity and thermal conductivity has been reported by Palani and Kim (2010)**.** The Crank-Nicolson type of finite difference scheme was used in his work. An isothermal vertical plate was immersed in a fluid with free convection. The effects of various parameters on velocity, temperature, shear stress and rate of heat transfer were discussed.

In all the above studies the effects of variable viscosity and thermal conductivity with Joule heating have not been considered. The present study is used to deal with this problem. It is known that the physical properties of the fluid (thermal conductivity and viscosity) may change significantly with temperature. Numerical results of the local skin friction coefficient and the surface temperature profile for different values of $\varepsilon$ are presented in tabular form. In the following section derivation of the governing equations for the flow and heat transfer and the method of solutions along with the results and discussion are presented.

## 2. Physical Model

A steady, two-dimensional natural convection flow of an electrically conducting, viscous and incompressible fluid with variable viscosity and thermal conductivity along a semi-infinite vertical flat plate of thickness $b$ (**Fig.1**) has been considered here. It is assumed that heat is transferred from the outside surface of the plate, which is maintained at a constant temperature $T_b$, where $T_b > T_\infty$, temperature of the ambient fluid. A uniform magnetic field of strength $H_0$ is imposed along the $\bar{y}$ -axis i.e. normal direction to the surface and $\bar{x}$ - axis is taken along the flat plate.

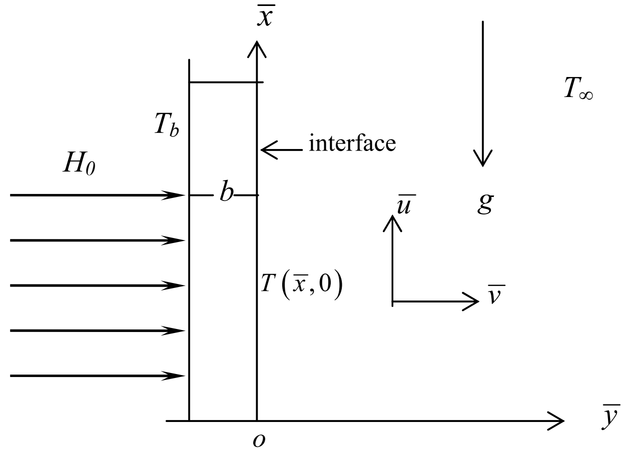


**Fig. 1**: Physical model

## 3. Mathematical Formulation

The governing equations of such laminar flow with variable viscosity and thermal conductivity along a vertical flat plate under the Boussinesq approximations can be written as

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0 \tag{1}$$

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right)=\frac{1}{\rho}\frac{\partial}{\partial \bar{y}}\left(\mu\frac{\partial \bar{u}}{\partial \bar{y}}\right)+g\beta(T_f-T_\infty)-\frac{\sigma H_0^2\,\bar{u}}{\rho} \tag{2}$$

$$\bar{u}\frac{\partial T_f}{\partial \bar{x}}+\bar{v}\frac{\partial T_f}{\partial \bar{y}}=\frac{1}{\rho C_p}\frac{\partial}{\partial \bar{y}}(\kappa_f\frac{\partial T_f}{\partial \bar{y}})+\frac{\sigma H_0^2}{\rho C_p}\bar{u}^2 \tag{3}$$

Here $\beta=-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T_f}\right)_p$ is coefficient of thermal expansion.

The variable thermal conductivity, which is used by Rahman et al. (2008), as follows

$$\kappa_f=\kappa_\infty[1+\delta(T_f-T_\infty)]$$

where $\kappa_\infty$ is thermal conductivity of the ambient fluid and $\delta$ is defined as $\delta=\frac{1}{\kappa_f}\left(\frac{\partial \kappa}{\partial T}\right)_f$.

The appropriate boundary conditions are

$$\left.\begin{array}{ll}\bar{u}=0, & \bar{v}=0\\ T_f=T(\bar{x},0), & \frac{\partial T_f}{\partial \bar{y}}=\frac{\kappa_s}{b\kappa_f}(T_f-T_b)\end{array}\right\}\quad on\quad \bar{y}=0,\ \bar{x}>0 \tag{4}$$

$$\bar{u}\to 0,\ T_f\to T_\infty\ \ as\quad \bar{y}\to\infty,\quad \bar{x}>0$$

The viscosity variation chosen in this study is

$$\mu=\frac{\mu_\infty}{1+\alpha\left(T_f-T_\infty\right)},\ \text{(see Molla et al. (2009))} \tag{5}$$

Here $\alpha=\frac{1}{\mu_f}\left(\frac{\partial \mu}{\partial T}\right)_f$ and $\mu_f$ , $\mu_\infty$ are viscosity of the fluid and ambient fluid.

It is observed that Equations (2) and (3) together with the boundary conditions (4) are non-linear partial differential equations. Then Equations (1) to (3) will be non-dimensionalized by using the dimensionless variables given below

$$x=\frac{\bar{x}}{L},\ y=\frac{\bar{y}}{L}Gr^{1/4},\ u=\frac{\bar{u}L}{\nu}Gr^{-1/2},\ v=\frac{\bar{v}L}{\nu}Gr^{-1/4},\ \theta=\frac{T_f-T_\infty}{T_b-T_\infty},\ Gr=\frac{g\beta L^3(T_b-T_\infty)}{\nu^2}$$

where $Gr$ is the Grashof number, $L=\dfrac{\nu^{2/3}}{g^{1/3}}$ is reference length , $\theta$ is the dimensionless temperature, $\nu=\mu/\rho$ is kinematic viscosity.

Substituting the above relations into Equations (1) to (3), the non-dimensional equations are written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+Mu=\frac{-\varepsilon}{(1+\varepsilon\theta)^2}\frac{\partial u}{\partial y}\frac{\partial\theta}{\partial y}+\frac{1}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial y^2}+\theta \tag{7}$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{1}{Pr}(1+\gamma\theta)\frac{\partial^2\theta}{\partial y^2}+\frac{\gamma}{Pr}\left(\frac{\partial\theta}{\partial y}\right)^2+Ju^2 \tag{8}$$

where $Pr=\dfrac{\mu\, C_p}{\kappa_\infty}$ is the Prandtl number, $M=\dfrac{\sigma\, {H_0}^2L^2}{\mu\, Gr^{1/2}}$ is the magnetic parameter, $\gamma=\delta(T_b-T_\infty)$ is the thermal conductivity variation parameter and $J=\dfrac{\sigma\, {H_0}^2\,\nu\, Gr^{1/2}}{\rho C_p(T_b-T_\infty)}$ is the Joule heating parameter.

Equation (5) becomes $\dfrac{\mu}{\mu_\infty}=\dfrac{1}{1+\varepsilon\theta}$ where viscosity variation parameter is $\varepsilon=\dfrac{1}{\mu_f}\left(\dfrac{\partial\mu}{\partial T}\right)_f(T_b-T_\infty)$.

The corresponding boundary conditions (4) then obtain the following form

$$\begin{aligned}&u=0,\ v=0,\ \theta-1=(1+\gamma\,\theta)\ p\frac{\partial\theta}{\partial y}\quad on\ \ y=0,\ x>0\\ &u\to 0,\ \theta\to 0\quad as\ \ y\to\infty,\ x>0\end{aligned} \tag{9}$$

where $p=\left(\dfrac{\kappa_\infty b}{\kappa_s L}\right)Gr^{1/4}$ is the conjugate conduction parameter.

The magnitude of $p$ governs as magnetic parameter and thermal conductivity variation parameter in this problem.

To solve Equations (7) and (8) subject to the boundary conditions (9) the transformations are proposed by Merkin & Pop (1996) written below

$$\begin{aligned}\psi&=x^{4/5}\,(1+x)^{-1/20}f(x,\eta)\\ \eta&=y\,x^{-1/5}\,(1+x)^{-1/20}\\ \theta&=x^{1/5}\,(1+x)^{-1/5}h(x,\eta)\end{aligned} \tag{10}$$

here $\eta$ is the similarity variable and $\psi$ is the stream function which satisfies the continuity equation and is related to the velocity components in the usual way as $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$. Thus the following equations are obtained

$$\begin{aligned}&\frac{1}{1+\varepsilon\{x/(1+x)\}^{1/5}h}f'''-\frac{\varepsilon\{x/(1+x)\}^{1/5}}{\left(1+\varepsilon\{x/(1+x)\}^{1/5}h\right)^2}h'f''+\frac{16+15x}{20(1+x)}ff''-\frac{6+5x}{10(1+x)}f'^2\\ &-Mx^{2/5}(1+x)^{1/10}f'+h=x\left(f'\frac{\partial f'}{\partial x}-f''\frac{\partial f}{\partial x}\right)\end{aligned} \tag{11}$$

$$\frac{1}{Pr}h'' + \frac{\gamma}{Pr}\left(\frac{x}{1+x}\right)^{1/5} h\,h'' + \frac{\gamma}{Pr}\left(\frac{x}{1+x}\right)^{1/5} h'^{\,2} + \frac{16+15x}{20(1+x)} f\,h'$$
$$+ J\,x^{7/5}(1+x)^{1/10} f'^{2} - \frac{1}{5(1+x)} f'h = x\left(f'\frac{\partial h}{\partial x} - h'\frac{\partial f}{\partial x}\right) \quad (12)$$

where prime denotes partial differentiation with respect to $\eta$. The boundary conditions as mentioned in Equation (9) then take the form given below

$$f(x,0) = f'(x,0) = 0$$
$$h'(x,0) = \frac{x^{1/5}(1+x)^{-1/5} h(x,0) - 1}{(1+x)^{-1/4} + \gamma\, x^{1/5}(1+x)^{-9/20} h(x,0)} \quad (13)$$
$$f'(x,\infty) \to 0,\; h(x,\infty) \to 0$$

## 4. Local Skin Friction Coefficient and Surface Temperature Profile

From the process of numerical computation, it is important to calculate the values of the surface shear stress in terms of the skin friction coefficient. This can be written in the following non-dimensional form

$$C_f = \frac{Gr^{-3/4} L^2}{\mu\nu}\tau_w \quad (14)$$

where $\tau_w [= \mu(\partial\bar{u}/\partial\bar{y})_{\bar{y}=0}]$ is the shearing stress. Using the transformation described in (10), the local skin friction co-efficient can be written as

$$C_{fx} = x^{2/5}(1+x)^{-3/20} f''(x,0) \quad (15)$$

The numerical values of the surface temperature profile are obtained from the relation given below

$$\theta(x,0) = x^{1/5}(1+x)^{-1/5} h(x,0) \quad (16)$$

## 5. Method of Solution

Along with the boundary condition (13), the solution of the parabolic differential equations (11) and (12) will be found by using the implicit finite difference method together with Keller- box scheme (1978) which is well documented by Cebeci and Bradshaw (1984) and widely used by Hossain et al. (2002). To apply the aforementioned method, Equations (11) and (12) are first converted into the following system of first order differential equations with dependent variables $u(\xi,\eta)$, $v(\xi,\eta)$, $p(\xi,\eta)$, $g(\xi,\eta)$ as $f' = u,\ u' = v,\ g' = p$ and

$$P_7 v' - P_8 p v + P_1 f\, v - P_2 u^2 - P_4 u + g = \xi\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) \quad (17)$$

$$\frac{1}{Pr}p' + P_1 f\, p - P_3 u\, g + \frac{P_5}{Pr} g\, p' + \frac{P_5}{Pr} p^2 + P_6 u^2 = \xi\left(u\frac{\partial g}{\partial \xi} - p\frac{\partial f}{\partial \xi}\right) \quad (18)$$

Here $\xi = x$, $h = g$ and

$$P_1 = \frac{16+15x}{20(1+x)},\ P_2 = \frac{6+5x}{10(1+x)},\ P_3 = \frac{1}{5(1+x)},\ P_4 = M\, x^{\frac{2}{5}}\,(1+x)^{\frac{1}{10}},\ P_5 = \left(\frac{x}{1+x}\right)^{\frac{1}{5}}\gamma,$$

$$P_6 = J\, x^{7/5}\left(1+x\right)^{1/10},\ P_7 = \frac{1}{1+\xi\, x^{1/5}\left(1+x\right)^{-1/5} g}\ ,\ P_8 = \frac{\xi\, x^{1/5}\left(1+x\right)^{-1/5}}{\left(1+\xi\, x^{1/5}\left(1+x\right)^{-1/5} g\right)^2}$$

The finite difference approximation according to box method then reduced the system of equation into non-linear system of difference equation. These are as follows

$$\frac{P_7}{2}\left(\frac{v_j^n - v_{j-1}^n}{h_j} + \frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) - P_8\left(pv\right)_{j-1/2}^{n-1/2} + P_1\left(fv\right)_{j-1/2}^{n-1/2} - P_2\left(u^2\right)_{j-1/2}^{n-1/2}$$
$$- P_4\, u_{j-1/2}^{n-1/2} + g_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\ \frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\ \frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right) \tag{19}$$

$$\frac{1}{2Pr}\left(\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + P_1\left(fp\right)_{j-1/2}^{n-1/2} - P_3\left(ug\right)_{j-1/2}^{n-1/2} +$$
$$+ \frac{P_5}{2Pr} g_{j-1/2}^{n-1/2}\left(\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + \frac{P_5}{Pr}\left(P^2\right)_{j-1/2}^{n-1/2} + P_6\left(u^2\right)_{j-1/2}^{n-1/2} \tag{20}$$
$$= \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\ \frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - p_{j-1/2}^{n-1/2}\ \frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)$$

The above equations are to be linearized by using Newton's Quasi-linearization method. Then linear algebraic equations can be written in a block matrix which forms a coefficient matrix. The whole procedure, namely reduction to first order followed by central difference approximations, Newton's Quasi linearization method and the block Thomas algorithm, is well known as Keller-box method.

## 6. Results and Discussion

The velocity, the temperature, the local skin friction coefficient and the surface temperature profiles obtained from the solutions of Equations (19) and (20) are depicted in Fig.2 to Fig.11. The values of the Prandtl number are considered to be 0.73, 1.73, 2.97 and 4.24. Numerical computation are carried out for a wide range of magnetic parameter $M$ = 0.01, 1.0, 2.5, 5.0, Joule heating parameter $J$ = 0.01, 0.5, 1.0, 1.3, thermal conductivity variation parameter $\gamma$ = 0.01, 0.31, 0.51, 0.77 and viscosity variation parameter $\varepsilon$ = 0.01, 2.0, 4.0, 6.0.

The interaction of the magnetic field and moving electric charge carried by the flowing fluid induces a force, which tends to oppose the fluid motion. In Fig. 2(a), it is shown that the magnetic field acting along the horizontal direction retards the fluid velocity with $\gamma$ = 0.3, $J$ = 0.01, $\varepsilon$ = 1.1 and $Pr$ = 0.73. Here position of peak velocity moves toward the interface with the increasing $M$. From Fig. 2(b), it can be observed that the temperature within the boundary layer increases for the increasing values of $M$. It is well known that thermal conductivity of solid is higher than fluid. For this reason, the magnetic field decreases the temperature gradient at the wall and increases the temperature in the flow region.

The variation of the local skin friction coefficient and the surface temperature for different values of $M$ at different positions are illustrated in Figs. 3(a) and (b), respectively. It is observed from Fig. 3(a) that the increased value of the magnetic parameter $M$ leads to a decrease the skin friction factor of 83.46%. Again Fig. 3(b) shows that the surface temperature increases due to the increasing values of $M$. The magnetic field acts against the flow and reduces the skin friction and produces the temperature at the interface.

The analysis of the effect of thermal conductivity variation parameter $\gamma$ on the velocity and the temperature profiles within the boundary layer with $M$ = 0.01, $J$ = 0.01, $\varepsilon$ = 1.0 and $Pr$ = 1.73 are provided in Figs. 4(a) and (b), respectively. As $\gamma = \delta(T_b - T_\infty)$, so growing values of $\gamma$ increase the temperature difference between outside the plate and ambient fluid. Then heat is transferred rapidly from plate to fluid. Thus velocity and the temperature profiles increase within the boundary layer with the increase of $\gamma$. It means that the velocity boundary layer and the thermal boundary layer thickness raise for higher value of $\gamma$. The maximum values of the velocity are 0.4131, 0.4397, 0.4563 and 0.4766 for $\gamma$ = 0.01, 0.31, 0.51 and 0.77, respectively and each of which occurs at $\eta$ = 0.9981. It is observed that the velocity increases by 13.32% when $\gamma$ grows up from 0.01 to 0.77. Furthermore, the highest values of the temperature are 0.7639, 0.7850, 0.7960 and 0.8076 for $\gamma$ = 0.01, 0.31, 0.51, and 0.77 respectively and each of which occurs at the surface. It is observed that the temperature profile increases by 5.41% due to raise of $\gamma$.

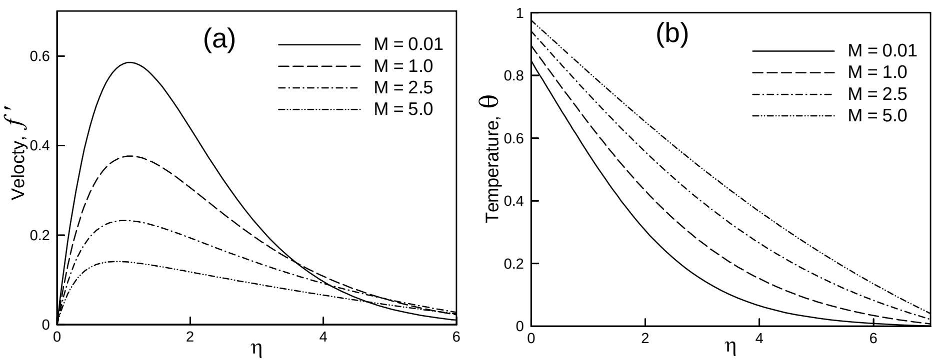


**Fig. 2.** (a) Velocity and (b) temperature profiles for different values of $M$ against $\eta$ with $\gamma$ = 0.3, $J$ = 0.01, $\varepsilon$ = 1.1 and $Pr$ = 0.73.

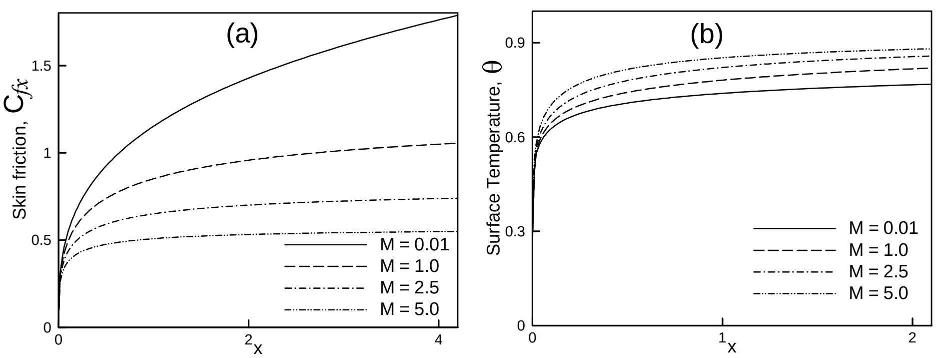


**Fig. 3.** (a) Skin friction and (b) surface temperature profile for varying of $M$ against $x$ with $\gamma$ = 0.3, $J$ = 0.01, $\varepsilon$ = 1.1 and $Pr$ = 0.73.

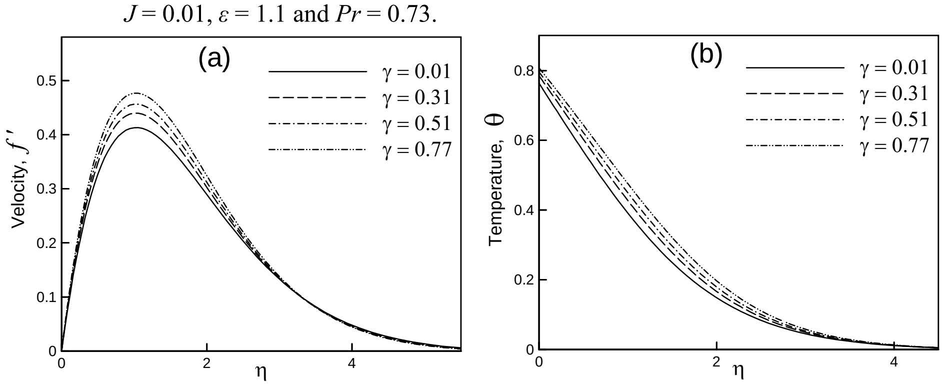


**Fig. 4.** (a) Velocity and (b) temperature profiles for varying of $\gamma$ against $\eta$ with $M$ = 0.01, $J$ = 0.01, $\varepsilon$ = 1.0 and $Pr$ = 1.73

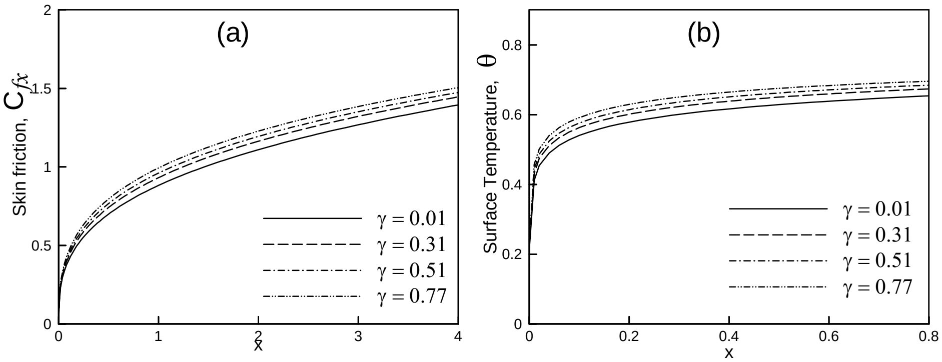


**Fig. 5.** (a) Skin friction and (b) surface temperature for varying of $\gamma$ against x with $M = 0.01$, $J = 0.01$, $\varepsilon = 1.0$ and $Pr = 1.73$.

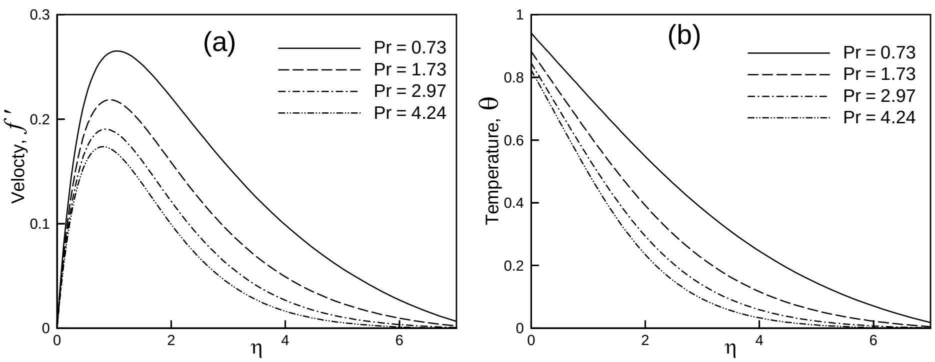


**Fig. 6.** (a) Velocity and (b) temperature profiles for different values of $Pr$ against $\eta$ with $\gamma = 0.3$, $J = 0.3$, $\varepsilon = 1.1$ and $M = 2.1$.

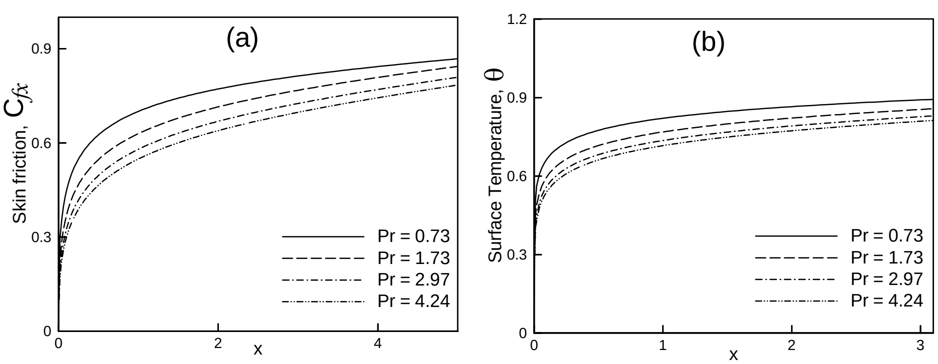


**Fig. 7.** (a) Skin friction and (b) surface temperature profile for different values of $Pr$ against $x$ with, $\gamma = 0.3$, $J = 0.3$, $\varepsilon = 1.1$ and $M = 2.1$

Figs. 5(a) & (b) demonstrate the influence of $\gamma$ with the same controlling parameters on the skin friction coefficient and the surface temperature profile. It is seen that skin friction coefficient raises monotonically along the upward direction of the plate for a particular value of $\gamma$. It is also observed that skin friction coefficient and surface temperature profile are enlarged by 0.97% and 6.03% respectively for the increasing values of $\gamma$ from 0.01 to 0.77. This is to be expected because the higher value for $\gamma$ accelerates the fluid flow and increases the temperature as mentioned in Figs. 4(a) & (b) correspondingly.

The velocity and the temperature profiles for influence of $Pr$ with $M = 2.1$, $J = 0.3$, $\varepsilon = 1.1$ and $\gamma = 0.3$ are displayed by Figs. 6(a) and (b). It is known that, $Pr$ is the ratio of viscous force and thermal force. So if $Pr$ increases then viscous force also enhances and this force acts against fluid motion. From Fig. 6(a), it can be observed that the velocity reduces as well as its position moves toward the interface with the increase in $Pr$. It is seen that it decreases by 34.51% when $Pr$ increases from 0.73 to 4.24. From Fig. 6(b), it is noticed that the temperature profiles modify downward with the increase in *Pr*. Again it devalues by 12.6% for escalating values of *Pr* at the interface. It is also observed that the temperature at the interface varies due to the conduction within the plate.

The effects of *Pr* on the skin friction coefficient and the surface temperature against *x* are exposed in Figs. 7(a), (b) respectively. It can be seen from Fig. 7(a) that skin friction coefficient increases monotonically for a particular value of *Pr*. Fig. 7(b) states that surface temperature decreases owing to the increase of *Pr*. It can also be noted that $C_{fx}$ and $\theta\,(x,\,0)$ decrease by 49.15% and 39.64% for the increasing values of *Pr* from 0.73 to 4.24 respectively.

The belongings of Joule heating parameter *J* on the velocity and the temperature profiles within the boundary layer with $M$ = 5.0, $\gamma$ = 0.3, $\varepsilon$ = 1.0 and $Pr$ = 0.73 are shown in Figs. 8 (a) and (b) respectively. Joule heating is the heating effects of conductors carrying currents. That's why the temperature of solid wall increases. So velocity and the temperature increase within the boundary layer with the increasing values of *J.* Moreover, the highest values of the velocity are 0.1404, 0.1426, 0.1450 and 0.1464 for $J$ = 0.01, 0.50, 1.0 and 1.3 respectively and each of which occurs at $\eta$ = 0.9423. It is known that Joule heating effect is very little in fluid. Thus velocity increases by only 4.1% when *J* increases from 0.01 to 1.3. Furthermore, the maximum values of the temperature are 0.9768, 0.9840, 0.9918 and 0.9967 for $J$ = 0.01, 0.5, 1.0, and 1.3 respectively and each of which occurs at the surface. It is pragmatic that it increases by 2% due to increase *J*.

Figs. 9(a) and (b) deal with manipulate of *J* on the skin friction coefficient and the surface temperature against *x*. It can be observed that the local skin friction coefficient and surface temperature profile increase monotonically for each value of *J*.

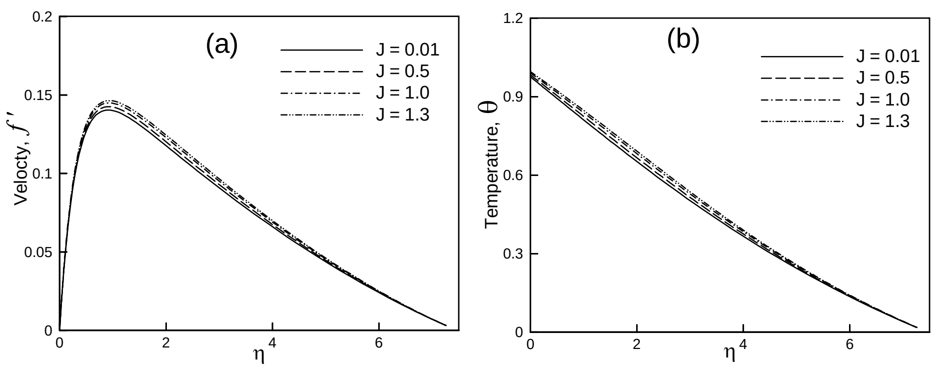


**Fig. 8.** (a) Velocity and (b) temperature profiles for different values of *J* against $\eta$ with $\gamma$ = 0.3, $Pr$ = 0.73, $\varepsilon$ = 1.0 and $M$ = 5.0.

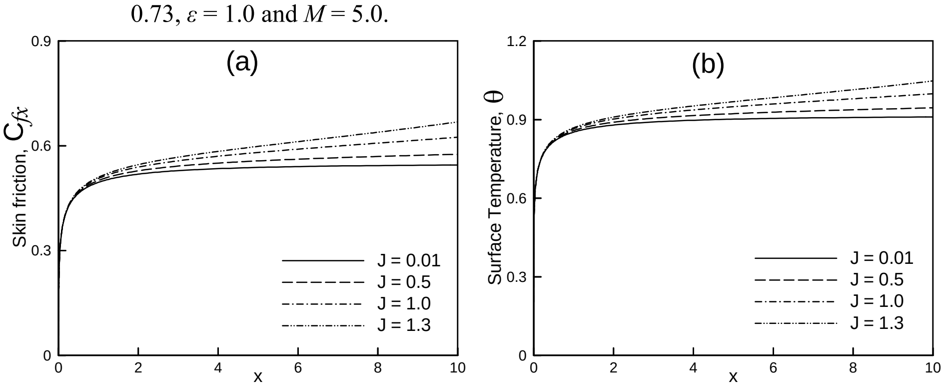


**Fig. 9.** (a) Skin friction and (b) surface temperature profile for different values of *J* against *x* with, $\gamma$ = 0.3, $Pr$ = 0.73, $\varepsilon$ = 1.0 and $M$ = 5.0.

The effects of viscosity variation parameter $\varepsilon$ on the velocity and the temperature profiles within the boundary layer with $M$ = 1.1, $J$ = 0.1, $\gamma$ = 0.1 and $Pr$ = 0.73 are revealed in Figs. 10(a) and (b), respectively. The highest values of velocity are 0.3123, 0.3755, 0.4056, 0.4247 for $\varepsilon$ = 0.01, 2.0, 4.0, 6.0 respectively. The velocity of the fluid increases along $\eta$ direction and reaches at utmost values which occurs between $\eta$ = 0.7 to 1.4, then all the profiles steadily decrease, cross each other near the point $\eta = 2.3$ and finally approach to zero, the asymptotic value. It is observed that the velocity increases by 26.5% when $\varepsilon$ enhances from 0.01 to 6.0 near to the surface. Again the peak values of temperature are 0.9081, 0.8920, 0.8858 and 0.8822 for $\varepsilon$ = 0.01, 2.0, 4.0 and 6.0 respectively and each of which occurs at the surface. In this case, temperature devalues by 2.85%.

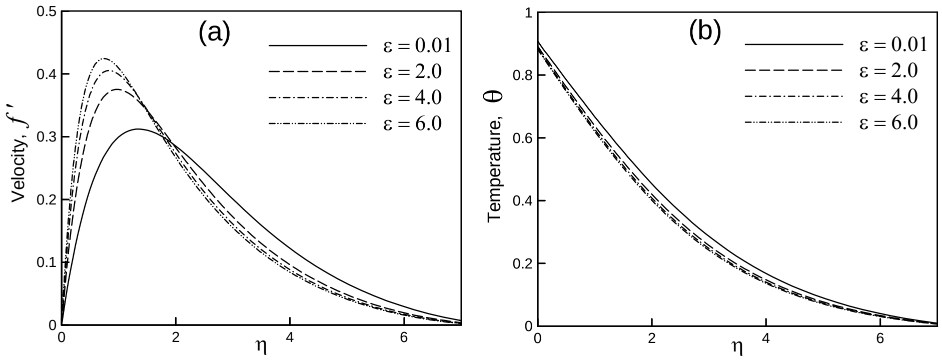


**Fig. 10.** (a) Velocity and (b) temperature profiles for varying of $\varepsilon$ against $\eta$ with $\gamma$ = 0.1, $Pr$ = 0.73, $J$ = 0.1 and $M$ = 1.1.

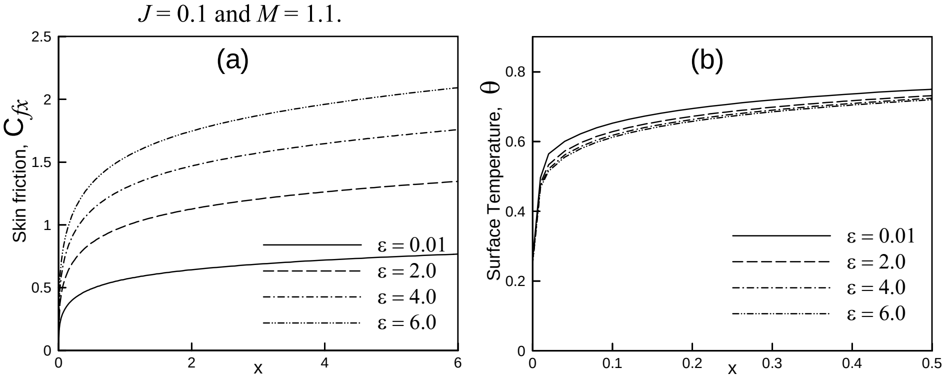


**Fig. 11.** (a) Skin friction and (b) surface temperature for varying of $\varepsilon$ against $\eta$ with$\gamma$ = 0.1, $Pr$ = 0.73, $J$ = 0.1 and $M$ = 1.1.

The influence of $\varepsilon$ on the skin friction coefficient and the surface temperature profile against x for the present problem is shown in Figs. 11(a), (b). It is established that increasing values of temperature decrease viscosity of the fluid. If heat is given in water, then it rises and becomes steam. Thus when $\varepsilon$ increases, then temperature and surface temperature profiles decrease. Also velocity profile increases, becomes highest and then decreases. It is seen that the skin friction coefficient increases monotonically along the upward direction of the plate. It is observed that $C_{fx}$ increase by 64.85% and $\theta(x, 0)$ decreases by 5.12% for increasing values of $\varepsilon$.

Table 1 demonstrates the numerical values of $C_{fx}$ and $\theta(x, 0)$ against $x$ for different values of viscous dissipation parameter $\varepsilon$ while $\gamma$ = 0.1, $M$ = 1.1, $J$ = 0.1 and $Pr$ = 0.73. It is observed that the values of the skin friction coefficient increases hurriedly at different location of $x$ for $\varepsilon$ = 0.01, 2.0, 4.0, 6.0. Near the axial position $x$ = 3.7803, the rate of increase of the local skin friction coefficient is 63.28% as $\varepsilon$ changes from 0.01 to 6.0. Also it is seen that the numerical values of the surface temperature profile reduce very slowly as $\varepsilon$ grows up. At the same axial position of $x$, the rate of decrease of the surface temperature profile is 0.63% as $\varepsilon$ changes from 0.01 to 6.0. It is obvious that $\varepsilon$ has significant effect on local skin friction coefficient and surface temperature profile.

## 7. Comparison with previous work and program validation

A comparison of the present numerical upshot of the skin friction coefficient and the surface temperature with those obtained by Pozzi and Lupo (1988) and Merkin and Pop (1996) is depicted in table 2. Here, the parameters $M$, $\gamma$, $\varepsilon$ and $J$ are ignored and the Prandtl number $Pr$ = 0.733 with $x^{1/5} = \xi$ is chosen. It is evidently seen that there is an tremendous concurrence among the current results with the solutions of Pozzi and Lupo (1988) and Merkin and Pop (1996).

**Table 1:** Skin friction coefficient and surface temperature profile against x for different values of $\varepsilon$ with $\gamma$ = 0.1, $M$ = 1.1, $J$ = 0.1 and $Pr$ = 0.73

| | $\varepsilon$ = 0.01 | | $\varepsilon$ = 2.0 | | $\varepsilon$ = 4.0 | | $\varepsilon$ = 6.0 | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $C_{fx}$ | $\theta$ | $C_{fx}$ | $\theta$ | $C_{fx}$ | $\theta$ | $C_{fx}$ | $\theta$ |
| 0.0100 | 0.1661 | 0.4966 | 0.2364 | 0.4810 | 0.2804 | 0.4735 | 0.3159 | 0.4682 |
| 0.8881 | 0.5544 | 0.7845 | 0.9687 | 0.7693 | 1.2630 | 0.7634 | 1.5021 | 0.7600 |
| 1.3356 | 0.5989 | 0.8087 | 1.0484 | 0.7962 | 1.3668 | 0.7915 | 1.6255 | 0.7888 |
| 2.5346 | 0.6691 | 0.8464 | 1.1733 | 0.8389 | 1.5301 | 0.8364 | 1.8200 | 0.8350 |
| 3.7803 | 0.7137 | 0.8704 | 1.2520 | 0.8665 | 1.6333 | 0.8654 | 1.9434 | 0.8649 |
| 4.9370 | 0.7444 | 0.8869 | 1.3062 | 0.8857 | 1.7045 | 0.8856 | 2.0286 | 0.8858 |
| 5.5785 | 0.7589 | 0.8948 | 1.3317 | 0.8948 | 1.7382 | 0.8953 | 2.0689 | 0.8957 |
| 6.6947 | 0.7814 | 0.9071 | 1.3714 | 0.9091 | 1.7905 | 0.9104 | 2.1316 | 0.9113 |
| 7.8683 | 0.8023 | 0.9187 | 1.4085 | 0.9226 | 1.8396 | 0.9246 | 2.1905 | 0.9259 |
| 8.7021 | 0.8161 | 0.9263 | 1.4329 | 0.9315 | 1.8720 | 0.9340 | 2.2295 | 0.9356 |

**Table 2:** Comparison of the current numerical results of skin friction coefficient and surface temperature with those obtained by Pozzi and Lupo and Merkin and Pop

| | Skin friction coefficient $C_{fx}$ | | | Surface temperature $\theta\,(x, 0)$ | | |
|---|---|---|---|---|---|---|
| $x^{1/5} = \xi$ | Pozzi and Lupo (1988) | Merkin and Pop (1996) | Present result | Pozzi and Lupo (1988) | Merkin and Pop (1996) | Present result |
| 0.7 | 0.430 | 0.430 | 0.424 | 0.651 | 0.651 | 0.651 |
| 0.8 | 0.530 | 0.530 | 0.529 | 0.684 | 0.686 | 0.688 |
| 0.9 | 0.635 | 0.635 | 0.634 | 0.708 | 0.715 | 0.716 |
| 1.0 | 0.741 | 0.745 | 0.743 | 0.717 | 0.741 | 0.741 |
| 1.1 | 0.829 | 0.859 | 0.859 | 0.699 | 0.762 | 0.764 |
| 1.2 | 0.817 | 0.972 | 0.973 | 0.640 | 0.781 | 0.781 |

## 8. Conclusion

A computational analysis of the effect of viscosity and thermal conductivity variation owing to temperature on the combination of conduction and Joule heating with MHD natural convection flow along a vertical flat plate has been offered. The subsequent outcome may be drawn as:

1. The velocity profile within the boundary layer radically increases for decreasing values of the magnetic parameter $M$, viscosity variation parameter $\varepsilon$, Prandtl number $Pr$ and increasing values of the thermal conductivity variation parameter $\gamma$ and the Joule heating parameter $J$.
2. The temperature profile within the boundary film raises considerably for the increasing values of $M$, $\gamma$, $J$ and falling values of $\varepsilon$ and $Pr$.
3. The local skin friction coefficient decreases noticeably for the growing values of $M$, $Pr$ and decreasing values of $\gamma$, $\varepsilon$ and $J$.
4. An increase in the values of $\gamma$, $J$ and $M$ guides to a significant increase in the surface temperature. Moreover this profile decreases due to increase of $Pr$ and $\varepsilon$.

## References

Ahmad, N., Zaidi, H. N. (2004): Magnetic Effect on Oberbeck Convection Through Vertical Stratum, Proc. 2nd BSME-ASME Int. Conf. on Thermal Eng., pp.157-168.

Alim, M. A., Alam, M. and Mamun, A. A. (2007): Joule Heating Effect on the Coupling of Conduction with MHD Free Convection Flow from a Vertical Flat Plate, Nonlinear Analysis: Modell. and Cont., Vol. 12, No. 3, pp. 307-316.

Alim, M. A., Alam, Md. M., Mamun, A. A. and Hossain, Md. B. (2008): Combined Effect of Viscous Dissipation & Joule Heating on the Coupling of Conduction & Free Convection along a Vertical Flat Plate, Int. Commun. in Heat & Mass Transfer, Vol. 35, No. 3, pp. 338-346.

Cebeci, T. and Bradshaw, P. (1984): Physical and Computational Aspects of Convective Heat Transfer, Springer, New York.

Chowdhury, M. K. and Islam, M. N. (2000): MHD Free Convection Flow of Visco-elastic Fluid past an Infinite Porous Plate, Heat Mass Transfer, Vol. 36, No. 5, pp. 439-447. doi:10.1007/s002310000103

Elbashbeshy, E. M. A. (2000): Free Convection Flow with Variable Viscosity and Thermal Diffusivity along a Vertical Plate in the Presence of Magnetic Field, Int. J. Eng. Science, Vol. 38, No. 2, pp. 207-213. doi:10.1016/S0020-7225(99)00021-X

Hossain, M. A.(1992): The Viscous and Joule Heating Effects on MHD Free Convection Flow with Variable Plate Temperature, Int. J. Heat Mass Transfer, Vol. 35, No. 12, pp. 3485-3487. doi:10.1016/0017-9310(92)90234-J

Hossain, M. A., Kabir, S. and Rees, D. A. S.(2002): Natural Convection Of Fluid With Variable Viscosity From A Heated Vertical Wavy Surface, Z. angew. Math. Phys., Vol. 53, pp. 48-52.

Keller, H. B. (1978): Numerical Methods in Boundary Layer Theory, Annual Rev. Fluid Mechanics, Vol. 10, pp. 417-433. doi:10.1146/annurev.fl.10.010178.002221

Mamun, A. A., Azim, N. H. Md. and Maleque, Md. A. (2007): Combined Effect of Conduction and Viscous Dissipation on MHD Free Convection Flow along a Vertical Flat Plate, Journal of Naval Archit. and Marine Eng., Vol. 4, No. 2, pp. 87-98.

Merkin, J. H. and Pop, I. (1996): Conjugate Free Convection on a Vertical Surface, Int. J. Heat Mass Transfer, Vol. 39, pp.1527-1534. doi:10.1016/0017-9310(95)00238-3

Molla, Md. M., Hossain, M. A., Reddy Gorla, R. S. (2009): Natural Convection Laminar Flow with Temperature Dependent Viscosity and Thermal Conductivity Along a Vertical Wavy Surface, Int. J. of Fluid Mech. Research, Vol. 36, pp. 272-288.

Palani, G., Kim, Kwang-Yong (2010): Numerical Study On A Vertical Plate With Variable Viscosity And Thermal Conductivity, Arch Appl. Mech., Vol. 80, No. 7, pp. 711-725.

Pozzi, A. and Lupo, M. (1988): The Coupling of Conduction with Laminar Convection along a Flat Plate, Int. J. of Heat and Mass Transfer, Vol. 31, No. 9, pp. 1807-1814. doi:10.1016/0017-9310(88)90195-0

Rahman, M. M., Mamun, A. A., Azim, M. A. and Alim, M. A. (2008): Effects of Temperature Dependent Thermal Conductivity on MHD Free Convection Flow along a Vertical Flat Plate with Heat Conduction, Nonlinear Analysis: Modell. and Cont., Vol. 13, No. 4, pp. 513-524.