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INVENTORY OPTIMIZATION MODEL OF DETERIORATING ITEMS WITH NONLINEAR RAMPED TYPE DEMAND FUNCTION

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Abstract:

Inventory models for deteriorating items of a supply chain is of critical essence to the maritime industry. This study proposes a real and economically efficient multicriteria inventory policy of inventory models for deteriorating items of a supply chain with nonlinear ramp functions and permissible delay in payment under inflation, two conditions Shortages Followed by Inventory (SFI) and Inventory Followed by Shortages (IFS) have been considered for the formulation of models. Both the models consist of ordering cost, unit cost, deterioration cost, shortage cost and holding cost with replenishment where delay in payments is allowed. The development of these models is to minimize the total average cost per unit time. In order to validate the models, numerical examples have been considered and the sensitivity of several major parameters of exponential and quadratic functions is analyzed. From the numerical result, it is clear that the cost per unit time of IFS model decreases with the increment of the values of parameters of the quadratic function and holding cost whereas, with the increment of the values of parameters of the exponential function as well as holding cost per unite time, the cost per unit time of SFI model decreases. However, if the values of the parameters of the exponential functions as well as ordering cost and shortage cost increase, the cost per unit time of IFS model increases. On the other hand, it will also increase considering SFI condition, if the values of parameters of the quadratic function as well as ordering cost and shortage cost increase. Finally, it is observed that the model considering IFS case works better than SFI model up to a certain level.

Keywords: Inventory, Ramped type demand, Exponential function, Deterioration, Holding Cost, Cost minimization.

NOMENCLATURE

T	Length of the replenishment cycle (year)
m	Permissible delay in setting the account (year)
I(t)	Inventory level at time t
k	Ordering cost of inventory per order ($/order)
h	Unit holding cost per unit time excluding interest charged ($/unit/unit time)
P	Unit purchase cost per item ($/item)
s	Unit shortage cost per item ($/item unit short)
θ	Deteriorating rate for inventory where $0 \le \theta < 1$
I_e	Interest earned per year ($/year)
I_r	Annual rate for the unsold stock to be financed
C_6	Unit cost incurred from the deterioration of one item ($/unit/unit time) considering IFS condition
C_3	Unit cost incurred from the deterioration of one item ($/unit/unit time) considering SFI condition
D(t)	Time dependent demand rate (unit/year)

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1. Introduction

Global freight transportation system that includes ocean and coastal routes, inland waterways, railways, roads, intermodal, and air freight ensures shipment efficiently. However, international maritime transportation is more commonly a complement to other modes of transportation. The maritime industry plays a very significant role in the socio-economic development of any country. Land locked countries greatly depend on countries with ports and harbour facilities to either import or export their goods. Inventory management is another significant for maritime industry. It plays important role to development the efficient freight transport as a well as to make the supply chain smooth. Different types of inventory model are used in literature to improve the development of freight transport.

Due to the growing burden for remaining competitive in the worldwide market place, inventory management has become a major challenge for companies to reduce costs and to improve customer service. Most manufacturing and service companies large or small face some type of inventory. The items in inventory are there to make the day-to-day operations of the company more efficient and the flow of goods smoother. While too much inventory reduces capital that can be directed into other aspects of the company's operations, not enough causes other serious problems such as loss of sales and customer frustration. Many physical goods are deteriorated over time. Electronics goods, radio-active substances, photographic film, grain etc. deteriorate through a gradual loss of potential with the passage of time. Fruits, vegetables, foodstuff etc. suffer depletion by direct spoilage when they are stored for a long time. Gasoline, petroleum, alcohol, etc. are highly perishable items. Thus, decay or deterioration of physical goods held in stock is a very realistic area of researchers.

An inventory model with a constant deterioration rate was first proposed in (Ghare and Schrader, 1963). With the consideration of a variable deterioration rate, this model was further extended, and modified in (Covert and Philip, 1973). Shah (Shah, 1977) proposed a generalized economic order quantity (EOQ) model with a variable deterioration rate, and the allowance of the complete backlogging of the unsatisfied demand. A two- warehouse inventory model with a constant demand rate for deteriorating items under inflation was proposed in (Yang, 2004) by considering two alternative situations. In the first situation, the author assumed a model which started with an instant order, and ended with shortages. In the second situation, the model began with shortages, and terminated without shortages. Considering shortages follows inventory (SFI) condition, this model was extended by Yang (Yang, 2006) with the incorporation of partial backlogging. The authors of (Jaggi et al., 2011) proposed an inventory model considering a linear time- dependent demand rate for deteriorating items, inflation and partial backlogging rate in a two- warehouse system. An extensive and comprehensive review on advancements in the field of inventory control of deteriorating items was proposed in (Bakker et al., 2012). In (Yang, 2012), two- warehouse partial backlogging inventory models were developed with three- parameters Weibull distribution deterioration under inflation. Considering partial backlogging, single-warehouse inventory models were developed in (Taleizadeh et al., 2012; Taleizadeh et al., 2013a; Taleizadeh et al., 2013b). On the other hand, Yang and Chang (Yang and Chang, 2013) proposed a two -warehouse inventory model for deteriorating items with permissible delay in payment under inflation. Considering, discount and without discount policy, inventory models for deteriorating items with nonlinear demand were estimated in (Moussawi-Haider et al., 2014). However, in (Mondal et al., 2016), inventory analysis was studied for deteriorating items with nonlinear demand function. Recently, the authors (Ziming et al., 2018) proposed a hybrid nonlinear regression and support vector machine (SVM) model. In this work, Off-peak hours, peak hours in peak months and peak hours in off-peak months are distinguished and different methods are designed to improve the forecast accuracy. The authors (Mahdi and Seyed, 2022) presented a multi-item Dynamic inventory model for deteriorating items with limited carrier capacity. The proposed research considered the carrier, which transports the order has limited capacity and the quantity of orders cannot be infinite.
The inventory model, taking the conditions inventory followed by shortages (IFS) and shortages followed by inventory (SFI), has been developed with the help of a pair of ramp type functions also known as piece-wise functions. It has been developed for deterioration items with nonlinear trend in demand, shortages, and permissible delay-in-payment, because in a competitive market, the demand of some product may increase due to the consumer's preference on some eye-catching product such as fashionable clothes, electronic equipment, and delicious foods.

In this study, exponential, and quadratic functions are assumed as demand functions where quadratic function comes after exponential function. For the IFS condition, the developed model starts with zero inventory level and starts increasing as well as ends with a specific level of inventory after a certain time from where inventory starts decreasing to reach at the zero-inventory level from which it started to increase. Deterioration cost, and holding cost have been evaluated for positive stock for quadratic function, whereas both of these costs have not been considered for the exponential function. Due to the existence of inventory, the quadratic function's shortage cost has been ignored, whereas it has been calculated for exponential function. On the other hand, for the SFI condition, the model starts with a specific level of inventory and ends with zero-inventory after a certain time from where inventory starts accumulating to reach at the same specific level from which it started decreasing. Deterioration cost, and holding cost have been evaluated for positive stock for exponential function, whereas both of these costs have not been considered for the quadratic function. Shortage cost has been calculated for quadratic function, whereas it has been ignored for the exponential function due to existence of inventory.

However, for both cases, the model has determined permissible delay period within the credit period which is settled by the whole-seller or distributor to the retailer or customer. Meanwhile, the annual unsold stock financed has also been evaluated beyond the credit period for permissible delay payment. In SECTION 2, the derivation of mathematical models considering IFS and SFI conditions is evaluated with the proof of appropriate theorem. Numerical example has been considered in SECTION 3 for illustrating the developed model setting the goal to minimize the total average cost per unit time and sensitivity analysis has also been studied on account of major parameters for the comparison between IFS and SFI model. In SECTION 4, the numerical results have been upheld by sufficient graphical presentations consisted of the effects of major parameters on cost per unit time.

2.Mathematical Model Formulation

With the help of a pair of nonlinear demand functions, the mathematical models considering IFS and SFI have been formulated. For the generation of models, few assumptions have been taken and a Theorem has been established for the validation of our developed models that both of them are successfully minimize the total average cost per unit time.

2.1 Assumptions

The following assumptions are made for the development of the models:

- The inventory system involves single type of item.
- Replenishment occurs instantaneously.
- The demand rate for the item is represented by a nonlinear, and continuous function of time.
- Shortages are considered, and completely backlogged.
- Delay-in-payments are considered.
- No interest is to be charged after commencement of shortages.
- Interest is to be earned for the unsold stock by a buyer after permissible delay-in-payment for the positive stock.
- The planning period is of infinite length.

2.2 IFS model derivation

Under the category inventory followed by shortages (IFS) model:

- Exponential function has been taken at the first stage in Fig. 1 over the time interval $[0, T_1)$ with initial inventory level zero at the beginning of time, whereas quadratic function has been taken at the second stage over the time interval $[T_1, T]$ with inventory level I_0 at T_1 after accumulation.
- Deteriorated items have been evaluated at the second stage in Fig. 1 over the time interval $[T_1, T]$, whereas it has been ignored at the first stage.
- Shortage cost (SC) has been evaluated at the first stage in Fig. 1 over the time interval $[0, T_1)$. On the other hand, it is zero at the second stage as inventory exists.
- Negative signs are given in front of exponential, and quadratic demand functions in differential equation to express shortage, and decrease respectively in inventory.
- Holding cost (HC) has been considered at the second stage in Fig. 1, whereas it has been avoided at the first stage in Fig. 1 due to inventory shortages.

- Credit period for permissible delay in payment has been considered within positive stock over the time interval [T_1, m] in Fig. 1.
- Unsold stock is financed with an annual rate I_r over the time interval [m, T] in Fig. 1 which is beyond the credit period [T_1, m] with the positive stock.

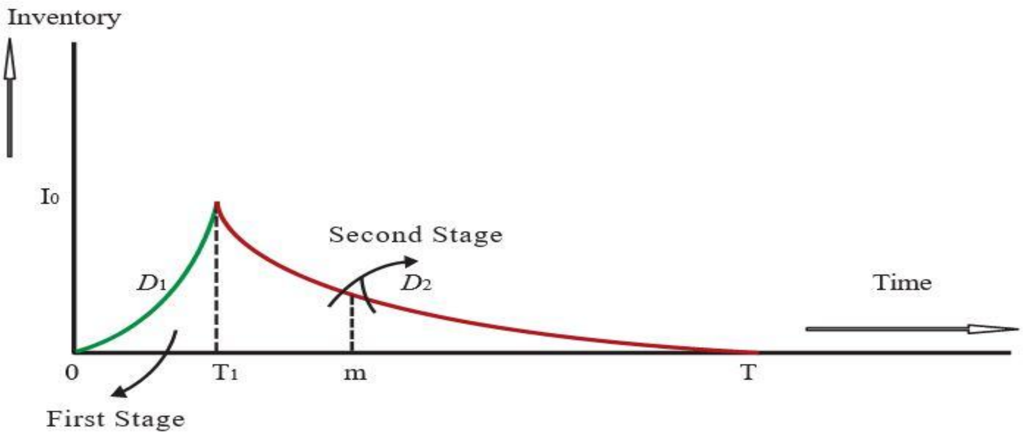

Fig. 1: Geometrical interpretation of inventory followed by shortages (IFS) model

Considering the initial stock as zero and accumulating the shortages over [0, T_1], the replenishment takes place at T_1. The inventory gradually decreases during [T1, T], and after adjusting existing demand, and deterioration it becomes zero at the end of cycle time T. The instantaneous state of the inventory level I(t) at any time t is governed by the differential equations:

$$\frac{dI(t)}{dt} = -Ae^{at}, 0 \le t < T_1, I(0) = 0 \tag{1}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -(b + ct + dt^2), T_1 \le t \le T \text{ and } I(T_1) = I_0 \tag{2}$$

$$I(t) = -\frac{A}{a}e^{at} + c_4 \tag{3}$$

Negative sign before the exponential function indicates shortages of inventory and negative sign before the quadratic function indicates decrease of inventory.

When t = 0, then I (0) = 0, and $c_4 = \frac{A}{a}$, Equation (3) becomes

$$I(t) = -\frac{A}{a}e^{at} + \frac{A}{a} \tag{4}$$

Now,

$$\frac{dI(t)}{dt} + \theta I(t) = -(b + ct + dt^2) \tag{5}$$

$$I.F = e^{\int \theta dt} = e^{\theta t} \tag{6}$$

With the help of Equation (6), Equation (5) becomes

$$I(t)e^{\theta t} = -b\int e^{\theta t}\,dt - c\int te^{\theta t}\,dt - d\int t^2e^{\theta t}\,dt + c_5$$

$$= -\frac{b}{\theta}e^{\theta t} - c\left[\frac{t}{\theta}e^{\theta t} - \frac{1}{\theta^2}e^{\theta t}\right] - d\left[\frac{t^2}{\theta}e^{\theta t} - \frac{2t}{\theta^2}e^{\theta t} + \frac{2}{\theta^3}e^{\theta t}\right] + c_5$$

$$= -\frac{b}{\theta}e^{\theta t} - \frac{c}{\theta}te^{\theta t} + \frac{c}{\theta^2}e^{\theta t} - \frac{d}{\theta}t^2e^{\theta t} + \frac{2dt}{\theta^2}e^{\theta t} - \frac{2d}{\theta^3}e^{\theta t} + c_5$$

$$\therefore\ I(t) = -\frac{b}{\theta} - \frac{c}{\theta}t + \frac{c}{\theta^2} - \frac{d}{\theta}t^2 + \frac{2dt}{\theta^2} - \frac{2d}{\theta^3} + c_5e^{-\theta t} \tag{7}$$

When t = T_1, then I(T_1) = I_0

$$I(T_1) = I_0 = -\frac{b}{\theta} - \frac{c}{\theta}T_1 + \frac{c}{\theta^2} - \frac{d}{\theta}T_1^2 + \frac{2dT_1}{\theta^2} - \frac{2d}{\theta^3} + c_5 e^{-\theta T_1}$$

$$c_5 e^{-\theta T_1} = I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}$$

$$c_5 = e^{\theta T_1}[I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}]$$

Therefore, plugging in the value of c_5 Equation (7) becomes

$$I(t) = -\frac{b}{\theta} - \frac{c}{\theta}t + \frac{c}{\theta^2} - \frac{d}{\theta}t^2 + \frac{2dt}{\theta^2} - \frac{2d}{\theta^3} + e^{\theta(T_1-t)}[I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}] \quad (8)$$

Therefore, the total inventory from Equation (4) and Equation (8) is

$$I_2(t) = I(t) + I(t)$$

$$I_2(t) = -\frac{A}{a}e^{at} + \frac{A}{a} + e^{\theta(T_1-t)}[I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}] - \frac{b}{\theta} - \frac{c}{\theta}t + \frac{c}{\theta^2} - \frac{d}{\theta}t^2 + \frac{2dt}{\theta^2} - \frac{2d}{\theta^3} \quad (9)$$

The total deteriorated items are

$$I_0 - \int_{T_1}^{T}(b + ct + dt^2)dt = I_0 - [bt + \frac{1}{2}ct^2 + \frac{1}{3}dt^3]_{T_1}^{T}$$

$$= I_0 - [bT + \frac{1}{2}cT^2 + \frac{1}{3}dT^3 - bT_1 - \frac{1}{2}cT_1^2 - \frac{1}{3}dT_1^3]$$

$$I_0 - \int_{T_1}^{T}(b + ct + dt^2)dt = I_0 + b(T_1 - T) + \frac{c}{2}(T_1^2 - T^2) + \frac{d}{3}(T_1^3 - T^3) \quad (10)$$

With the help of Equation (4), the shortages cost (SC) over time interval [0, T_1] is

$$SC = -s\int_0^{T_1} I(t)dt$$

$$= -s\int_0^{T_1}[-\frac{A}{a}e^{at} + \frac{A}{a}]dt$$

$$= -s\left[-\frac{A}{a^2}e^{at} + \frac{A}{a}t\right]_0^{T_1}$$

$$= -s[-\frac{A}{a^2}e^{aT_1} + \frac{A}{a}T_1 + \frac{A}{a^2}]$$

$$\therefore SC = s[\frac{A}{a^2}e^{aT_1} - \frac{A}{a}T_1 - \frac{A}{a^2}] \quad (11)$$

For holding cost (HC) on [T_1, T], using Equation (8)

$$\int_{T_1}^{T} I(t)dt = \int_{T_1}^{T} -\frac{b}{\theta} - \frac{c}{\theta}t + \frac{c}{\theta^2} - \frac{d}{\theta}t^2 + \frac{2dt}{\theta^2} - \frac{2d}{\theta^3} + e^{\theta(T_1-t)}[I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}]dt$$

$$= [-\frac{b}{\theta}t - \frac{c}{2\theta}t^2 + \frac{c}{\theta^2}t - \frac{d}{3\theta}t^3 + \frac{dt^2}{\theta^2} - \frac{2d}{\theta^3}t - \frac{1}{\theta}e^{\theta(T_1-t)}\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\}]_{T_1}^{T}$$

$$= [-\frac{b}{\theta}T - \frac{c}{2\theta}T^2 + \frac{c}{\theta^2}T - \frac{d}{3\theta}T^3 + \frac{d}{\theta^2}T^2 - \frac{2d}{\theta^3}T - \frac{1}{\theta}e^{\theta(T_1-T)}\left\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\right\} - \left\{-\frac{b}{\theta}T_1 - \frac{c}{2\theta}{T_1}^2 + \frac{c}{\theta^2}T_1 - \frac{d}{3\theta}{T_1}^3 + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1\right\} + \frac{1}{\theta}\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\}]$$

$$\therefore \int_{T_1}^{T} I(t)dt = \frac{b}{\theta}(T_1 - T) + \frac{c}{2\theta}(T_1^2 - T^2) + \frac{d}{3\theta}(T_1^3 - T^3) + \frac{2d}{\theta^3}(T_1 - T) + \frac{c}{\theta^2}(T - T_1) + \frac{d}{\theta^2}(T^2 - T_1^2) + \frac{1}{\theta}\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\} - \frac{1}{\theta}e^{\theta(T_1 - T)}\left\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\right\} \tag{12}$$

Therefore, the holding cost (HC) on $[T_1, T]$ stands

$$HC = h\int_{T_1}^{T} I(t)dt$$

$$= h[\frac{b}{\theta}(T_1 - T) + \frac{c}{2\theta}(T_1^2 - T^2) + \frac{d}{3\theta}(T_1^3 - T^3) + \frac{2d}{\theta^3}(T_1 - T) + \frac{c}{\theta^2}(T - T_1) + \frac{d}{\theta^2}(T^2 - T_1^2) + \frac{1}{\theta}\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\} - \frac{1}{\theta}e^{\theta(T_1 - T)}\left\{I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3}\right\}]$$

$$\therefore HC = h[\frac{b}{\theta}(T_1 - T) + \frac{c}{2\theta}(T_1^2 - T^2) + \frac{d}{3\theta}(T_1^3 - T^3) + \frac{2d}{\theta^3}(T_1 - T) + \frac{c}{\theta^2}(T - T_1) + \frac{d}{\theta^2}(T^2 - T_1^2)] + h[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}] - he^{\theta(T_1 - T)}[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}] \tag{13}$$

With the help of Equation (10), the total deterioration cost (DC) is given by

$$DC = c_6[I_0 + b(T_1 - T) + \frac{c}{2}(T_1^2 - T^2) + \frac{d}{3}(T_1^3 - T^3)] \tag{14}$$

The permissible delay period m is settled by the whole-seller or distributor to the retailer or customer. Credit period permissible delay-payment has been considered within positive stock over the time interval [T1, m] where a buyer can use sales revenue to earn interest E at an annual rate Ie in $[T_1, m]$.

$$E = pI_e\int_{T_1}^{m} tD(t)dt$$

$$= pI_e\int_{T_1}^{m} t(b + ct + dt^2)dt$$

$$= pI_e\left[\frac{1}{2}bt^2 + \frac{1}{3}ct^3 + \frac{1}{4}dt^4\right]_{T_1}^{m}$$

$$= pI_e\left[\left\{\frac{1}{2}bm^2 + \frac{1}{3}cm^3 + \frac{1}{4}dm^4\right\} - \left\{\frac{1}{2}bT_1^2 + \frac{1}{3}cT_1^3 + \frac{1}{4}dT_1^4\right\}\right]$$

$$\therefore E = \left[pI_e\left\{\frac{1}{2}bm^2 + \frac{1}{3}cm^3 + \frac{1}{4}dm^4\right\} - pI_e\left\{\frac{1}{2}bT_1^2 + \frac{1}{3}cT_1^3 + \frac{1}{4}dT_1^4\right\}\right] \tag{15}$$

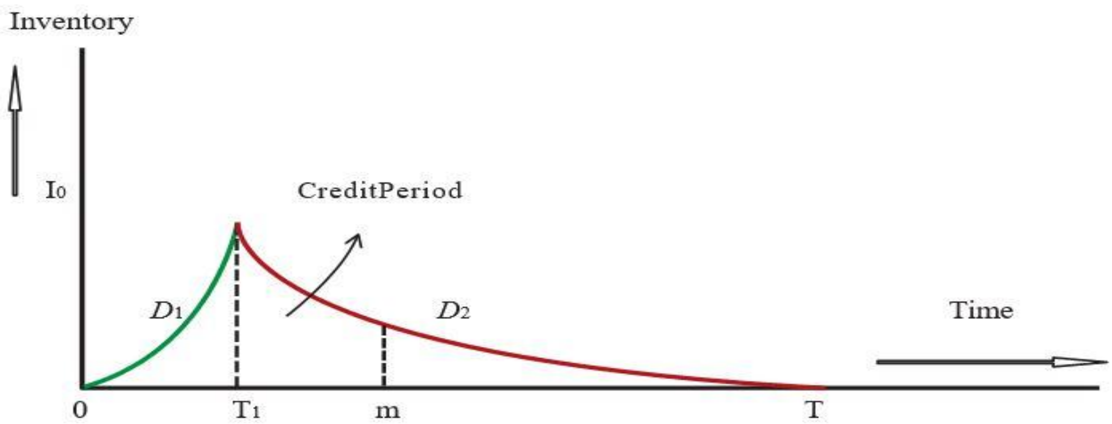

Fig. 2: Inventory followed by shortages (IFS) curve with credit period

Beyond the credit period over the time [m, T], unsold stock L is supposed to be financed with an annual rate I_r.

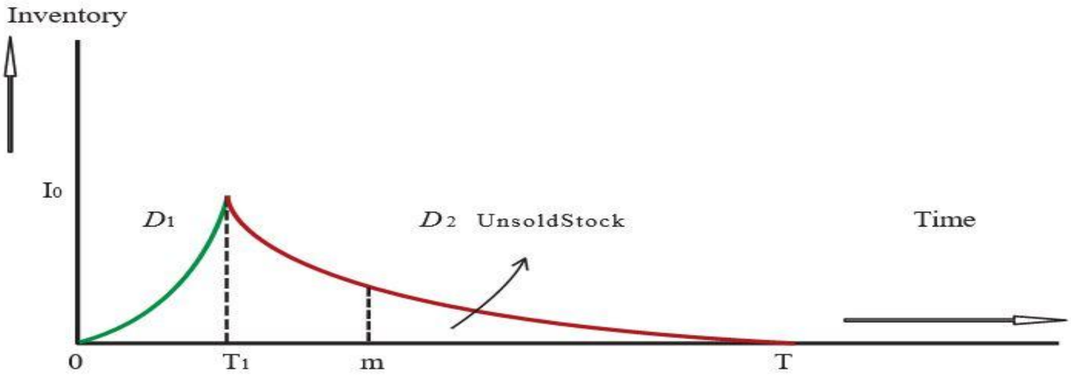

Fig. 3: Inventory followed by shortages (IFS) curve of unsold stock

$$L = pI_r \int_m^T I(t)dt$$

$$= pI_r \int_m^T \left[-\frac{b}{\theta} - \frac{c}{\theta}t + \frac{c}{\theta^2} - \frac{d}{\theta}t^2 + \frac{2dt}{\theta^2} - \frac{2d}{\theta^3} + e^{\theta(T_1 - t)} \left[I_0 + \frac{b}{\theta} + \frac{c}{\theta}T_1 - \frac{c}{\theta^2} + \frac{d}{\theta}T_1^2 - \frac{2dT_1}{\theta^2} + \frac{2d}{\theta^3} \right] \right] dt \quad (16)$$

$$\therefore L = pI_r \left[\frac{b}{\theta}(m - T) + \frac{c}{2\theta}(m^2 - T^2) + \frac{d}{3\theta}(m^3 - T^3) \right] + pI_r \left[\frac{2d}{\theta^3}(m - T) + \frac{c}{\theta^2}(T - m) + \frac{d}{\theta^2}(T^2 - m^2) \right] + pI_r \left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}m - \frac{c}{\theta^3} + \frac{d}{\theta^2}m^2 - \frac{2d}{\theta^3}m + \frac{2d}{\theta^4} \right] - pI_r e^{\theta(m-T)} \left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}m - \frac{c}{\theta^3} + \frac{d}{\theta^2}m^2 - \frac{2d}{\theta^3}m + \frac{2d}{\theta^4} \right] \quad (17)$$

Therefore, the total average cost per unit time is given by

$$V(T_1, T) = \frac{1}{T}[k + DC + HC + SC + L - E] = \frac{1}{T}[G(T_1, T)]$$

where, $G(T_1, T) = [k + DC + HC + SC + L - E$

$$\therefore G(T_1, T) = [k + c_6 \left[I_0 + b(T_1 - T) + \frac{c}{2}(T_1^2 - T^2) + \frac{d}{3}(T_1^3 - T^3) \right] + h[\frac{b}{\theta}(T_1 - T) + \frac{c}{2\theta}(T_1^2 - T^2) + \frac{d}{3\theta}(T_1^3 - T^3) + \frac{2d}{\theta^3}(T_1 - T) + \frac{c}{\theta^2}(T - T_1) + \frac{d}{\theta^2}(T^2 - T_1^2)] + h\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4} \right] - he^{\theta(T_1 - T)} \left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4} \right] + s\left[\frac{A}{a^2}e^{aT_1} - \frac{A}{a}T_1 - \frac{A}{a^2} \right] + pI_r \left[\frac{b}{\theta}(m - T) + \frac{c}{2\theta}(m^2 - T^2) + \frac{d}{3\theta}(m^3 - T^3) \right] + pI_r \left[\frac{2d}{\theta^3}(m - T) + \frac{c}{\theta^2}(T - m) + \frac{d}{\theta^2}(T^2 - m^2) \right] + pI_r \left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}m - \frac{c}{\theta^3} + \frac{d}{\theta^2}m^2 - \frac{2d}{\theta^3}m + \frac{2d}{\theta^4} \right] - pI_r e^{\theta(m-T)} \left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}m - \frac{c}{\theta^3} + \frac{d}{\theta^2}m^2 - \frac{2d}{\theta^3}m + \frac{2d}{\theta^4} \right] - pI_e\left\{ \frac{1}{2}bm^2 + \frac{1}{3}cm^3 + \frac{1}{4}dm^4 \right\} - pI_e\left\{ \frac{1}{2}bT_1^2 + \frac{1}{3}cT_1^3 + \frac{1}{4}dT_1^4 \right\}] \quad (18)$$

2.2.1 Relevant theorem

If a function $\boldsymbol{V(T_1,T)=\frac{1}{T}G(T_1,T)}$ where $\boldsymbol{G(T_1,T)}$ admits continuous partial derivatives of second order, $\boldsymbol{V(T_1,T)}$ is minimum at $T_1 = T_1^*$, $T = T^*$ if all principal minors are positive definite, i.e. , if $\frac{\partial^2 G(T_1,T)}{\partial T_1^2} > 0$, and

$$\begin{vmatrix} \frac{\partial^2 G(T_1,T)}{\partial T_1^2} & \frac{\partial^2 G(T_1,T)}{\partial T_1 \partial T} \\ \frac{\partial^2 G(T_1,T)}{\partial T \partial T_1} & \frac{\partial^2 G(T_1,T)}{\partial T^2} \end{vmatrix} > 0$$

$$\begin{aligned}\frac{\partial G}{\partial T_1} = {} & \left[c_6\left\{b + cT_1 + dT_1^2\right\}\right] + h\left[\left(\frac{b}{\theta} + \frac{c}{\theta}T_1 + \frac{d}{\theta}T_1^2\right) + \frac{2d}{\theta^3} - \frac{c}{\theta^2} - \frac{2d}{\theta^2}T_1\right] \\ & + h\left[\frac{c}{\theta^2} + \frac{2d}{\theta^2}T_1 - \frac{2d}{\theta^3}\right] \\ & - h\theta e^{\theta(T_1-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}\right] \\ & - he^{\theta(T_1-T)}\left[\frac{c}{\theta^2} + \frac{2d}{\theta^2}T_1 - \frac{2d}{\theta^3}\right] + s\left[\frac{A}{a}e^{aT_1} - \frac{A}{a}\right] - pI_e[bT_1 + cT_1^2 \\ & + dT_1^3]\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial^2 G}{\partial T_1^2} = {} & \left[c_6\left\{c + 2dT_1\right\}\right] + h\left[\left(\frac{c}{\theta} + \frac{2d}{\theta}T_1\right) - \frac{2d}{\theta^2}\right] + h\left[\frac{2d}{\theta^2}\right] \\ & - h\theta^2 e^{\theta(T_1-T)}\left[\frac{I_0}{\theta^2} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}\right] \\ & - h\theta e^{\theta(T_1-T)}\left[\frac{c}{\theta^2} + \frac{2d}{\theta^2}T_1 - \frac{2d}{\theta^3}\right] - h\theta e^{\theta(T_1-T)}\left[\frac{c}{\theta^2} + \frac{2d}{\theta^2}T_1 - \frac{2d}{\theta^3}\right] \\ & - he^{\theta(T_1-T)}\left[\frac{2d}{\theta^2}\right] + s\left[Ae^{aT_1}\right] - pI_e[b + 2cT_1 + 3dT_1^2]\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial^2 G}{\partial T_1 \partial T} = {} & h\theta^2 e^{\theta(T_1-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}\right] \\ & + h\theta e^{\theta(T_1-T)}\left[\frac{c}{\theta^2} + \frac{2d}{\theta^2}T_1 - \frac{2d}{\theta^3}\right]\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial G}{\partial T} = {} & \left[c_6\left\{-b - cT - dT^2\right\}\right] + h\left[-\frac{b}{\theta} - \frac{c}{\theta}T - \frac{d}{\theta}T^2 - \frac{2d}{\theta^3} + \frac{c}{\theta^2} + \frac{2d}{\theta^2}T\right] \\ & + h\theta e^{\theta(T_1-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}\right] \\ & + pI_r\left[-\frac{b}{\theta} - \frac{c}{\theta}T - \frac{d}{\theta}T^2\right] + pI_r\left[-\frac{2d}{\theta^3} + \frac{c}{\theta^2} + \frac{2d}{\theta^2}T\right] \\ & + p\theta I_r e^{\theta(m-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}m - \frac{c}{\theta^3} + \frac{d}{\theta^2}m^2 - \frac{2d}{\theta^3}m + \frac{2d}{\theta^4}\right]\end{aligned} \tag{22}$$

$$\therefore \frac{\partial^2 G}{\partial T^2} = [c_6\{-c - 2dT\}] + h\left[-\frac{c}{\theta} - \frac{2d}{\theta}T + \frac{2d}{\theta^2}\right] - h\theta^2 e^{\theta(T_1-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}\right] + pI_r\left[-\frac{c}{\theta} - \frac{2d}{\theta}T\right] + pI_r\left[\frac{2d}{\theta^2}\right] - p\theta^2 I_r e^{\theta(m-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}m - \frac{c}{\theta^3} + \frac{d}{\theta^2}m^2 - \frac{2d}{\theta^3}m + \frac{2d}{\theta^4}\right] \tag{23}$$

$$\therefore \frac{\partial^2 G}{\partial T \partial T_1} = h\theta^2 e^{\theta(T_1-T)}\left[\frac{I_0}{\theta} + \frac{b}{\theta^2} + \frac{c}{\theta^2}T_1 - \frac{c}{\theta^3} + \frac{d}{\theta^2}T_1^2 - \frac{2d}{\theta^3}T_1 + \frac{2d}{\theta^4}\right] + h\theta e^{\theta(T_1-T)}\left[\frac{c}{\theta^2} + \frac{2d}{\theta^2}T_1 - \frac{2d}{\theta^3}\right] \tag{24}$$

With the help of Equation (21) and Equation (24), it is noted that

$$\frac{\partial^2 G}{\partial T_1 \partial T} = \frac{\partial^2 G}{\partial T \partial T_1}$$

By putting $\frac{\partial G}{\partial T_1} = \mathbf{0}$ and $\frac{\partial G}{\partial T} = \mathbf{0}$, Equation (19) and (22) become a pair of simultaneous equations. Solution of simultaneous equations give optimum values of T and T_1 which satisfy the condition $\frac{\partial^2 G(T_1,T)}{\partial T_1^2} > \mathbf{0}$ and it proves that the total average cost per unit time $V(T_1, T)$ is minimum.

2.3 SFI model derivation

Under the category shortages followed by inventory (SFI) model:

- Exponential function has been taken at the first stage in Fig. 4 over the time interval $[0, T_1)$ with initial inventory level I_0 at the beginning of time, whereas quadratic function has been taken at the second stage over the time interval $[T_1, T]$ with inventory shortages.
- Deteriorated items have been evaluated at the first stage in Fig. 4 over the time interval $[0, T_1)$, whereas it has been ignored at the second stage in Fig. 4 over the time interval $[T_1, T]$ due to inventory shortages.
- Shortage cost (SC) has been evaluated at the second stage in Fig. 4 over the time interval $[T_1, T]$. On the other hand, it is zero at the first stage as inventory exists.
- Negative signs are given in front of exponential, and quadratic demand functions in differential equation to express decrease, and shortage respectively in inventory.
- Holding cost (HC) has been considered at the first stage in Fig. 4, whereas it has been avoided at the second stage in Fig. 4 due to inventory shortages.
- Credit period for permissible delay in payment has been considered within positive stock over the time interval [0, m] in Fig. 4.
- Unsold stock is financed with an annual rate I_r over the time interval $[m, T_1)$ in Fig. 4 which is beyond the credit period [0, m] with the positive stock.

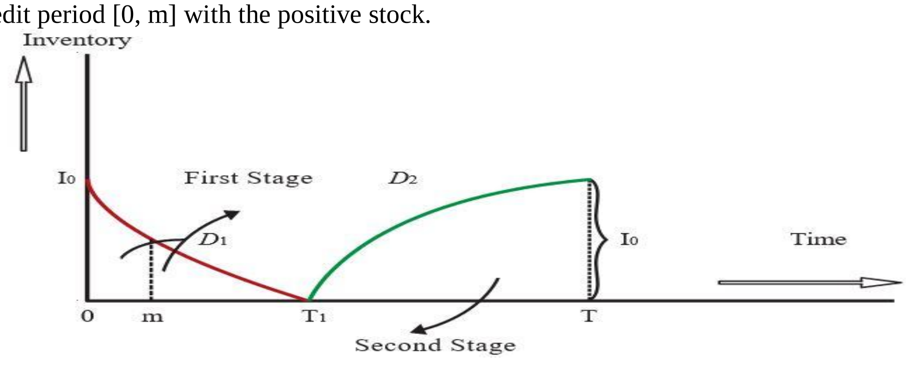

Fig. 4: Geometrical interpretation of shortages followed by inventory (SFI) model

The inventory level I(t) at time t generally decreases from I_0 to meet market's demand, and product's deterioration reaches to zero at T_1. Thus, shortages accumulate over the time interval [T_1, T]. Hence, the variation of inventory with respect to time can be described by the governing differential equation:

$$\frac{dI(t)}{dt} + \theta I(t) = f(t) \tag{25}$$

$$f(t) = \begin{cases} D_1(t), 0 \le t < T_1 \\ D_2(t), T_1 \le t \le T \end{cases} \tag{26}$$

$$\begin{pmatrix} D_1(t) = Ae^{at}, A > 0, a > 0 \\ D_2(t) = (b + ct + dt^2),\ \ b, c, d > 0 \end{pmatrix} \tag{27}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -Ae^{at}, 0 \le t < T_1, I(0) = I_0 \tag{28}$$

Negative sign before the exponential function indicates the decrement of inventory.

$$I.F = e^{\int \theta dt} = e^{\theta t} \tag{29}$$

With the help of Equation (28), Equation (29) becomes

$$\therefore I(t)e^{\theta t} = -\int Ae^{(a+\theta)t}dt + c_1$$

$$= -\frac{A}{a+\theta}e^{(a+\theta)t} + c_1$$

$$I(t) = -\frac{A}{a+\theta}e^{at} + c_1 e^{-\theta t} \tag{30}$$

When t = 0, then I (0) = I_0

$$I(0) = -\frac{A}{a+\theta}e^{0} + c_1 e^{0}$$

$$I(0) = -\frac{A}{a+\theta} + c_1$$

$$c_1 = I_0 + \frac{A}{a+\theta}$$

Therefore, Equation (30) becomes

$$I(t) = -\frac{A}{a+\theta}e^{at} + \left(I_0 + \frac{A}{a+\theta}\right)e^{-\theta t}$$

$$I(t) = -\frac{A}{a+\theta}e^{at} + \frac{A}{a+\theta}e^{-\theta t} + I_0 e^{-\theta t} \tag{31}$$

$$\frac{dI}{dt} = -(b + ct + dt^2); \textit{when } t = T_1, I(T_1) = 0 \tag{32}$$

Negative sign before the quadratic function indicates shortages of inventory. After integration, Equation (32) becomes

$$I(t) = -(bt + \frac{1}{2}ct^2 + \frac{1}{3}dt^3) + c_2 \tag{33}$$

When t = T_1, I (T_1) = 0, Equation (33) becomes

$$I(T_1) = -\left(bT_1 + \frac{1}{2}cT_1^2 + \frac{1}{3}dT_1^3\right) + c_2 = 0$$

$$c_2 = \left(bT_1 + \frac{1}{2}cT_1^2 + \frac{1}{3}dT_1^3\right)$$

Therefore, Equation (33) can be written as

$$I(t) = b(T_1 - t) + \frac{c}{2}(T_1^2 - t^2) + \frac{d}{3}(T_1^3 - t^3) \tag{34}$$

Therefore, total inventory from Equation (31) and Equation (34) is given by

$$I_1(t) = I(t) + I(t)$$

$$I_1(t) = -\frac{A}{a+\theta}e^{at} + \frac{A}{a+\theta}e^{-\theta t} + I_0 e^{-\theta t} + b(T_1 - t) + \frac{c}{2}(T_1^2 - t^2) + \frac{d}{3}(T_1^3 - t^3) \tag{35}$$

The total deteriorated items are

$$I_0 - \int_0^{T_1} D_1(t)dt = I_0 - \int_0^{T_1} Ae^{at}dt$$
$$= I_0 - \frac{A}{a}[e^{at}]_0^{T_1}$$
$$= I_0 - \frac{A}{a}[e^{aT_1} - e^0]$$
$$= I_0 - \frac{A}{a}e^{aT_1} + \frac{A}{a}$$

Therefore, the total deteriorated items can be written as

$$I_0 - \int_0^{T_1} D_1(t)dt = I_0 + \frac{A}{a} - \frac{A}{a}e^{aT_1} \tag{36}$$

With the help of Equation (34), the shortage cost (SC) over the time interval [T_1, T] is

$$SC = -s\int_{T_1}^{T} I(t)dt$$
$$= -s\int_{T_1}^{T}\left[b(T_1 - t) + \frac{c}{2}(T_1^2 - t^2) + \frac{d}{3}(T_1^3 - t^3)\right]dt$$
$$= -s\left[bT_1t - \frac{b}{2}t^2 + \frac{c}{2}T_1^2t - \frac{c}{6}t^3 + \frac{d}{3}T_1^3t - \frac{d}{12}t^4\right]_{T_1}^{T}$$
$$= -s[bT_1T - \frac{b}{2}T^2 + \frac{c}{2}T_1^2T - \frac{c}{6}T^3 + \frac{d}{3}T_1^3T - \frac{d}{12}T^4 - bT_1^2 - \frac{b}{2}T_1^2 - \frac{c}{2}T_1^3 - \frac{c}{6}T_1^3 + \frac{d}{3}T_1^4 - \frac{d}{12}T_1^4]$$

Therefore, the shortage cost (SC) is

$$SC = s\left\{\frac{b}{2}(T^2 - T_1^2) + \frac{c}{6}(T^3 - T_1^3) + \frac{d}{12}(T^4 - T_1^4)\right\} + s\{b(T_1^2 - T_1T) + \frac{c}{2}(T_1^3 - T_1^2T) + \frac{d}{3}(T_1^4 - T_1^3T)\} \tag{37}$$

With the help of Equation (31), the holding cost (HC) on [0, T_1]

$$HC = h\int_0^{T_1} I(t)dt = h\int_0^{T_1}\left(-\frac{A}{a+\theta}e^{at} + \frac{A}{a+\theta}e^{-\theta t} + I_0e^{-\theta t}\right)dt$$
$$= h\left[-\frac{A}{a(a+\theta)}e^{at} - \frac{A}{\theta(a+\theta)}e^{-\theta t} - \frac{I_0}{\theta}e^{-\theta t}\right]_0^{T_1}$$

$$\therefore HC = h\left\{\frac{A}{a(a+\theta)} + \frac{A}{\theta(a+\theta)} + \frac{I_0}{\theta}\right\} - h\{\frac{A}{a(a+\theta)}e^{aT_1} + \frac{A}{\theta(a+\theta)}e^{-\theta T_1} + \frac{I_0}{\theta}e^{-\theta T_1}\} \tag{38}$$

The total deterioration cost (DC) stands by using Equation (36)

$$DC = c_3(I_0 + \frac{A}{a} - \frac{A}{a}e^{aT_1}) \tag{39}$$

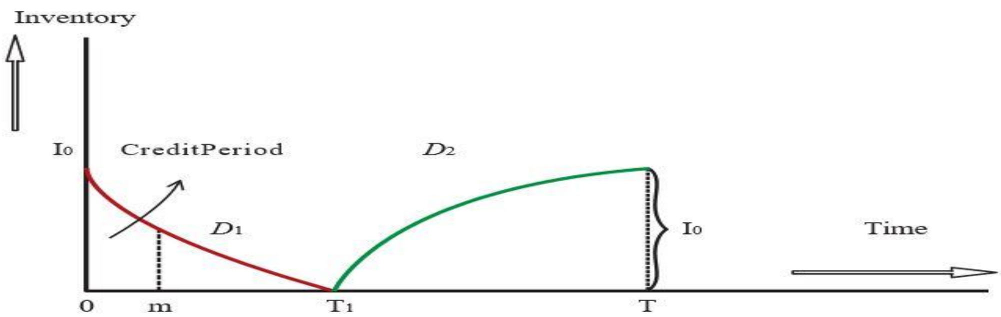

Fig. 5: Shortages followed by inventory (SFI) curve with credit period

The permissible delay period m is settled by the whole-seller or distributor to the retailer or customer. Credit period permissible delay-payment has been considered within positive stock over the time interval [0, m] where a buyer can use sales revenue to earn interest E at an annual rate I_e in [0, m].

$$E = pI_e \int_0^m tD(t)dt$$
$$= pI_e \int_0^m te^{at}dt$$
$$= pI_e \left[\frac{t}{a}e^{at} - \frac{1}{2a}e^{at}\right]_0^m$$
$$= pI_e[\left\{\frac{m}{a}e^{am} - \frac{1}{2a}e^{am}\right\} - \{-\frac{1}{2a}e^{0t}\}]$$
$$\therefore E = pI_e(\frac{m}{a}e^{am} - \frac{1}{2a}e^{am} + \frac{1}{2a}) \tag{40}$$

Beyond the credit period over the time [m, T_1], unsold stock L is supposed to be financed with an annual rate I_r.

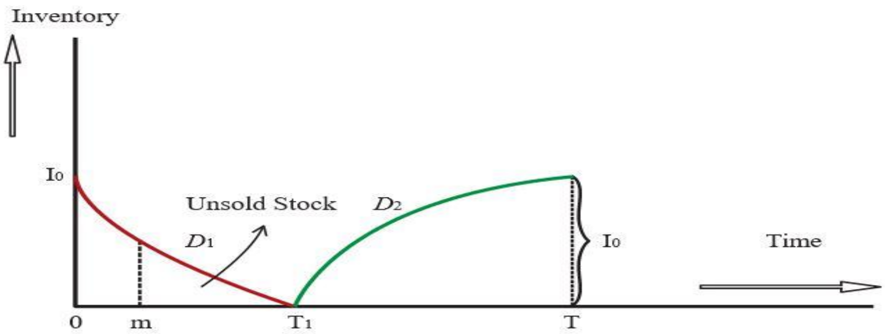

Fig. 6: Shortages followed by inventory (SFI) curve of unsold stock

$$L = pI_r \int_m^{T_1} I(t)dt$$
$$= pI_r \int_m^{T_1} \left(-\frac{A}{a+\theta}e^{at} + \frac{A}{a+\theta}e^{-\theta t} + I_0 e^{-\theta t}\right)dt$$
$$= PI_r \left[-\frac{A}{a(a+\theta)}e^{at} - \frac{A}{\theta(a+\theta)}e^{-\theta t} - \frac{I_0}{\theta}e^{-\theta t}\right]_m^{T_1}$$
$$L = pI_r\left\{-\frac{A}{a(a+\theta)}e^{aT_1} - \frac{A}{\theta(a+\theta)}e^{-\theta T_1} - \frac{I_0}{\theta}e^{-\theta T_1}\right\} - PI_r\left\{-\frac{A}{a(a+\theta)}e^{am} - \frac{A}{\theta(a+\theta)}e^{-\theta m} - \frac{I_0}{\theta}e^{-\theta m}\right\} \tag{41}$$

Therefore, the total average cost per unit time is given by

$$V(T_1, T) = \frac{1}{T}[k + DC + HC + SC + L - E] = \frac{1}{T}[G(T_1, T)]$$

Where, $G(T_1, T) = [k + DC + HC + SC + L - E]$

$$
\begin{aligned}
(\boldsymbol{T_1}, \boldsymbol{T}) = [\boldsymbol{k} &+ \left\{\boldsymbol{c_3}\left(\boldsymbol{I_0} + \frac{\boldsymbol{A}}{\boldsymbol{a}} - \frac{\boldsymbol{A}}{\boldsymbol{a}}\boldsymbol{e}^{\boldsymbol{a}\boldsymbol{T_1}}\right)\right\} \\
&+ \left\{\boldsymbol{h}\left\{\frac{\boldsymbol{A}}{\boldsymbol{a}(\boldsymbol{a}+\boldsymbol{\theta})} + \frac{\boldsymbol{A}}{\boldsymbol{\theta}(\boldsymbol{a}+\boldsymbol{\theta})} + \frac{\boldsymbol{I_0}}{\boldsymbol{\theta}}\right\}\right. \\
&\left. - \boldsymbol{h}\left\{\frac{\boldsymbol{A}}{\boldsymbol{a}(\boldsymbol{a}+\boldsymbol{\theta})}\boldsymbol{e}^{\boldsymbol{a}\boldsymbol{T_1}} + \frac{\boldsymbol{A}}{\boldsymbol{\theta}(\boldsymbol{a}+\boldsymbol{\theta})}\boldsymbol{e}^{-\boldsymbol{\theta}\boldsymbol{T_1}} + \frac{\boldsymbol{I_0}}{\boldsymbol{\theta}}\boldsymbol{e}^{-\boldsymbol{\theta}\boldsymbol{T_1}}\right\}\right\} \\
&+ \left\{\boldsymbol{s}\left\{\frac{\boldsymbol{b}}{\boldsymbol{2}}(\boldsymbol{T}^2 - \boldsymbol{T_1^2}) + \frac{\boldsymbol{c}}{\boldsymbol{6}}(\boldsymbol{T}^3 - \boldsymbol{T_1^3}) + \frac{\boldsymbol{d}}{\boldsymbol{12}}(\boldsymbol{T}^4 - \boldsymbol{T_1^4})\right\}\right. \\
&\left. + \boldsymbol{s}\left\{\boldsymbol{b}(\boldsymbol{T_1^2} - \boldsymbol{T_1}\boldsymbol{T}) + \frac{\boldsymbol{c}}{\boldsymbol{2}}(\boldsymbol{T_1^3} - \boldsymbol{T_1^2}\boldsymbol{T}) + \frac{\boldsymbol{d}}{\boldsymbol{3}}(\boldsymbol{T_1^4} - \boldsymbol{T_1^3}\boldsymbol{T})\right\}\right\} \\
&+ \left\{\boldsymbol{pI_r}\left\{-\frac{\boldsymbol{A}}{\boldsymbol{a}(\boldsymbol{a}+\boldsymbol{\theta})}\boldsymbol{e}^{\boldsymbol{a}\boldsymbol{T_1}} - \frac{\boldsymbol{A}}{\boldsymbol{\theta}(\boldsymbol{a}+\boldsymbol{\theta})}\boldsymbol{e}^{-\boldsymbol{\theta}\boldsymbol{T_1}} - \frac{\boldsymbol{I_0}}{\boldsymbol{\theta}}\boldsymbol{e}^{-\boldsymbol{\theta}\boldsymbol{T_1}}\right\}\right. \\
&\left. - \boldsymbol{PI_r}\left\{-\frac{\boldsymbol{A}}{\boldsymbol{a}(\boldsymbol{a}+\boldsymbol{\theta})}\boldsymbol{e}^{\boldsymbol{am}} - \frac{\boldsymbol{A}}{\boldsymbol{\theta}(\boldsymbol{a}+\boldsymbol{\theta})}\boldsymbol{e}^{-\boldsymbol{\theta m}} - \frac{\boldsymbol{I_0}}{\boldsymbol{\theta}}\boldsymbol{e}^{-\boldsymbol{\theta m}}\right\}\right\} \\
&- \{\boldsymbol{pI_e}(\frac{\boldsymbol{m}}{\boldsymbol{a}}\boldsymbol{e}^{\boldsymbol{am}} - \frac{\boldsymbol{1}}{\boldsymbol{2a}}\boldsymbol{e}^{\boldsymbol{am}} + \frac{\boldsymbol{1}}{\boldsymbol{2a}})\}]
\end{aligned}
\tag{42}
$$

With the help of Theorem 2.3.1, it can also be proven that the total average cost per unit time $\boldsymbol{V}(\boldsymbol{T_1}, \boldsymbol{T})$ is minimum.

3. Results and Discussion

This study has solved the formulated model analytically and to validate the models several values of the key parameters are also considered. In addition, comparative sensitivity analysis has been studied to sought out the best model among these proposed models.

3.1 Numerical Examples

3.1.1 Inventory Model Imposing IFS Condition

Taking the values of key parameters as A=1000; a=12; k=10; b=5; c=10; d=15; I_0=0.11; I_r=0.015; I_e=0.013; s= 0.35; c_6=3.3; m=0.010; p=0.1; h=0.02; θ =0.001; T=0.30 year, and T_1 = 0.13 year. Hence, the minimum average cost becomes $V(T_1, T)$= $ 37.6166.

3.1.2 Inventory Model Imposing SFI Condition

Considering the values of the parameters A=1000; a=12; k=10; b=5; c=10; d=15; I_0=0.11; I_r=0.015; I_e=0.013; s= 0.35; c_3=0.0001; m=0.010; p=0.1; h=0.02; θ =0.001; T=0.25 year, and T_1 = 0.15 year. Hence, the minimum average cost becomes $V(T_1, T)$= $ 37.9422.

3.2 Comparative Sensitivity Analysis

The sensitivity analysis for total cost per unit time TC is carried out with respect to the changes in the values of the parameters of exponential function, parameters of quadratic function, until holding cost per unit time h, ordering cost k, and unit shortage cost per item s and is performed by considering variation in each one of the above parameters by 5% change in stipulated standard value, keeping all other remaining parameters as fixed. The present works have illustrated the comparative numerical outcomes based on the effects of generating parameters on total cost per unit time as shown in Figs.7-14. The x-axis of each figure represents the considered numerical values of corresponding parameters, whereas the y-axis consists of total cost per unit time. Every figure represents the rate of change of total cost per unit time with respect to the particular parameters.

Figs. 7 and 8 show the comparison between the total costs per unit time of IFS and SFI models for parameters A and a. From Figs. 7 and 8, it is clear that with the increment of the value of parameters A and a, the total cost per unit time of IFS model gradually increases whereas it steadily decreases in SFI model. However, the line graphs as shown in in Figs. 9 and 10 give exactly reverse outcome on the effects of parameters b and c.

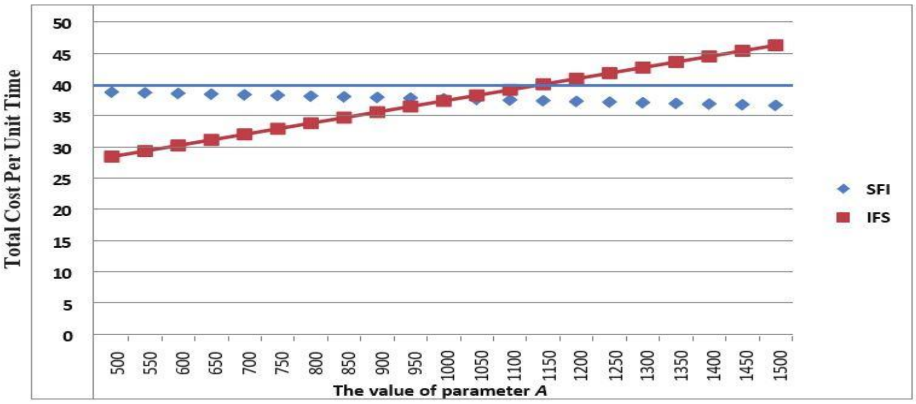

Fig. 7: Comparison between IFS and SFI model for parameter A

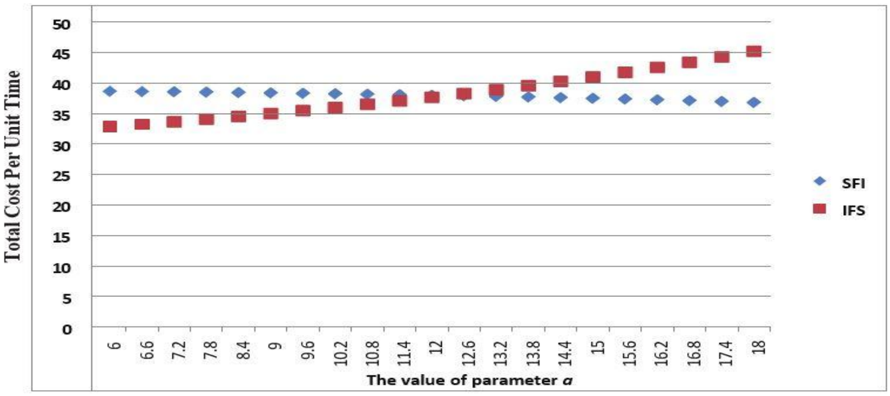

Fig. 8: Comparison between IFS and SFI model for parameter a

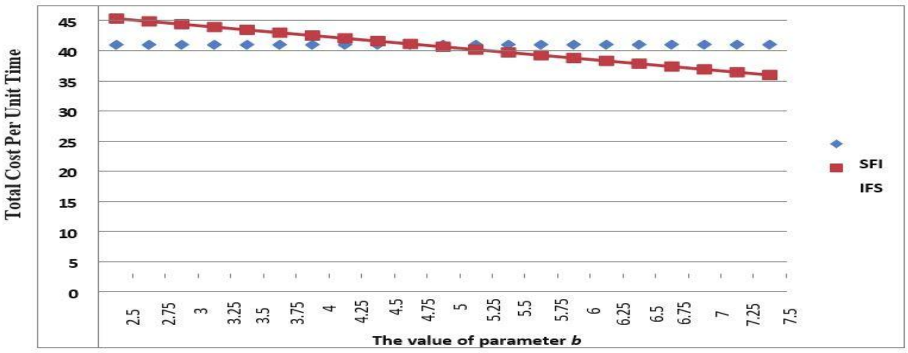

Fig. 9: Comparison between IFS and SFI model for parameter b

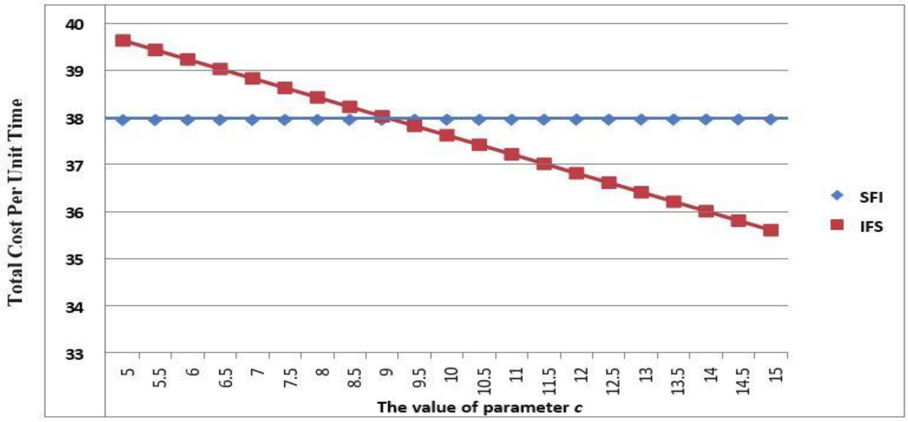

Fig. 10: Comparison between IFS and SFI model for parameter c

From Fig. 11 and 12, it is observed that the cost per unit time sharply decreases in IFS and SFI model with the increment of parameter d and holding cost h respectively.

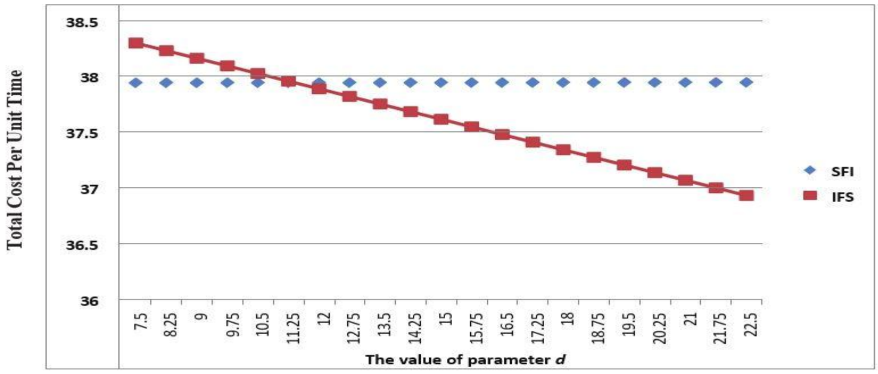

Fig. 11: Comparison between IFS and SFI model for parameter d

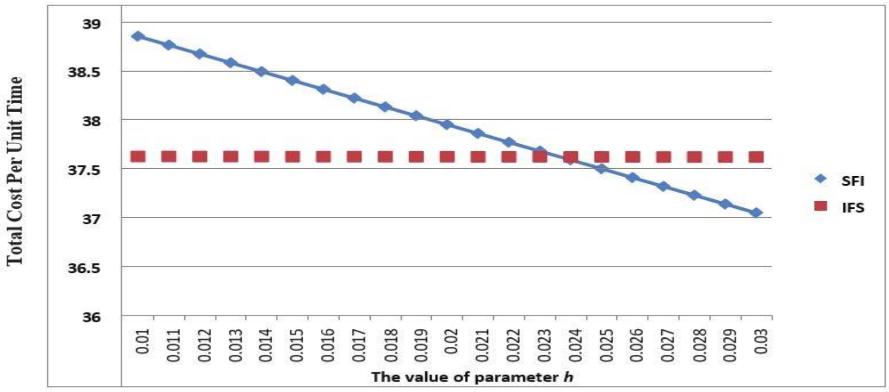

Fig. 12: Comparison between IFS and SFI model for holding cost h

The effect of ordering cost on total cost of IFS and SFI models are almost same by the Fig. 13 and from Fig. 14, it has been seen that the increment of shortage costs increases the entire cost per unit time of IFS model. The cost per unit time decreases with the increase of values of parameters of the exponential function as well as holding cost per unit time which are noted from the numerical results of SFI model, whereas cost per unit time increases with the increase of values of parameters of the quadratic function as well as ordering cost and shortage cost. The cost per unit time increases when the values of parameters of the exponential function as well as ordering cost and shortage cost increase which is observed from the numerical results of IFS model. But cost per unit time decreases while the values of parameters of the quadratic function as well as holding cost increase. Comparison between SFI model and IFS model on optimum total cost per unit time exposes that SFI model is better than IFS model for certain intervals of parameters. On the other hand, IFS model is better than SFI model for remaining intervals of the same parameters.

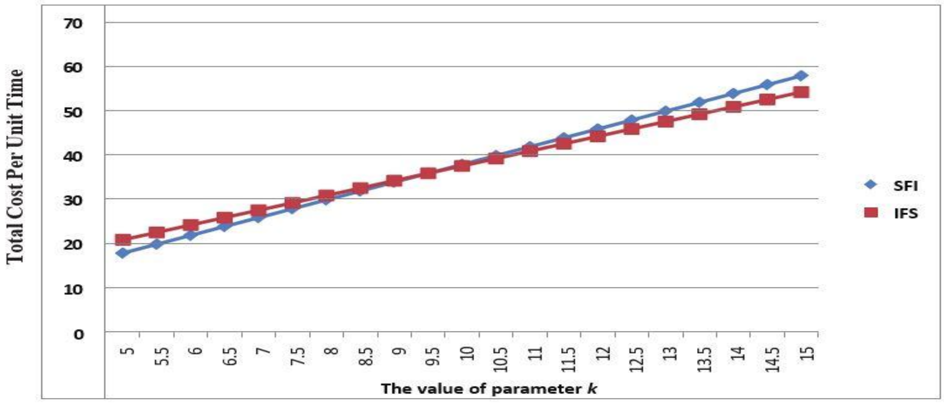

Fig. 13: Comparison between IFS and SFI model for ordering cost k

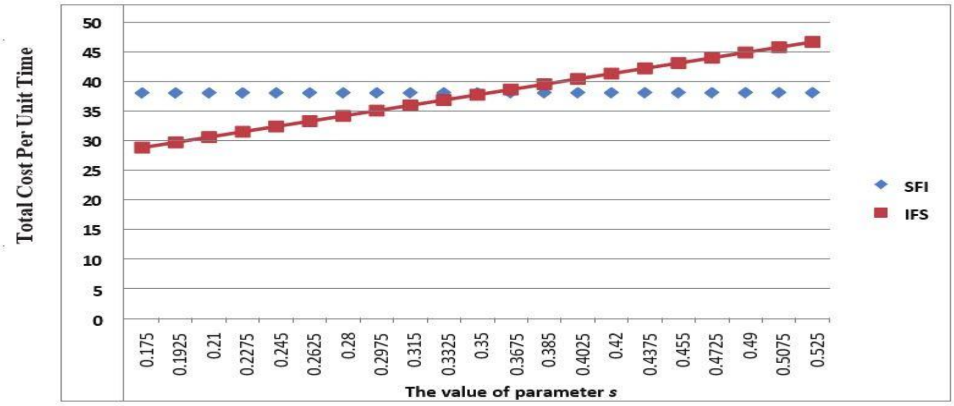

Fig. 14: Comparison between IFS and SFI model for shortage costs

4. Conclusion

Inventory management has a dynamic function that has pivotal role in basic engineering management. When a company handles numerous products in their inventory and the demand is random the common approaches cannot be used but rather need to be adapted to better fit the environment to assist the company in their inventory management**.**

The inventory model has been upheld for two types of practical oriented condition inventory followed by shortages (IFS) and shortages followed by inventory (SFI) with deterioration rate, and infinite replenishment with permissible delay in payment. This proposed model can aid the manage in concisely determining the effect of cost per unit time for variation of different parameters. We bring our conclusion evidently saying after a thorough analysis that for the IFS condition, when the values of parameters of exponential function, ordering cost, and shortage cost increase, the total cost per unit time increases. Additionally, when the values of parameters of quadratic function, and holding cost increase, total cost per unit time decreases. However, imposing the SFI condition, we conclude saying that when the values of parameters of quadratic function, ordering cost, and shortage cost increase, the total cost per unit time increases and meanwhile, total cost per unit time decreases when the values of parameters of exponential function, and unit holding cost increase. Finally, comparing IFS and SFI model on optimum total cost per unit time, it exposes that SFI model is better than IFS model for certain intervals of parameters, while IFS model is also better than SFI model for remaining intervals of the same parameters. From the findings above, it can be notice that inventory model with deteriorating items places a very critical role in the success of many organizations in almost every industry with the supply chain, freight and maritime industry inclusive.

In consideration of the present research for inventory of deteriorating items of a supply chain the recommendations for future works are to incorporate holding costs for the entire time cycle to evaluate the total cost for SFI model and IFS model, and cost per unit time in the long run. Further, this work can apply as a case study.

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